A General Formulation of Normal Modes for Limited-Area Models: Application to Initialization

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ABSTRACT

A formulation of normal modes for a limited-area model is proposed. The case of shallow water equations on a conformal projection is considered. This formulation is a generalization of Brière's proposal. It can handle the full variation of the Coriolis parameter and of the map scale factor; it is written in physical-space variables and does not need a rectangular domain to be applied as in Brière's scheme. It gives rise to stationary Rossby modes and gravity modes fully identified and easily separated on the basis of their frequency. By applying Machenhauer's initialization scheme, we rigorously deduce the vertical mode initialization proposed and demonstrated by Bourke and McGregor for a limited-area model.

1. Introduction

Nonlinear Normal Mode Initialization (NNMI) is now considered an efficient technique for the initialization of primitive equation models. The formulation developed by Machenhauer (1977) and Baer (1977) has been applied successfully to global models for spectral formulation (Daley, 1979), as well as for gridpoint discretization (Temperton and Williamson, 1981). This initialization successfully eliminates unwanted gravity wave oscillations of these models.

The application of NNMI to a limited-area model, however, is not straightforward. The reason is that it requires the construction of the normal modes of a linearized set of the model equations. In the case of a periodic domain, hemispheric or global, Laplace tidal equations provide a linearized set of equations giving rise to eigenfunctions separable in latitude and longitude. However, for a limited-area model on a conformal projection, the problem is no longer separable due to the variation of the Coriolis factor, f, and of the map scale factor, m. This problem is further complicated by the problem of boundary conditions.

Brière (1982) has proposed a technique that uses constant values of these factors (f and m) in the linearized equations and, by using a zero-value boundary condition on a rectangular domain, can deal with double sinusoidal expansions for normal mode decomposition. This technique has been applied successfully to the initialization of a limited-area model (Craplet, 1983, 1985).

The question naturally arises of the validity of such hypotheses (f and m constants), for example, if the domain crosses the equator (Verner and Benoit, 1984). Moreover, for a nonrectangular domain or for a finite element formulation, it is not obvious how to apply Brière's technique.

Thus, some proposals have been made for initialization of limited-area models without use of any normal mode decomposition because of the problems mentioned. Bourke and McGregor (1983) developed a physical-space initialization that appears to be very similar to Brière's method but does not need the mentioned hypotheses (constant values for f and m). Recently, Lynch (1985) developed a general procedure based on the Laplace transform, which can filter out high-frequency oscillations during the model integration. Our purpose here is not to ignore any normal mode decomposition but rather to exhibit a normal mode formulation for a limited-area model without the restrictions of the Brière scheme and then to show that the application of the Machenhauer scheme leads exactly to the Bourke and McGregor scheme.

This paper is organized as follows. Section 2 proposes a general formulation of normal modes for a limited-area shallow water model on a conformal projection without the restrictions of using constant values for f and m in the linearized equations and of using a rectangular domain. Section 3 formulates linear initialization and Machenhauer's nonlinear initialization for this normal mode formulation and transforms these schemes in the physical space. Some additional remarks concerning the definition of variables, the boundary conditions, and the relation of this nonlinear normal mode initialization to Brière's scheme are given in section 4. Section 5 discusses the extension of the formulations proposed in sections 2 and 3 to the case of a latitude–longitude limited-area model. Section 6, the conclusion, discusses the possible application of such physical-space formulations to other problems. This study is restricted to the shallow water equations, which embody the essentials of the normal-mode approach for primitive equation models.

2. Definition of normal modes in the case of a conformal projection

a. *The linearized system*

We consider the shallow water equations for a conformal projection of map scale factor m. The divergence D and vorticity ζ are defined by

$$D = m^2\left[\frac{\partial}{\partial x}\left(\frac{u}{m}\right) + \frac{\partial}{\partial y}\left(\frac{v}{m}\right)\right]$$

$$\zeta = m^2\left[\frac{\partial}{\partial x}\left(\frac{v}{m}\right) - \frac{\partial}{\partial y}\left(\frac{u}{m}\right)\right]$$

where u, v are the horizontal velocity components. We write the time rate of change equations for the quantities D, ζ and the geopotential ϕ and retain the following linearization:

$$\frac{\partial D}{\partial t} - f\zeta + m^2\nabla^2\phi = 0 \qquad (1a)$$

$$\frac{\partial \zeta}{\partial t} + fD = 0 \qquad (1b)$$

$$\frac{\partial \phi}{\partial t} + \bar{\phi}D = 0 \qquad (1c)$$

where $\bar{\phi}$ is a mean geopotential height, and the Coriolis parameter f and the map scale factor m are known functions of the Cartesian coordinates (x, y). The limited area domain $\mathscr{D}$ is somewhat arbitrary, with regular boundary $\mathscr{C}$, but it is not necessarily a rectangle.

