A Fully Implicit Scheme for the Barotropic Primitive Equations

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ABSTRACT

An efficient implicit finite-difference method is developed and tested for a global barotropic model. The scheme requires at each time step the solution of only one-dimensional block-tridiagonal linear systems. This additional computation is offset by the use of a time step chosen independently of the mesh spacing. The method is second-order accurate in time and fourth-order accurate in space. Our experience indicates that this implicit method is practical for numerical simulation on fine meshes.

1. Introduction

The time step of current methods for integrating the global baroclinic primitive equations, including explicit and semi-implicit schemes based on finite-difference, finite-element and spectral methods, is limited by a numerical stability criterion. For explicit finite-difference schemes, the Courant–Friedrichs–Lewy (CFL) stability criterion requires that the space-time grid must resolve all wave motions governed by the differential equations. Thus the time step Δt is restricted by the inequality

$$\Delta t \leqslant \Delta s_{\min}/c_{\max}, \tag{1.1}$$

where $\Delta s_{\min}$ denotes the minimum distance between grid points and $c_{\max}$ is the maximum phase speed. Suitable versions of this inequality hold for other current methods. By controlling the size of the time step, the CFL condition limits the efficiency of numerical models.

The CFL condition is particularly restrictive for explicit finite-difference schemes for two reasons. First, on a uniform latitude–longitude grid the mesh spacing becomes very small near the poles. The time step is therefore limited by the minuteness of Δs over only a small region. Second, the waves with the largest phase speeds are external gravity waves, which propagate approximately ten times faster than the waves which carry most of the atmosphere's synoptic-scale energy. Several techniques are used in practice to increase the permissible time step. Some numerical schemes use Fourier filtering near the poles to avoid the restriction due to grid-point convergence. Semi-implicit schemes remove the limitation imposed by gravity waves, by treating implicitly the terms in the differential equations which give rise to these waves.

An alternative approach is to use a fully implicit numerical method. Implicit methods are usually unconditionally stable: the size of the time step is not limited by any stability criterion. Instead, the time step is chosen solely on the basis of accuracy requirements. That is, if ΔT is the smallest time interval over which significant changes occur in the flow, then ΔT also suffices as the time step; the maximum time step given by formula (1.1) is generally far smaller than this ΔT. If a time step much larger than ΔT is used, of course, one cannot expect an accurate integration (Beam and Warming, 1976).

Accurate forecasting also requires fine spatial meshes, regardless of the numerical scheme, and herein lies the potential advantage of implicit methods. Explicit and semi-implicit methods require a decreasing time step as spatial resolution is increased, while for implicit methods the time step can be held fixed. For fine meshes, one can expect implicit schemes to be more efficient than alternative schemes.

The difficulty with implicit methods is that they generally require the solution of a large system of algebraic equations at each time step. The size of the system is usually equal to the number of degrees of freedom of the model. For an implicit scheme to be practical for numerical weather prediction, efficient computational methods must be developed. This has been accomplished in many areas of application by use of alternating-direction-implicit (ADI) methods (Douglas, 1955; Peaceman and Rachford, 1955; Gustafsson, 1971) and fractional-step methods (Strang, 1968; Yanenko, 1971; Marchuk, 1974). In these methods, the dimensionality of the problem is reduced by separating it into smaller problems, each involving only one spatial dimension.

Beam and Warming (1976) and Briley and McDonald (1977) have introduced variants of the ADI technique which are simpler to apply in practice than the conventional ADI method and generally more accurate than fractional-step methods. These newer methods are based on a factorization of the implicit operator into one-dimensional operators, and yield second-order accuracy in time and fourth-order accuracy in space. The schemes are efficient because they require only the solution of one-dimensional block-tridiagonal linear systems. The applicability of these factorization methods has been demonstrated for a variety of meteorological models. For example, Navon and Riphagen (1979) have developed a Beam–Warming-like scheme for a limited-area shallow-water model. Gilliland (1981) has developed a global barotropic model based on the formulation of Briley and McDonald (1977). Isaacson *et al.* (1979) and Isaacson and Marchesin (1980) have reported results on preliminary versions of our scheme for the global barotropic equations. Factorization methods are less straightforward to apply to hydrostatic baroclinic models because of the nonlocal vertical coupling of layers introduced by the hydrostatic assumption. Marchesin (1984) describes a factorization technique for a hydrostatic baroclinic model of a height-latitude atmospheric cross section.

Many of the difficulties in developing efficient implicit methods for the global baroclinic equations arise in two horizontal dimensions. The purpose of this paper is to show how to resolve these difficulties. We introduce an efficient and accurate implicit method for the global shallow-water equations. Our approach is based on that of Beam and Warming (1976), hereinafter the BW method.

In Section 2, we describe the essentials of the BW method. The shallow-water model is developed in Section 3, and in Section 4 numerical experiments demonstrating accuracy and stability are reported. Conclusions are drawn in Section 5.

2. Some simple examples

a. A one-dimensional scalar equation

To explain the essential features and properties of the BW method, it is illustrated first by means of examples. We begin by considering the simple scalar equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \tag{2.1}$$

on the interval $0 \le x \le L$, with the periodic boundary condition

$$u(0, t) = u(L, t). \tag{2.2}$$

In Eq. (2.1), $f(u)$ denotes an arbitrary nonlinear function of u. If $f(u) = u^2/2$, for example, then Eq. (2.1) represents the simple nonlinear advection equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0. \tag{2.3}$$

The version of the BW scheme considered in this paper can be regarded as a variant of the standard Crank–Nicolson scheme. The latter is an implicit time-discretization method, and perhaps the simplest one with second-order accuracy in time. The Crank–Nicolson discretization for Eq. (2.1) is written

$$\frac{u^{n+1} - u^n}{\Delta t} + \frac{\partial}{\partial x}\left[\frac{f(u^{n+1}) + f(u^n)}{2}\right] = 0, \tag{2.4}$$

or

$$u^{n+1} + \frac{1}{2}\Delta t \frac{\partial}{\partial x} f(u^{n+1}) = u^n - \frac{1}{2}\Delta t \frac{\partial}{\partial x} f(u^n), \tag{2.5}$$

where the superscript n is the time index and Δt the time increment, $t = n\Delta t$. It is clear that the scheme is second-order accurate in time, since the difference and the average in Eq. (2.4) are centered at the same time, $(n + ½)\Delta t$. Discretizing the spatial differential operator $\partial/\partial x$ in Eq. (2.5) yields a system of algebraic equations to be solved for u^{n+1} at each time step; in explicit methods, u^{n+1} is determined directly from u^m at previous times $m \le n$. In the case of (2.5), the system would be nonlinear, since the function $f(u)$ is generally nonlinear. An appropriate linearization of scheme (2.5) is desirable, because the necessity of solving nonlinear systems would render the scheme impractical compared to alternative explicit methods.

In one spatial dimension, the BW scheme is simply a linearization of the standard Crank–Nicolson scheme which preserves second-order temporal accuracy. Since

$$f(u^{n+1}) = f(u^n) + a^n(u^{n+1} - u^n) + O(\Delta t^2), \tag{2.6}$$

provided f and u are sufficiently differentiable, where a^n denotes

$$a^n = \left.\frac{df(u)}{du}\right|_{u=u^n}, \tag{2.7}$$

it follows from Eq. (2.4) that

$$\frac{u^{n+1} - u^n}{\Delta t} + \frac{\partial}{\partial x}\left[\frac{1}{2} a^n(u^{n+1} - u^n) + f(u^n) + O(\Delta t^2)\right] = 0, \tag{2.8}$$

or

$$(u^{n+1} - u^n) + \frac{\Delta t}{2}\frac{\partial}{\partial x}[a^n(u^{n+1} - u^n)] = -\Delta t \frac{\partial}{\partial x} f(u^n) + O(\Delta t^3). \tag{2.9}$$

The resulting scheme,

$$u^{n+1} + \frac{\Delta t}{2}\frac{\partial}{\partial x}[a^n u^{n+1}] = u^n + \frac{\Delta t}{2}\frac{\partial}{\partial x}[a^n u^n] - \Delta t \frac{\partial}{\partial x} f(u^n) \tag{2.10}$$

is second-order accurate since only an $O(\Delta t^3)$ term is being neglected. This scheme requires only the solution of *linear* algebraic equations, because the left-hand side of Eq. (2.10) is linear in u^{n+1}.