To deal with the hyperbolic system (1), we need boundary conditions for the functions D and ϕ for the following reason. By taking the time derivative of the divergence equation and by using Eqs. (1b) and (1c) we obtain the second-order equation for divergence

$$\frac{\partial^2 D}{\partial t^2} - \mathscr{H}D = 0$$

involving the elliptic operator $\mathscr{H} = -f^2 + \bar{\phi}m^2\nabla^2$. By eliminating the divergence D between (1b) and (1c) we get the result that the quantity $\bar{\phi}\zeta - f\phi$ is independent of time and thus is a given function $C(x, y)$. By taking the time derivative of Eq. (1c) and with the use of the preceding relation, we get the second-order equation for ϕ

$$\frac{\partial^2 \phi}{\partial t^2} - \mathscr{H}\phi = -fC(x, y).$$

Therefore, the whole system (1) will be well posed if we use, for example, Dirichlet boundary conditions for D and ϕ to get these functions and if we deduce the vorticity from the relation $\bar{\phi}\zeta - f\phi = C(x, y)$.

We only consider the case of constant boundary conditions, i.e., independent of time. For the sake of simplicity but without limiting the generality, as will be seen below, we assume that D is zero on the boundary $\mathscr{C}$ and that ϕ has the Dirichlet boundary condition on $\mathscr{C}$ held constant in time. We define a deviated geopotential $\phi' = \phi - \phi_1$ with the function ϕ_1 equal to ϕ at the boundary $\mathscr{C}$ and solution of the Laplace equation: $\nabla^2\phi_1 = 0$. For the system (1) written in terms of D, ζ, ϕ' there is no supplementary term, since the function ϕ_1 is constant in time.

A nonhomogeneous boundary condition (constant in time) for the divergence D can be dealt with in a similar way, by defining a deviated divergence $D' = D - D_1$ with D_1 equal to D at the boundary $\mathscr{C}$ and the solution of the Laplace equation $\nabla^2 D_1 = 0$, for example. In this case, the system (1) written in terms of D', ζ, ϕ' introduces two supplementary terms involving D_1 in Eqs. (1b) and (1c) that will be taken into account with the forcing in the complete nonlinear system. It will be shown in (3c), however, that the nonlinear initialization scheme will not explicitly involve such a function D_1, if constant in time. Therefore, we will assume for simplicity that D is zero on the boundary $\mathscr{C}$ in the following.

In order to simplify the subsequent calculations, we consider the new variable $\eta = f\zeta$, which is equivalent to the vorticity ζ. We then write the system (1) in terms of this variable by multiplying the second equation by f. Dropping the prime we have the following linear system:

$$\frac{\partial \mathbf{E}}{\partial t} + \mathbf{ME} = 0$$

with the three-component vector function $\mathbf{E} = (D, \eta, \phi)^t$ (t denotes the transpose of a vector) defined on the domain $\mathscr{D}$, such that D and ϕ are zero at the boundary $\mathscr{C}$ and such that $\eta = f\zeta$, and with the matrix operator $\mathbf{M}$

$$\mathbf{M} = \begin{pmatrix} 0 & -1 & m^2\nabla^2 \\ f^2 & 0 & 0 \\ \bar{\phi} & 0 & 0 \end{pmatrix}$$

which contains the nonconstant coefficients $f(x, y)$ and $m(x, y)$.

b. *The normal modes*

A normal mode is defined as a three-component vector function $\mathbf{E}$ such that

$$\mathbf{ME} = \lambda\mathbf{E}$$

for the eigenvalue λ. The modes are defined as eigenfunctions of the matrix operator $\mathbf{M}$ and therefore are not dependent on a particular discretization. The three functions D, η, ϕ satisfy the system:

$$\left.\begin{aligned}-\eta+m^2\nabla^2\phi&=\lambda D\\ f^2D&=\lambda\eta\\ \bar{\phi}D&=\lambda\phi\end{aligned}\right\}. \qquad (2)$$

To solve this system, we isolate the case $\lambda = 0$, for which the nontrivial solution is defined as

$$D=0, \quad \eta=f\zeta=m^2\nabla^2\phi$$

with an arbitrary deviated geopotential ϕ such that $\phi = 0$ on the boundary $\mathscr{C}$. It is a stationary nondivergent mode, and we call it a Rossby mode. It also implies for this Rossby mode that $\nabla^2\phi = 0$ at the equator.

For $\lambda \neq 0$ we eliminate D and ζ in the system (2) and obtain

$$\mathscr{H}\phi \equiv (-f^2+\bar{\phi}m^2\nabla^2)\phi=\lambda^2\phi. \qquad (3)$$

This relation means that ϕ is an eigenfunction of the elliptic operator $\mathscr{H}$ with the boundary condition $\phi = 0$, for the eigenvalue λ^2. It is a Sturm–Liouville problem (Courant and Hilbert, 1953) for which there is a denumerable family of functions $\phi_k(x, y)$ satisfying this relation, with negative eigenvalues λ_k^2, so that $\lambda_k = \pm i\sigma_k$, with real number σ_k. This family is a complete orthogonal system for the scalar product

$$(\phi_1,\phi_2)=\int_{\mathscr{D}}\phi_1\phi_2\frac{dxdy}{m^2}.$$

(See the Appendix for details.)

Then, returning to the system (2), we obtain the divergence and the vorticity of the kth-mode $\mathbf{E}_k$ with deviated geopotential ϕ_k:

$$D_k=\pm i\sigma_k\frac{\phi_k}{\bar{\phi}}, \quad \zeta_k=f\frac{\phi_k}{\bar{\phi}}$$

for the eigenvalue $\pm i\sigma_k$. They consist of a pair of complex conjugate oscillatory modes with opposite frequencies:

$$\text{if} \quad \lambda_k=+i\sigma_k, \mathbf{E}_k^+=\begin{pmatrix} i\sigma_k\\ f^2\\ \bar{\phi}\end{pmatrix}\frac{\phi_k}{\bar{\phi}};$$

$$\text{if} \quad \lambda_k=-i\sigma_k, \mathbf{E}_k^-=\begin{pmatrix} -i\sigma_k\\ f^2\\ \bar{\phi}\end{pmatrix}\frac{\phi_k}{\bar{\phi}}=\mathbf{E}_k^{\mp}$$

with the overbar meaning complex conjugation.

These modes are the fast gravity modes. We also notice that, since ϕ_k is a deviated geopotential, their quasi-geostrophic potential vorticity (Haltiner and Williams, 1980)

$$q_k=\zeta_k-f\frac{\phi_k}{\bar{\phi}}$$

is zero. Thus, the divergence and the vorticity of the kth gravity mode are closely related to the geopotential, independently of the complicated horizontal structure given by the eigenfunctions $\phi_k(x, y)$. These relations will be very useful later for initialization.

We notice the following main difference from Brière's (1982) method: the functions $\phi_k(x, y)$ are not separable in the coordinates x and y, unlike the double sinusoidal functions used in Brière's method. Because of their nonseparability, it is difficult to use such functions $\phi_k(x, y)$ as a basis for practical decomposition. This would require a very large memory. However, due to the mentioned properties of the complete orthogonal system, we may use them as a basis for formal decomposition. It will be done in the following paragraph by developing the geopotential of the Rossby mode on this basis. The time-dependent solution of system (1) will then appear fully developed on this basis, in order to prepare the application of initialization schemes.

The separation between stationary mode and gravity modes results from the fact that the sequence $|\sigma_k|$ has a positive minimum as shown in the Appendix.