The linearized scheme can be written in a much cleaner way by noticing that u^{n+1} appears in Eq. (2.8) only as the difference $u^{n+1} - u^n$. Defining the forward difference operator Δ by

$$\Delta u^n = u^{n+1} - u^n, \tag{2.11}$$

we can rewrite Eq. (2.10) as

$$\Delta u^n + \frac{\Delta t}{2}\frac{\partial}{\partial x}[a^n \Delta u^n] = -\Delta t \frac{\partial}{\partial x} f(u^n). \tag{2.12}$$

Finally, in operator notation, using I as the identity operator,

$$\left[I + \frac{\Delta t}{2}\frac{\partial}{\partial x} a^n\right]\Delta u^n = -\Delta t \frac{\partial}{\partial x} f(u^n), \tag{2.13}$$

where the operator $\partial/\partial x$ always implies an operation on all factors appearing to its right. Together with the update equation

$$u^{n+1} = u^n + \Delta u^n, \tag{2.14}$$

Eq. (2.13) represents the time discretization of the BW scheme for Eq. (2.1). The BW scheme uses a Δ-*formulation:* it is Δu^n which is solved for, rather than u^{n+1} itself.

To define completely the BW scheme for Eq. (2.1), it remains to discretize the operator $\partial/\partial x$. Suppose, first of all, that we use the standard second-order explicit formula

$$\frac{\partial}{\partial x} w_i \sim \frac{1}{2\Delta x}(w_{i+1} - w_{i-1}), \tag{2.15}$$

where i is the spatial index and Δx the spatial increment, $\Delta x = L/I$, with I being the number of grid points. Then the linear system represented by Eq. (2.13) is

$$\mathbf{My} = \mathbf{b}, \tag{2.16}$$

where $\mathbf{M}$ is the cyclic tridiagonal matrix

$$\mathbf{M} = \begin{bmatrix} \alpha_1 & \gamma_1 & & & \beta_1 \\ \beta_2 & \alpha_2 & \gamma_2 & 0 & \\ & \beta_3 & \ddots & \ddots & \\ & 0 & \ddots & \ddots & \gamma_{I-1} \\ \gamma_I & & & \beta_I & \alpha_I \end{bmatrix}, \tag{2.17}$$

with elements

$$\alpha_i = 1, \tag{2.18a}$$

$$\gamma_i = \frac{1}{4}\frac{\Delta t}{\Delta x}\frac{df(u)}{du}\bigg|_{u=u_{i+1}^n}, \tag{2.18b}$$

$$\beta_i = -\frac{1}{4}\frac{\Delta t}{\Delta x}\frac{df(u)}{du}\bigg|_{u=u_{i-1}^n}, \tag{2.18c}$$

for $i = 1, 2, \ldots, I$, and where the vectors $\mathbf{y}$ and $\mathbf{b}$ have components

$$y_i = \Delta u_i^n, \quad i = 1, 2, \ldots, I, \tag{2.19}$$

$$b_1 = -\frac{1}{2}\frac{\Delta t}{\Delta x}[f(u_2^n) - f(u_I^n)], \tag{2.20a}$$

$$b_i = -\frac{1}{2}\frac{\Delta t}{\Delta x}[f(u_{i+1}^n) - f(u_{i-1}^n)],$$
$$i = 2, 3, \ldots, I-1, \tag{2.20b}$$

$$b_I = -\frac{1}{2}\frac{\Delta t}{\Delta x}[f(u_1) - f(u_{I-1})]. \tag{2.20c}$$

The elements β_1 and γ_I appearing in the corners of the matrix $\mathbf{M}$, Eq. (2.17), arise from the periodic boundary condition (2.2). The cyclic tridiagonal system (2.16) can be efficiently solved by LU decomposition in only $O(I)$ operations (Isaacson and Keller, 1966).

How efficient is the scheme we have defined? Comparing Eqs. (2.13, 2.14) with the standard leapfrog scheme, which for Eq. (2.1) can be written

$$u^{n+1} = u^{n-1} - 2\Delta t \frac{\partial}{\partial x} f(u^n), \tag{2.21}$$

we see that both schemes require calculation of the tendency $\Delta t \partial f(u^n)/\partial x$. If the same discretization of $\partial/\partial x$ is used for each scheme, then the only difference between the two schemes is that the BW algorithm requires setting up the matrix elements (2.18) and solving the resulting cyclic tridiagonal system (2.16). Both of these calculations involve $O(I)$ operations, as does calculation of the tendency.

Thus, the ratio between the amount of arithmetic per time step required by the implicit and explicit schemes is small and independent of the number of grid points I. The key to the wide applicability of the BW scheme is that it also has this property for equations in more than one dimension, as will be seen in the following subsection. The underlying idea, of course, is that the increased time step permitted by the implicit scheme should offset the increased computational effort involved per time step.

Suppose now that we would like to achieve fourth-order spatial accuracy. The usual way to do this for explicit schemes is to use the fourth-order accurate formula

$$\frac{\partial}{\partial x} w_i \sim \frac{4}{3}\left[\frac{1}{2\Delta x}(w_{i+1} - w_{i-1})\right] - \frac{1}{3}\left[\frac{1}{4\Delta x}(w_{i+2} - w_{i-2})\right]. \tag{2.22}$$

Use of this formula in the leapfrog scheme (2.21) would involve about twice the computational effort as use of the second-order formula (2.15). The situation appears much worse for the implicit scheme: since formula (2.22) involves five grid points, it would yield a cyclic *pentadiagonal* matrix in place of the cyclic tridiagonal matrix **M** in Eq. (2.17). Solution of pentadiagonal systems generally requires about four times the arithmetic that solution of tridiagonal systems requires.

A remarkable fact about many implicit schemes, however, is that their spatial accuracy can actually be increased with almost no additional computational effort. This is accomplished by implicit spatial discretization. To achieve fourth-order spatial accuracy for the implicit scheme (2.13) in an efficient manner, one can use the fourth-order accurate *Padé derivative formula*

$$\frac{1}{6}\frac{\partial}{\partial x}w_{i+1}+\frac{2}{3}\frac{\partial}{\partial x}w_i+\frac{1}{6}\frac{\partial}{\partial x}w_{i-1}\sim\frac{1}{2\Delta x}(w_{i+1}-w_{i-1}); \quad (2.23)$$

[cf. Beam and Warming (1976)]. This formula is compact in that it achieves fourth-order accuracy with the fewest possible grid points, namely, three. It is also equivalent to finite-element discretization based on the one-dimensional chapeau function (Staniforth and Daley, 1979, and references therein; Swartz and Wendroff, 1974).

As a discretization of the operator $\partial/\partial x$, the Padé formula is implicit: calculation of the derivative of a grid function requires the solution of a constant-coefficient tridiagonal linear system. For a periodic function, the system would be cyclic tridiagonal, of the form (2.16, 2.17), with

$$\alpha_i=\frac{2}{3},\quad \beta_i=\gamma_i=\frac{1}{6}, \quad (2.24)$$

for

$$i=1,2,\ldots,I,$$

and

$$b_1=\frac{1}{2\Delta x}(w_2-w_I), \quad (2.25a)$$

$$b_I=\frac{1}{2\Delta x}(w_1-w_{I-1}), \quad (2.25b)$$

$$b_i=\frac{1}{2\Delta x}(w_{i+1}-w_{i-1}), \quad (2.25c)$$

for $i=2,3,\ldots,I-1$. This system can be solved very efficiently, as shown by Temperton (1975).

The Padé formula is highly efficient when implemented for the BW scheme. In one dimension, in fact, it is not even necessary to solve a system to calculate $\partial f(u^n)/\partial x$. To demonstrate this, we first write the formula in operator notation. We define the averaging and differencing operators μ and δ as

$$\mu w_i=\frac{1}{2}(w_{i+1/2}+w_{i-1/2}), \quad (2.26a)$$

$$\delta w_i=w_{i+1/2}-w_{i-1/2}. \quad (2.26b)$$

Equation (2.23) can be written symbolically as

$$(1+\delta^2/6)\frac{\partial}{\partial x}w_i\sim\frac{1}{\Delta x}\mu\delta w_i, \quad (2.27)$$

or, more concisely,

$$\frac{\partial}{\partial x}\sim\frac{1}{\Delta x}\frac{\mu\delta}{1+\delta^2/6}. \quad (2.28)$$

Substituting Eq. (2.28) into the implicit scheme (2.13), we have

$$\left[I+\frac{\Delta t}{2\Delta x}\frac{\mu\delta}{1+\delta^2/6}a^n\right]\Delta u^n=-\frac{\Delta t}{\Delta x}\frac{\mu\delta}{1+\delta^2/6}f(u^n), \quad (2.29)$$

or, clearing the operator denominator,

$$\left[I+\frac{\delta^2}{6}+\frac{\Delta t}{2\Delta x}\mu\delta a^n\right]\Delta u^n=-\frac{\Delta t}{\Delta x}\mu\delta f(u^n). \quad (2.30)$$

Observe that the operators apply to all factors appearing to their right. Substitution of the definitions (2.26) of μ and δ shows that the fourth-order accurate scheme (2.30) is again a cyclic tridiagonal system (2.16–2.20), but with the matrix coefficients (2.18) replaced by

$$\alpha_i=\frac{2}{3}, \quad (2.31a)$$

$$\gamma_i=\frac{1}{6}+\frac{1}{4}\frac{\Delta t}{\Delta x}\frac{df(u)}{du}\bigg|_{u=u_{i+1}{}^n}, \quad (2.31b)$$

$$\beta_i=\frac{1}{6}-\frac{1}{4}\frac{\Delta t}{\Delta x}\frac{df(u)}{du}\bigg|_{u=u_{i-1}{}^n}. \quad (2.31c)$$

Owing to the fortuitous cancellation of the operator denominators in Eq. (2.29), for the one-dimensional case, the implicit scheme achieves fourth-order spatial accuracy at no additional cost over the second-order version.

Equation (2.30) is the complete BW scheme for the simple scalar equation (2.1). We have shown that it is second-order accurate in time and fourth-order accurate in space. Computational efficiency was achieved with no loss of temporal accuracy by the linearization (2.6), and fourth-order spatial accuracy obtained efficiently by use of the Padé derivative formula (2.28). The scheme is linearly stable since for linear equations it is identical to the Crank–Nicolson scheme, which is stable (Isaacson and Keller, 1966). Next we will show how the scheme generalizes to equations in more than one spatial dimension.

b. A two-dimensional scalar equation

Consider now the two-dimensional model equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) + \frac{\partial}{\partial y} g(u) = 0 \quad (2.32)$$

on the square $0 \le x \le L$, $0 \le y \le L$, again with periodic boundary conditions

$$u(0, y, t) = u(L, y, t), \quad (2.33a)$$

$$u(x, 0, t) = u(x, L, t). \quad (2.33b)$$

By analogy with Eq. (2.13), and using the linearization

$$f(u^{n+1}) = f^n + a^n(u^{n+1} - u^n) + O(\Delta t^2), \quad (2.34a)$$

$$g(u^{n+1}) = g^n + b^n(u^{n+1} - u^n) + O(\Delta t^2), \quad (2.34b)$$

where

$$f^n = f(u^n), \qquad g^n = g(u^n), \quad (2.35a,b)$$

$$a^n = \left.\frac{df(u)}{du}\right|_{u=u^n}, \quad b^n = \left.\frac{dg(u)}{du}\right|_{u=u^n}. \quad (2.36a,b)$$

A linearized Crank–Nicolson scheme for (2.32) having second-order temporal accuracy is

$$\left[I + \frac{\Delta t}{2}\frac{\partial}{\partial x} a^n + \frac{\Delta t}{2}\frac{\partial}{\partial y} b^n\right]\Delta u^n = -\Delta t\left(\frac{\partial}{\partial x} f^n + \frac{\partial}{\partial y} g^n\right). \quad (2.37)$$

The central difficulty here is that of computational complexity: when $\partial/\partial x$ and $\partial/\partial y$ are discretized in (2.37), the operator on the left-hand side becomes an $IJ \times IJ$ matrix, where I and J are the numbers of grid points in the x and y directions, respectively. Such a large linear system would render the scheme impractical, regardless of whether compact formulas are used to discretize $\partial/\partial x$ and $\partial/\partial y$. The crucial idea in the BW method is to factorize the operator on the left-hand side of (2.37) into two operators, each involving only one spatial dimension:

$$\left[I + \frac{\Delta t}{2}\frac{\partial}{\partial x} a^n + \frac{\Delta t}{2}\frac{\partial}{\partial y} b^n\right]\Delta u^n$$

$$= \left[I + \frac{\Delta t}{2}\frac{\partial}{\partial x} a^n\right]\left[I + \frac{\Delta t}{2}\frac{\partial}{\partial y} b^n\right]\Delta u^n - E^n, \quad (2.38a)$$

where

$$E^n = \left[\left(\frac{\Delta t}{2}\right)^2 \frac{\partial}{\partial x} a^n \frac{\partial}{\partial y} b^n\right]\Delta u^n. \quad (2.38b)$$

Since $\Delta u^n = O(\Delta t)$, the factorization error term E^n in Eq. (2.38) is $O(\Delta t^3)$. By neglecting this term, we obtain from Eq. (2.37) a scheme

$$\left[I + \frac{\Delta t}{2}\frac{\partial}{\partial x} a^n\right]\left[I + \frac{\Delta t}{2}\frac{\partial}{\partial y} b^n\right]\Delta u^n = -\Delta t\left(\frac{\partial}{\partial x} f^n + \frac{\partial}{\partial y} g^n\right), \quad (2.39)$$

which is still second-order accurate in time. This is the BW time discretization for the two-dimensional equation (2.32). Discretizing the operators $\partial/\partial x$ and $\partial/\partial y$ yields one-dimensional linear problems along lines of constant x and y, which can be solved efficiently, as already noted.

The factorized scheme is implemented by solving first

$$\left[I + \frac{\Delta t}{2}\frac{\partial}{\partial x} a^n\right]\widehat{\Delta u} = -\Delta t\left(\frac{\partial}{\partial x} f^n + \frac{\partial}{\partial y} g^n\right) \quad (2.40a)$$

for the auxiliary quantity $\widehat{\Delta u}$, and then solving

$$\left[I + \frac{\Delta t}{2}\frac{\partial}{\partial y} b^n\right]\Delta u^n = \widehat{\Delta u}. \quad (2.40b)$$

As in the one-dimensional case, the scheme can be made fourth-order accurate in space by using the Padé derivative formula. One first calculates $\partial f_{ij}{}^n/\partial x$ at each grid point (i, j) by solving J linear systems of dimension I, of the form (2.16, 2.17, 2.24, 2.25), and then calculates $\partial g_{ij}{}^n/\partial y$ at each grid point by solving I similar systems of dimension J. Adding these two results gives the tendency $-\partial u^n/\partial t$. Applying the Padé formula to discretize $\partial/\partial x$ on the left-hand-side of (2.40a), one then solves J cyclic tridiagonal systems

$$\left[I + \frac{\delta^2}{6} + \frac{\Delta t}{2\Delta x}\mu\delta a^n\right]\widehat{\Delta u} = \left[I + \frac{\delta^2}{6}\right]\Delta t\frac{\partial u^n}{\partial t}. \quad (2.41)$$

Similarly, the Padé formula is used to discretize $\partial/\partial y$ in Eq. (2.40b), yielding Δu^n at all grid points by solving I cyclic tridiagonal systems of dimension J.