A fact must be noted: the approximation f = constant has not been used in the linearization, and we have obtained a stationary Rossby mode. To obtain a nonstationary one, it is necessary to include the terms involving derivatives of f (usually called β-terms) in the linearization.

c. The time-dependent solution

It is easy to infer from these normal modes the general time-dependent solution of (1). It can be written in the form $e^{i\sigma t}\mathbf{V}(x, y)$, and we find that $\mathbf{V}$ is one of the three preceding normal modes. So we write the solution as the sum of two series of oscillatory gravity modes plus a stationary mode:

$$\begin{pmatrix} D\\ f\zeta\\ \phi\end{pmatrix}=\sum_k \alpha_k\begin{pmatrix} -i\sigma_k\\ f^2\\ \bar{\phi}\end{pmatrix}S_ke^{i\sigma_kt} + \bar{\alpha}_k\begin{pmatrix} i\sigma_k\\ f^2\\ \bar{\phi}\end{pmatrix}S_ke^{-i\sigma_kt}+\begin{pmatrix} 0\\ m^2\nabla^2\phi_a\\ \phi_a\end{pmatrix}$$

with an arbitrary function ϕ_a (such that $\phi_a = 0$ on $\mathscr{C}$) and an arbitrary complex sequence α_k. Here $\overline{\alpha_k}$ is the complex conjugate of α_k, which appears because the solution is real. Henceforth, S_k denotes the eigenfunction solution of (3), assumed normalized for the scalar product (,) for the sake of simplicity. By using the property of the family S_k, we expand the function ϕ_a on this basis:

$$\phi_a=\sum_k c_kS_k$$

with

$$c_k=(\phi_a,S_k).$$

Then the function $m^2\nabla^2\phi_a$ can be written $\sum c_k m^2\nabla^2 S_k$. By using the relation (3) for the eigenfunctions S_k we deduce

$$m^2\nabla^2 S_k = \frac{1}{\Phi}(f^2 - \sigma_k{}^2)S_k$$

without the ∇^2 operator. We denote γ_k as the sequence of functions defined as

$$\gamma_k(x, y) = \frac{1}{\Phi}[f^2(x, y) - \sigma_k{}^2].$$

So we get for the stationary mode

$$\begin{pmatrix} 0 \\ m^2\nabla^2\phi_a \\ \phi_a \end{pmatrix} = \sum_k c_k \begin{pmatrix} 0 \\ \gamma_k \\ 1 \end{pmatrix} S_k.$$

Then we get the fully developed solution

$$\begin{pmatrix} D \\ f\zeta \\ \phi \end{pmatrix} = \sum_k Q_k \begin{pmatrix} c_k \\ \alpha_k e^{i\sigma_k t} \\ \bar{\alpha}_k e^{-i\sigma_k t} \end{pmatrix} S_k \qquad (4)$$

with arbitrary real sequence c_k and arbitrary complex sequence α_k and with the following matrix

$$\mathbf{Q}_k = \begin{pmatrix} 0 & -i\sigma_k & +i\sigma_k \\ \gamma_k & f^2 & f^2 \\ 1 & \bar{\phi} & \bar{\phi} \end{pmatrix}$$

where the coefficients γ_k and f are dependent on the coordinates (x, y).

This form confirms the fact that the divergence contains only gravity modes and that the geopotential and the vorticity are strongly related (coefficients f and $\bar{\phi}$ in the matrix $\mathbf{Q}_k$, independent of k).

We then show how to compute the sequence of coefficients α_k (complex) and c_k (real) so that the solution (4) fits a prescribed initial state given by $(D_0, f\zeta_0, \phi_0)^t$. Setting $t = 0$ in (4), we write

$$\begin{pmatrix} D_0 \\ f\zeta_0 \\ \phi_0 \end{pmatrix} = \sum_k \mathbf{Q}_k \begin{pmatrix} c_k \\ \alpha_k \\ \bar{\alpha}_k \end{pmatrix} S_k. \qquad (5)$$

We use the notations: $\alpha_k = a_k + ib_k$, so $\bar{\alpha}_k = a_k - ib_k$, with a_k, b_k, c_k real numbers and expand the relation (5):

$$\left.\begin{aligned} D_0 &= \sum_k 2\sigma_k b_k S_k \\ f\zeta_0 &= \sum_k \gamma_k c_k S_k + f^2 \sum_k 2a_k S_k \\ \phi_0 &= \sum_k c_k S_k + \bar{\phi}\sum_k 2a_k S_k \end{aligned}\right\}.$$

The first relation immediately gives

$$b_k = \frac{1}{2\sigma_k}(D_0, S_k)$$

since the family S_k is an orthonormal system. In order to eliminate the terms involving a_k and c_k we respectively form the quantities $\bar{\phi}f\zeta_0 - f^2\phi_0$, $f\zeta_0 - m^2\nabla^2\phi_0$ and with a little algebra and the relation (3), we get

$$\left.\begin{aligned} c_k &= \frac{1}{\sigma_k{}^2}(f^2\phi_0 - \bar{\phi}f\zeta_0, S_k) \\ a_k &= \frac{1}{2\sigma_k{}^2}(f\zeta_0 - m^2\nabla^2\phi_0, S_k) \end{aligned}\right\}.$$

These formulas clearly show that the real coefficients a_k, b_k, c_k are only dependent on the quantities D_0, $\bar{\phi}f\zeta_0 - f^2\phi_0$ and $f\zeta_0 - m^2\nabla^2\phi_0$. Regardless of coefficients, they are the projection of such quantities on the S_k basis.