Linear stability of the two-dimensional BW scheme is easily demonstrated. If $f(u) = au$ and $g(u) = bu$, where a and b are constants, then the BW scheme is

$$\left[I + \frac{\Delta t}{2}\frac{\partial}{\partial x} a\right]\left[I + \frac{\Delta t}{2}\frac{\partial}{\partial y} b\right](u^{n+1} - u^n) = -\Delta t\left(a\frac{\partial}{\partial x} u^n + b\frac{\partial}{\partial y} u^n\right), \quad (2.42)$$

with $\partial/\partial x$ and $\partial/\partial y$ discretized by the Padé formula. The scheme is linearly stable if $|\rho| \le 1$ for all solutions of the form

$$u_{ij}{}^n = \rho^n e^{i(\alpha x_i + \beta y_j)}, \quad (2.43)$$

e.g., Richtmyer and Morton (1967). To see that this is the case, first notice that the Padé operator applied to $e^{i\alpha x_i}$ gives

$$\frac{1}{\Delta x}\frac{\mu\delta}{1+\delta^2/6}e^{i\alpha x_i} = i\xi e^{i\alpha x_i}, \quad (2.44)$$

where

$$\xi = \frac{\sin\alpha\Delta x}{\Delta x(2+\cos\alpha\Delta x)/3} \quad (2.45a)$$

is a real number. Similarly, we define

$$\eta = \frac{\sin\beta\Delta y}{\Delta y(2+\cos\beta\Delta y)/3}. \quad (2.45b)$$

Substituting (2.43) into (2.42) then yields the equation

$$\left(1+\frac{\Delta t}{2}i\xi a\right)\left(1+\frac{\Delta t}{2}i\eta b\right)(\rho-1) = -\Delta t(ai\xi + bi\eta). \quad (2.46)$$

Solving for the amplification factor ρ, one finds that

$$\rho = \frac{1-\Delta t^2 a\xi b\eta/4 - i\Delta t(a\xi+b\eta)/2}{1-\Delta t^2 a\xi b\eta/4 + i\Delta t(a\xi+b\eta)/2}. \quad (2.47)$$

The numerator and denominator are complex conjugates since ξ and η are real, and therefore $|\rho| = 1$; that is, the scheme is stable and nondissipative.

Actually, a more general statement can be made. Notice that the stability proof did not rely on the exact form of ξ and η, Eqs. (2.45a, b); it used only the fact that ξ and η are real. Therefore the scheme is stable for any skew-Hermitian matrices representing discrete approximations of the operators $\partial/\partial x$ and $\partial/\partial y$, i.e., for any discretizations D_x and D_y such that

$$D_x e^{i\alpha x} = i\xi e^{i\alpha x}, \quad (2.48a)$$

$$D_y e^{i\beta y} = i\eta e^{i\beta y}, \quad (2.48b)$$

with ξ and η real. This includes most discretizations used in practice, as they usually inherit the property that $\partial/\partial x$ itself is skew-Hermitian:

$$\frac{\partial}{\partial x}e^{i\alpha x} = i\alpha e^{i\alpha x}. \quad (2.49)$$

In particular, this includes the standard centered second-order discretization (2.15), i.e., $D_x = (\Delta x)^{-1}\mu\delta$, for which $\xi = (\Delta x)^{-1}\sin\alpha\Delta x$.

3. The shallow-water equations on the sphere

a. The Beam–Warming scheme

We now describe our implementation of the BW scheme for a global barotropic model. The shallow-water equations are written in flux form as

$$\frac{\partial h}{\partial t} + \frac{1}{a\cos\phi}\frac{\partial hu}{\partial\lambda} + \frac{1}{a}\frac{\partial hv}{\partial\phi} - \frac{\tan\phi}{a}hv = 0, \quad (3.1a)$$

$$\frac{\partial hu}{\partial t} + \frac{1}{a\cos\phi}\frac{\partial}{\partial\lambda}\left(hu^2+\frac{1}{2}gh^2\right) + \frac{1}{a}\frac{\partial huv}{\partial\phi} - fhv - \frac{2\tan\phi}{a}huv + \frac{gh}{a\cos\phi}\frac{\partial h_s}{\partial\lambda} = 0, \quad (3.1b)$$

$$\frac{\partial hv}{\partial t} + \frac{1}{a\cos\phi}\frac{\partial huv}{\partial\lambda} + \frac{1}{a}\frac{\partial}{\partial\phi}\left(hv^2+\frac{1}{2}gh^2\right) + fhu + \frac{\tan\phi}{a}(hu^2-hv^2) + \frac{gh}{a}\frac{\partial h_s}{\partial\phi} = 0. \quad (3.1c)$$

Here h is the thickness of the fluid layer, h_s the height of the ground, u and v the eastward and northward velocity components; ϕ is the latitude and λ the longitude; a is the radius of the earth, g the acceleration due to gravity, and $f = 2\Omega\sin\phi$ the Coriolis parameter. Note that $h + h_s$ is the height of the free surface.

For convenience we also write the equations in vector form. Introducing the momentum variables

$$U = hu, \quad V = hv, \quad (3.2a, b)$$

and the column vector

$$\mathbf{W} = (h, U, V)^T, \quad (3.3)$$

Eqs. (3.1) can be written

$$\frac{\partial}{\partial t}\mathbf{W} + \frac{\partial}{\partial\lambda}\mathbf{F}(\mathbf{W}) + \frac{\partial}{\partial\phi}\mathbf{G}(\mathbf{W}) + \mathbf{K}(\mathbf{W}) + \mathbf{L}(\mathbf{W}) = 0, \quad (3.4)$$

where

$$\mathbf{F} = \frac{1}{a\cos\phi}\begin{bmatrix} U \\ U^2/h + gh^2/2 \\ UV/h \end{bmatrix}, \quad (3.5a)$$

$$\mathbf{G} = \frac{1}{a}\begin{bmatrix} V \\ UV/h \\ V^2/h + gh^2/2 \end{bmatrix}, \quad (3.5b)$$

$$\mathbf{K} = \begin{bmatrix} -\frac{1}{a}(\tan\phi)V \\ -fV - \frac{2}{a}(\tan\phi)\frac{UV}{h} + \frac{gh}{a\cos\phi}\frac{\partial h_s}{\partial\lambda} \\ 0 \end{bmatrix}, \quad (3.6a)$$

$$\mathbf{L} = \begin{bmatrix} 0 \\ 0 \\ fU + \frac{1}{a}(\tan\phi)\frac{U^2-V^2}{h} + \frac{gh}{a}\frac{\partial h_s}{\partial\phi} \end{bmatrix}. \quad (3.6b)$$

Analogously with Eq. (2.5), the Crank–Nicolson scheme for Eq. (3.4) is

$$\mathbf{W}^{n+1} + \frac{1}{2}\Delta t\left(\frac{\partial}{\partial\lambda}\mathbf{F}^{n+1} + \frac{\partial}{\partial\phi}\mathbf{G}^{n+1} + \mathbf{K}^{n+1} + \mathbf{L}^{n+1}\right) = \mathbf{W}^n - \frac{1}{2}\Delta t\left(\frac{\partial}{\partial\lambda}\mathbf{F}^n + \frac{\partial}{\partial\phi}\mathbf{G}^n + \mathbf{K}^n + \mathbf{L}^n\right), \quad (3.7)$$

where $\mathbf{F}^{n+1} = \mathbf{F}(\mathbf{W}^{n+1}), \ldots$. The appropriate linearizations are

$$\mathbf{F}^{n+1} = \mathbf{F}^n + A^n(\mathbf{W}^{n+1} - \mathbf{W}^n) + O(\Delta t^2), \quad (3.8a)$$

$$\mathbf{G}^{n+1} = \mathbf{G}^n + B^n(\mathbf{W}^{n+1} - \mathbf{W}^n) + O(\Delta t^2), \quad (3.8b)$$

$$\mathbf{K}^{n+1} = \mathbf{K}^n + C^n(\mathbf{W}^{n+1} - \mathbf{W}^n) + O(\Delta t^2), \quad (3.8c)$$

$$\mathbf{L}^{n+1} = \mathbf{L}^n + D^n(\mathbf{W}^{n+1} - \mathbf{W}^n) + O(\Delta t^2), \quad (3.8d)$$

where $\mathbf{A}^n$, $\mathbf{B}^n$, $\mathbf{C}^n$ and $\mathbf{D}^n$ are the Jacobian matrices

$$(A^n)_{ij} = \left.\frac{\partial F_i}{\partial W_j}\right|_{\mathbf{W}=\mathbf{W}^n}, \quad (B^n)_{ij} = \left.\frac{\partial G_i}{\partial W_j}\right|_{\mathbf{W}=\mathbf{W}^n}, \quad (3.9a,b)$$

$$(C^n)_{ij} = \left.\frac{\partial K_i}{\partial W_j}\right|_{\mathbf{W}=\mathbf{W}^n}, \quad (D^n)_{ij} = \left.\frac{\partial L_i}{\partial W_j}\right|_{\mathbf{W}=\mathbf{W}^n}; \quad (3.9c,d)$$

for example, $\mathbf{A}$ is given by

$$\mathbf{A} = \frac{1}{a\cos\phi}\begin{bmatrix} 0 & 1 & 0 \\ -U^2/h^2 + gh & 2U/h & 0 \\ -UV/h^2 & V/h & U/h \end{bmatrix}. \quad (3.10)$$

By analogy with Eq. (2.37), the linearized Crank–Nicolson scheme in Δ-formulation is then

$$\left[I + \frac{\Delta t}{2}\left(\frac{\partial}{\partial\lambda}A^n + \frac{\partial}{\partial\phi}B^n + C^n + D^n\right)\right]\Delta\mathbf{W}^n = -\Delta t\left(\frac{\partial}{\partial\lambda}\mathbf{F}^n + \frac{\partial}{\partial\phi}\mathbf{G}^n + \mathbf{K}^n + \mathbf{L}^n\right), \quad (3.11)$$

where I is the 3×3 identity matrix. We introduce the factorization

$$\left[I + \frac{\Delta t}{2}\left(\frac{\partial}{\partial\lambda}A^n + C^n\right)\right]\left[I + \frac{\Delta t}{2}\left(\frac{\partial}{\partial\phi}B^n + D^n\right)\right]\Delta\mathbf{W}^n = -\Delta t\left(\frac{\partial}{\partial\lambda}\mathbf{F}^n + \frac{\partial}{\partial\phi}\mathbf{G}^n + \mathbf{K}^n + \mathbf{L}^n\right), \quad (3.12)$$

maintaining second-order temporal accuracy. As discussed in Section 2b, Eq. (3.12) represents two sets of linear systems to be solved: first the λ-sweep

$$\left[I + \frac{\Delta t}{2}\left(\frac{\partial}{\partial\lambda}A^n + C^n\right)\right]\widehat{\Delta\mathbf{W}} = -\Delta t\left(\frac{\partial}{\partial\lambda}\mathbf{F}^n + \frac{\partial}{\partial\phi}\mathbf{G}^n + \mathbf{K}^n + \mathbf{L}^n\right), \quad (3.13a)$$

and then the ϕ-sweep

$$\left[I + \frac{\Delta t}{2}\left(\frac{\partial}{\partial\phi}B^n + D^n\right)\right]\Delta\mathbf{W}^n = \widehat{\Delta\mathbf{W}}, \quad (3.13b)$$

after which $\mathbf{W}^{n+1}$ is obtained by the simple update

$$\mathbf{W}^{n+1} = \mathbf{W}^n + \Delta\mathbf{W}^n \quad (3.13c)$$

Equation (3.12), or (3.13), represents the BW scheme for the global shallow-water equations. Notice that some choices have been made in the grouping of terms in the factorization. The rationale of this factorization is discussed in Section 3c. Details of the fourth-order spatial discretization are given in Section 3d.

b. The filtered Beam–Warming scheme

The Beam and Warming (1976) scheme is unconditionally linearly stable for hyperbolic systems in one and two dimensions. The possibility of nonlinear instability still does exist for (3.12), and in fact was observed in our first numerical experiments. For this reason, we apply a Shapiro filter (Shapiro, 1970), in a way which does not change the order of accuracy of the scheme.

We apply the fourth-order Shapiro filter to the *increments* $\widehat{\Delta\mathbf{W}}$ and $\Delta\mathbf{W}^n$, rather than to $\mathbf{W}^{n+1}$ itself. That is, we replace algorithm (3.13) by

$$\left[I + \frac{\Delta t}{2}\left(\frac{\partial}{\partial\lambda}A^n + C^n\right)\right]\Delta\mathbf{W}_{(1)} = -\Delta t\left(\frac{\partial}{\partial\lambda}\mathbf{F}^n + \frac{\partial}{\partial\phi}\mathbf{G}^n + \mathbf{K}^n + \mathbf{L}^n\right), \quad (3.14a)$$

$$\Delta\mathbf{W}_{(2)} = \left(1 - \tfrac{1}{4}\delta_\lambda^2\right)\left(1 + \tfrac{1}{4}\delta_\lambda^2\right)\Delta\mathbf{W}_{(1)}, \quad (3.14b)$$

$$\left[I + \frac{\Delta t}{2}\left(\frac{\partial}{\partial\phi}B^n + D^n\right)\right]\Delta\mathbf{W}_{(3)} = \Delta\mathbf{W}_{(2)}, \quad (3.14c)$$

$$\Delta\mathbf{W}^n = \left(1 - \tfrac{1}{4}\delta_\phi^2\right)\left(1 + \tfrac{1}{4}\delta_\phi^2\right)\Delta\mathbf{W}_{(3)}, \quad (3.14d)$$

$$\mathbf{W}^{n+1} = \mathbf{W}^n + \Delta\mathbf{W}^n. \quad (3.14e)$$

Here δ_λ and δ_ϕ represent the difference operator δ, Eq. (2.26b), applied in the λ- and ϕ-directions, respectively. Thus the operators $(1 - \delta_\lambda^2/4)(1 + \delta_\lambda^2/4)$ and $(1 - \delta_\phi^2/4)(1 + \delta_\phi^2/4)$ are the usual fourth-order zonal and meridional Shapiro operators.

No stability problems were observed with the modified scheme (3.14), and hence we adopted it as the basic implicit scheme for the barotropic model. Fourier smoothing, which can have undesirable consequences (Takacs and Balgovind, 1983), is not necessary in our scheme.

The Shapiro smoothing applied in our scheme is mild, since it is applied to the increments rather than to $\mathbf{W}^{n+1}$ itself. In particular, it does not affect the spatial (or temporal) order of accuracy of the scheme: from Eqs. (3.14b, d) it follows that

$$\Delta\mathbf{W}_{(2)} = \Delta\mathbf{W}_{(1)} + O[(\Delta\lambda)^4\Delta t], \quad (3.15a)$$

$$\Delta\mathbf{W}^n = \Delta\mathbf{W}_{(3)} + O[(\Delta\phi)^4\Delta t], \quad (3.15b)$$

so that, assuming $\Delta t = O(\Delta\lambda) = O(\Delta\phi)$, fourth-order spatial accuracy is preserved. Were the Shapiro filter applied directly to $\mathbf{W}^{n+1}$, as is the common practice, a higher-order Shapiro filter would be needed to maintain the order of accuracy. A further advantage of applying the filter to the increments is that the filter then has no effect on the computation of steady states, and hence very little effect on low-frequency Rossby-type motions.

c. Choice of factorization

Several choices have been made in the factorization (3.12). First, the factors are ordered so that the λ-sweep precedes the ϕ-sweep, rather than vice versa. That is, we have chosen to factor the linearized Crank–Nicolson operator (3.11) as

$$\left[I + \frac{\Delta t}{2}\left(\frac{\partial}{\partial\lambda}A^n + \frac{\partial}{\partial\phi}B^n + C^n + D^n\right)\right]\Delta\mathbf{W}^n = \left[I + \frac{\Delta t}{2}\left(\frac{\partial}{\partial\lambda}A^n + C^n\right)\right]\left[I + \frac{\Delta t}{2}\left(\frac{\partial}{\partial\phi}B^n + D^n\right)\right] \times \Delta\mathbf{W}^n + O(\Delta t^3); \quad (3.16a)$$

an equally accurate factorization would appear to be

$$\left[I + \frac{\Delta t}{2}\left(\frac{\partial}{\partial\lambda}A^n + \frac{\partial}{\partial\phi}B^n + C^n + D^n\right)\right]\Delta\mathbf{W}^n = \left[I + \frac{\Delta t}{2}\left(\frac{\partial}{\partial\phi}B^n + D^n\right)\right]\left[I + \frac{\Delta t}{2}\left(\frac{\partial}{\partial\lambda}A^n + C^n\right)\right] \times \Delta\mathbf{W}^n + O(\Delta t^3), \quad (3.16b)$$

which would interchange the order of the λ-sweep and ϕ-sweep. Second, the undifferentiated terms $\mathbf{C}^n$ and $\mathbf{D}^n$ are grouped in a particular way: $\mathbf{C}^n$ with $\mathbf{A}^n$ and $\mathbf{D}^n$ with $\mathbf{B}^n$.

Actually, the choices we have made are essential for the stability and accuracy of the scheme, as our numerical experiments demonstrated. When the operator in (3.11) is factored, an error is introduced which, even though it is of the same order as the truncation error of the unfactored scheme, can still affect the accuracy and stability of the scheme. Our factorization was chosen with the intention of reducing this factorization error.

For example, the reversed factorization (3.16b) turned out to be unstable, regardless of whether the Shapiro filter was used. No stability problems were observed with scheme (3.14). These contrasting results are probably due to the singularity of the polar coordinate system. In Eq. (3.12), the right-hand side and the first of the two factors on the left-hand side are singular, containing the factor $1/\cos\phi$; the second factor is not singular. Thus, the singularities on the right and left sides of the λ-sweep (3.14a) cancel, producing a nonsingular $\Delta\mathbf{W}_{(1)}$, and then the ϕ-sweep (3.14c) is free of singularities. When the factorization (3.16b) is employed, the ϕ-sweep produces a singular $\Delta\mathbf{W}_{(1)}$, which the following λ-sweep is apparently unable to cancel exactly. This phenomenon is not at variance with the stability results of Beam and Warming (1976), which apply to linear systems on doubly infinite domains.

The grouping of undifferentiated terms was chosen so that the dominant terms in Eq. (3.1b, c), namely, the pressure-gradient and Coriolis terms, appear together. Thus the factorization error is reduced by grouping the zonal pressure gradient terms with the balancing Coriolis term in the λ-sweep (3.14a), and similarly by grouping the meridional pressure gradient terms with the balancing Coriolis term in the ϕ-sweep (3.14c). In our numerical experiments we monitored the time history of total mass, energy, and potential enstrophy, which are conserved by the continuous system (3.1), and verified that this grouping was the most accurate one.

The placement of the remaining undifferentiated terms, namely, the curvature terms involving $\tan\phi$ in $\mathbf{K}$ and $\mathbf{L}$, Eqs. (3.6a, b), was chosen mostly as a matter of convenience. Numerical experiments with other groupings showed no significant effect on the accuracy of the scheme, and no effect on the stability of the scheme. That the treatment of undifferentiated terms generally does not alter the stability of a finite-difference scheme is a standard result, e.g., Richtmyer and Morton (1967, Sections 4.7, 4.8, 5.3, 8.4).

d. Spatial discretization

We discretize the barotropic model on a uniform spherical mesh of $I \times J$ grid points, so that $\Delta\lambda = 2\pi/I$ and $\Delta\phi = \pi/J$. The poles are not grid points.

To implement the scheme efficiently and to avoid excessive searches for information located in widely separated computer memory cells (paging), a data structure which makes the domain appear doubly periodic was used. During operations in which the dependent variables are being accessed most rapidly in the λ-direction, the two-dimensional arrays h, U, V, Δh, ΔU, ΔV are stored with the λ-index first. In this storage order, which we call the IJ representation, the FORTRAN arrays have dimension (I, J). Then when an operation involving most rapid access in the ϕ-direction is required, each array is transposed in such a way that each column represents a full great circle of longitude. Also, to give the wind components the appropriate orientation, the signs are reversed on half the elements of the U, V, ΔU and ΔV arrays, namely, those elements "across the pole" in an "eastern" hemisphere. In the transposed storage order, called the JI representation, the arrays have dimension $(2J, I/2)$. A related technique, aimed toward parallel processors, has been described by Kalnay-Rivas and Takacs (1981).

In detail now, one time step of the implicit algorithm (3.14) is performed as follows. At the beginning of the time step, all arrays are in the JI representation. First, the 3×3 matrices $\mathbf{K}$ and $\mathbf{L}$ in (3.14a) are evaluated at each grid point and added. Next $\partial\mathbf{G}/\partial\phi$ is evaluated, since we are in the JI representation, by use of the Padé formula. As $\mathbf{G}$ is a 3-vector, this involves solving three sets of $I/2$ cyclic tridiagonal systems, each system of size $2J$. These systems can be solved efficiently, as we have already remarked. As the systems are solved one at a time, the ϕ-index is varying the most rapidly, and the current JI representation is the appropriate one.

Since the λ-index varies most rapidly in the next several steps, we now transpose the arrays to the IJ representation. First $\partial\mathbf{F}/\partial\lambda$ is computed at all grid points via the Padé formula, involving the solution of three sets of J cyclic tridiagonal systems, each of size I. Evaluation of the right-hand side of (3.14a), which we denote by $\Delta t\partial\mathbf{W}^n/\partial t$, is now complete.

The next step is to solve the system

$$\left[I + \frac{\Delta t}{2}\left(\frac{\partial}{\partial\lambda}A^n + C^n\right)\right]\Delta\mathbf{W}_{(1)} = \Delta t\,\frac{\partial\mathbf{W}^n}{\partial t} \quad (3.17)$$

which, after substitution of the Padé formula (2.28) for the λ-derivative, is written

$$\left[\left(I + \frac{\delta_\lambda{}^2}{6}\right)\left(I + \frac{\Delta t}{2}C^n\right) + \frac{\Delta t}{2\Delta\lambda}\mu_\lambda\delta_\lambda A^n\right]\Delta\mathbf{W}_{(1)} = \Delta t\left(I + \frac{\delta_\lambda{}^2}{6}\right)\frac{\partial\mathbf{W}^n}{\partial t}. \quad (3.18)$$

This equation represents J linear systems of the form (2.16, 2.17), but now with the elements (α_i, β_i, γ_i, $i = 1, 2, \ldots, I$) of the matrix $\mathbf{M}$ being 3×3 matrices:

$$\alpha_i = \frac{2}{3}\left(I + \frac{\Delta t}{2}C_i^n\right), \quad (3.19a)$$

$$\gamma_i = \frac{1}{6}\left(I + \frac{\Delta t}{2}C_{i+1}{}^n\right) + \frac{\Delta t}{4\Delta\lambda}A_{i+1}{}^n, \quad (3.19b)$$

$$\beta_i = \frac{1}{6}\left(I + \frac{\Delta t}{2}C_{i-1}{}^n\right) - \frac{\Delta t}{4\Delta\lambda}A_{i-1}{}^n. \quad (3.19c)$$

The components of $\mathbf{y}$ and $\mathbf{b}$ are 3-vectors $\mathbf{y}_i$, $\mathbf{b}_i$, given by

$$\mathbf{y}_i = (\Delta\mathbf{W}_{(1)})_i = \begin{bmatrix}\Delta h_{(1)}\\ \Delta U_{(1)}\\ \Delta V_{(1)}\end{bmatrix}_i, \quad i = 1, 2, \ldots, I, \quad (3.20)$$

$$\mathbf{b}_1 = \Delta t\left[\frac{1}{6}\left(\frac{\partial\mathbf{W}^n}{\partial t}\right)_I + \frac{2}{3}\left(\frac{\partial\mathbf{W}^n}{\partial t}\right)_1 + \frac{1}{6}\left(\frac{\partial\mathbf{W}^n}{\partial t}\right)_2\right], \quad (3.21a)$$

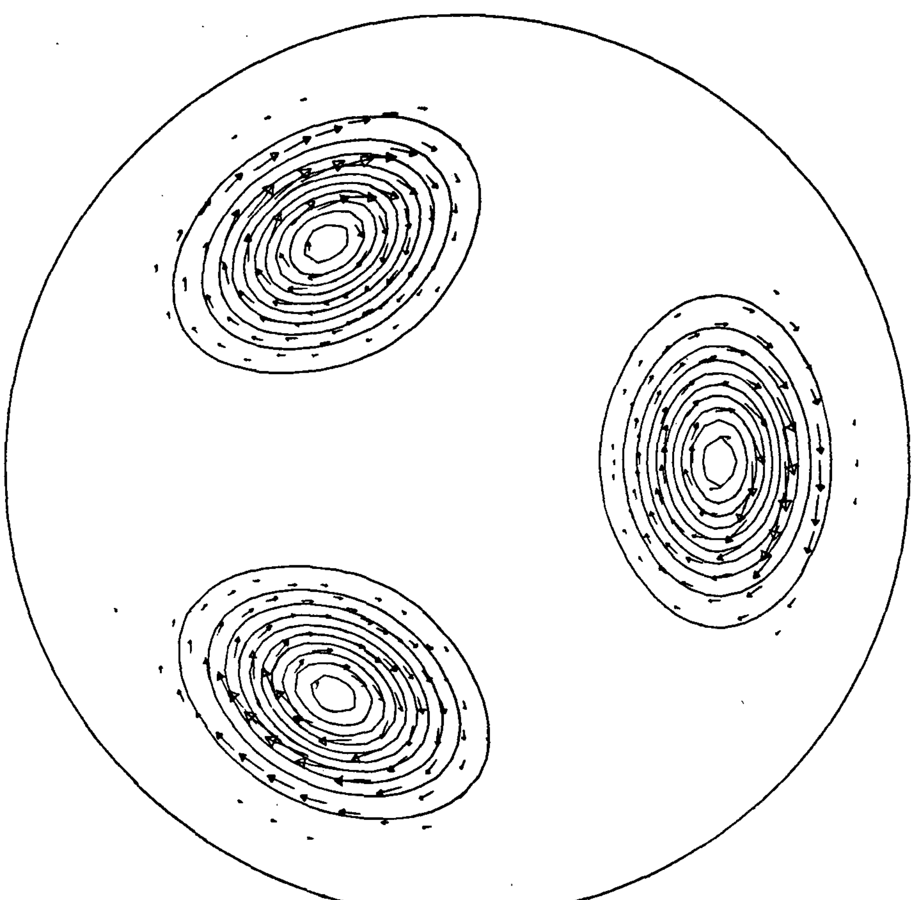

FIG. 1. Initial height contours, Eq. (4.1a), and wind vectors, Eqs. (4.1b, c), in a stereographic projection of the Northern Hemisphere.

$$\mathbf{b}_i = \Delta t\left[\frac{1}{6}\left(\frac{\partial \mathbf{W}^n}{\partial t}\right)_{i-1} + \frac{2}{3}\left(\frac{\partial \mathbf{W}^n}{\partial t}\right)_i + \frac{1}{6}\left(\frac{\partial \mathbf{W}^n}{\partial t}\right)_{i+1}\right],$$

$$i = 2, 3, \ldots, I-1, \tag{3.21b}$$

$$\mathbf{b}_I = \Delta t\left[\frac{1}{6}\left(\frac{\partial \mathbf{W}^n}{\partial t}\right)_{I-1} + \frac{2}{3}\left(\frac{\partial \mathbf{W}^n}{\partial t}\right)_I + \frac{1}{6}\left(\frac{\partial \mathbf{W}^n}{\partial t}\right)_1\right]. \tag{3.21c}$$

The subscript j, $j = 1, 2, \ldots, J$, has been omitted from Eqs. (3.19, 3.20, 3.21): each of the J systems has an identical form. These systems are solved by use of the block LU decomposition (Isaacson and Keller, 1966, Chap. 2, Section 3.3). Having obtained $\Delta\mathbf{W}_{(1)}$, we take the next step, the zonal Shapiro smoothing (3.14b). The current IJ representation is, of course, appropriate for this step.

Before solving the ϕ-sweep (3.14c), we first transpose the dependent variable arrays and the Δ-arrays back to the JI representation. Analogously to (3.18), the ϕ-sweep is performed by solving $I/2$ cyclic block-tridiagonal systems of $2J$ 3×3 blocks each. Finally, the meridional Shapiro filter (3.14d) is applied and the dependent variables are updated (3.14e), still in the JI representation. Each array has been transposed twice per time step.

4. Numerical experiments

We now describe numerical experiments illustrating the stability and accuracy properties of the BW scheme for the barotropic model.

As initial conditions, we prescribe subtropical highs centered at latitudes ±30° and longitudes 0, 120 and 240°. This initial total height field, $h_0 + h_s$, is given by

$$h_0 + h_s = h_1\left[1 + 0.01\cos^6\left(\frac{3}{2}\lambda\right) \times \sin^6\left(\frac{7}{2}|\phi| - \frac{3}{\pi}\phi^2\right)\right], \tag{4.1a}$$

with h_1 = 10 km. To obtain initial data which are reasonably free of fast wave components, we deduce the initial wind field (u_0, v_0) by requiring $Du_0/Dt = Dv_0/Dt = 0$, where D/Dt denotes the material derivative. Thus, from the advection form

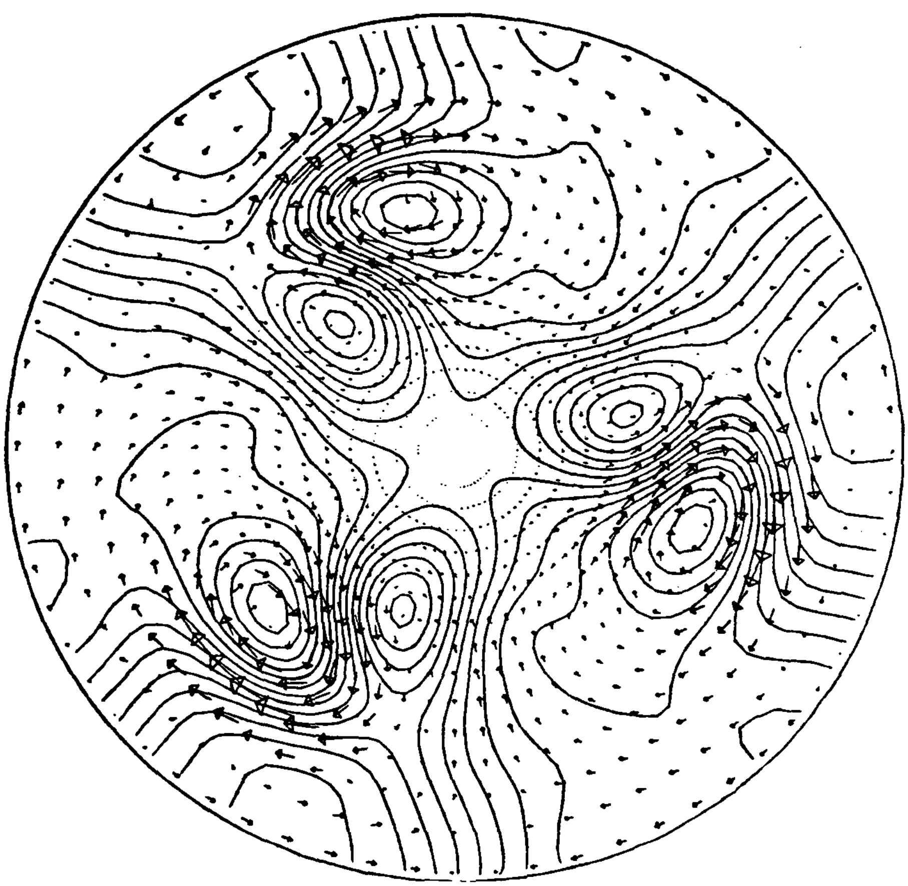

FIG. 2. Evolution of the initial three highs (Fig. 1) after 24 h, using a mesh of 128 × 64 points and a 7.5 min time step.

$$\frac{Du}{Dt} - \left(f + \frac{u}{a}\tan\phi\right)v + \frac{g}{a\cos\phi}\frac{\partial(h + h_s)}{\partial\lambda} = 0,$$

$$\frac{Dv}{Dt} + \left(f + \frac{u}{a}\tan\phi\right)u + \frac{g}{a}\frac{\partial(h + h_s)}{\partial\phi} = 0,$$

of the original momentum equations (3.1b, c), we obtain

$$u_0 = -a\Omega\cos\phi + [a^2\Omega^2\cos^2\phi - (g/\tan\phi)\partial(h_0 + h_s)/\partial\phi]^{1/2} \quad (4.1b)$$

$$v_0 = g(u_0\sin\phi + af\cos\phi)^{-1}\partial(h_0 + h_s)/\partial\lambda. \quad (4.1c)$$

These initial conditions are depicted in Fig. 1, which shows contours of $h_0 + h_s$ in a stereographic projection of the Northern Hemisphere. Arrows indicating the direction and magnitude of the wind field are superimposed at every other grid point.

Figure 2 shows for an experiment with no orography, $h_s = 0$, the result of a 24 hour integration on a mesh of 128 × 64 points, using a time step of 7.5 minutes. The flow has developed a reasonable degree of complexity, with the formation of large-scale cyclones adjacent to each of the original subtropical highs.

Figure 3 shows for an increased time step of 15 min the result of an experiment which is otherwise identical to the previous one. Figures 2 and 3 are nearly indistinguishable, from which we conclude that a 15 min time step can be taken with no loss of accuracy.

In Fig. 4 the time step has been increased to 30 min, again for an otherwise identical experiment. Comparing this figure with the previous ones, a slight loss of accuracy is apparent. Note, however, that the centers of vorticity have not been displaced.

Similar experiments were carried out using time steps of up to 2 h, and for integration periods of 60 days. In no case was computational instability observed, although a loss of accuracy was noted for time steps greater than 30 minutes.

As a more stringent test of stability, experiments were made with orography given by

$$h_s = \frac{1}{8}h_1\cos^6\left(\frac{3\lambda - \pi}{2}\right)\cos^2\phi. \quad (4.2)$$

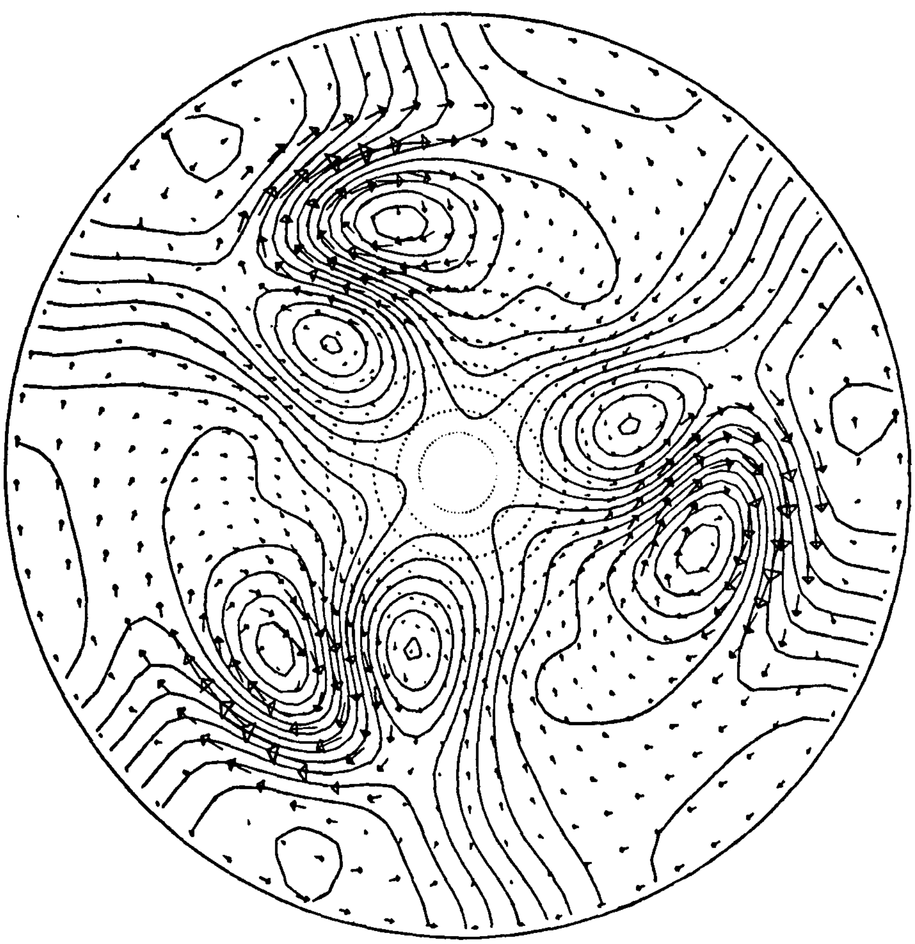

FIG. 3. As in Fig. 2, but using a 15 min time step.

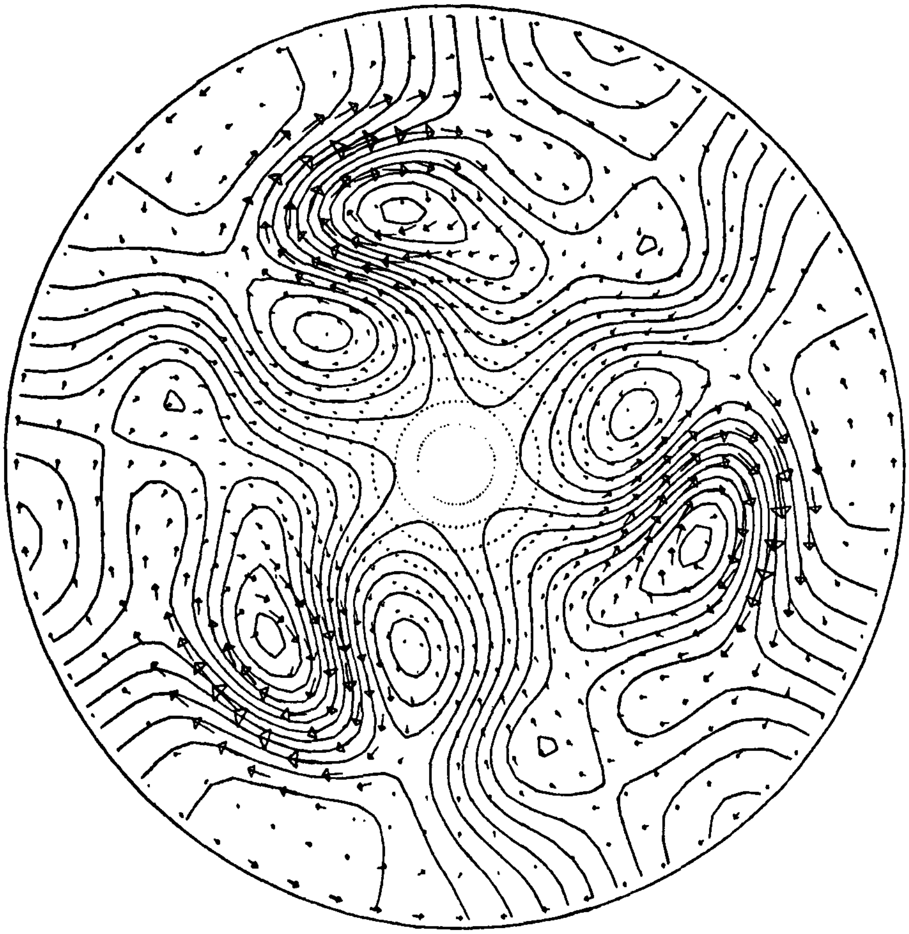

FIG. 4. As in Fig. 2, but using a 30 min time step.

Still, with time steps of up to 2 h and integration periods of 60 days, no stability problems were observed.

These experiments illustrate a central feature of the implicit scheme, namely, that the time step can be chosen solely on the basis of accuracy requirements. With a properly formulated implicit scheme, computational stability is maintained regardless of the time step. This property has an important practical consequence: as the mesh spacing is decreased, implicit schemes become more efficient relative to explicit and semi-implicit schemes. That is, in explicit and semi-implicit schemes, one must decrease the time step with decreasing mesh spacing, whereas in implicit schemes, the time step may be held fixed. Although implicit schemes involve more computational work per time step, they become relatively more efficient for fine meshes.

We have found that for the barotropic model, a time step of 15 min gives excellent accuracy and a time step of 30 min gives accuracy which might be regarded as acceptable. At the spatial resolution used in our experiments, an explicit scheme would require a much smaller time step, even with the use of Fourier filtering. Experiments comparing the implicit scheme with an explicit Williamson and Browning (1973) scheme indicate that the implicit scheme is more efficient than the explicit one already for a 128 $\times$ 64 mesh.

On the other hand, Robert (1982) has described a semi-Lagrangian and semi-implicit scheme that gives accurate forecasts using a time step of two hours. Experiments with our fully implicit scheme show unacceptably large phase errors for time steps exceeding about ½ hour. We have carried out an analysis of the implicit scheme which shows that for time steps larger than ½ hour, the factorization error E^n in Eq. (2.38) dominates the overall time discretization error, causing the Rossby mode phase speeds to decrease. The analysis also indicates remedies for this difficulty, and they will be the object of forthcoming work.

5. Conclusions

Implicit schemes have the desirable property of allowing a time step chosen solely to resolve the evolution of the atmospheric flow. Efficient and accurate implicit schemes can be obtained by considering a multidimensional problem as a succession of smaller, one-dimensional problems. We have tested this idea by developing a factorized scheme for a global barotropic model. Our results indicate that an implicit three-dimensional model is feasible, and would be well-suited for accurate forecasting on fine meshes.

The factorization scheme we have used introduces phase errors which become large for time steps exceeding approximately half an hour. Results using an improved factorization will be presented in a forthcoming work.

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