The sequence of coefficients α_k (complex) and c_k (real) are determined entirely in terms of the initial fields D_0, ζ_0, ϕ_0 by the formulas:

$$\alpha_k = \frac{1}{2\sigma_k{}^2}(f\zeta_0 - m^2\nabla^2\phi_0, S_k) + \frac{i}{2\sigma_k}(D_0, S_k)$$

$$c_k = \frac{1}{\sigma_k{}^2}(f^2\phi_0 - \bar{\phi}f\zeta_0, S_k).$$

Finally, we notice that no problem can occur with the denominator, which never reaches zero, even if the domain crosses the equator, as shown in the Appendix.

3. Application to initialization

In order to apply an initialization scheme using such a decomposition into stationary nondivergent modes and gravity modes, it is necessary to define the modal components themselves, i.e., the value of the projection of the vector state $(D, f\zeta, \phi)^t$ onto a Rossby or a gravity mode. Since we are not interested in the numerical value of the modal components and since we will only consider homogeneous initialization schemes, this will be done without using an orthogonal projection but in an "oblique" way consistent with the preceding decomposition. After formally writing linear and nonlinear normal mode initializations, we return to the physical space and discuss the results.

a. *Definition of the modal components*

For an arbitrary, time-dependent, three-component vector $(D, f\zeta, \phi)^t$ we define the Rossby component (x_{1k}) and two complex gravity components (x_{2k}, x_{3k} such that $x_{3k} = \overline{x_{2k}}$) of the order k by using the following relation:

$$\begin{pmatrix} D \\ f\zeta \\ \phi \end{pmatrix} = \sum_k Q_k \begin{pmatrix} x_{1k} \\ x_{2k} \\ x_{3k} \end{pmatrix} S_k,$$

which appears very similar to the expression (4) of the time-dependent solution. By applying, at a given time t, the algebra used in the preceding paragraph for the initial value problem, we deduce that the modal components of the order k are uniquely defined as a function of time as

$$x_{1k} = \frac{1}{\sigma_k^2}(f^2\phi - \bar{\phi}f\zeta, S_k)$$

$$x_{2k} = \frac{1}{2\sigma_k^2}(f\zeta - m^2\nabla^2\phi, S_k) + \frac{i}{2\sigma_k}(D, S_k)$$

$$x_{3k} = \bar{x}_{2k}.$$

The index k is dropped in the following.

In order to implement nonlinear formulation we also define for the complete nonlinear equations written in the form:

$$\frac{\partial \mathbf{E}}{\partial t} + \mathbf{M}E = \mathbf{N} = \begin{pmatrix} \mathbf{N}_D \\ N_\eta \\ N_\phi \end{pmatrix} \tag{6}$$

the nonlinear modal components N_1 (Rossby kth component), N_2, N_3 (with $N_3 = \bar{N}_2$, gravity kth component) in the same way:

$$\begin{pmatrix} N_D \\ N_\eta \\ N_\phi \end{pmatrix} = \sum_k \mathbf{Q}_k \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} S_k.$$

They are defined in terms of the nonlinear contributions N_D, N_η, N_ϕ with the same relations as the parameters x_1, x_2, x_3 are in terms of D, η, ϕ.

The time rate of change system (6) can be readily transformed into

$$\sum_k \mathbf{Q}_k \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} S_k + \mathbf{M}\sum_k \mathbf{Q}_k \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} S_k = \sum_k \mathbf{Q}_k \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} S_k. \tag{7}$$

by using the preceding definition of modal components. By using the property of the three eigenmodes appearing as the vector column of the matrix Q_kS_k, we have

$$\mathbf{MQ}_k\mathbf{S}_k = \mathbf{Q}_k\mathbf{S}_k \begin{pmatrix} 0 & 0 & 0 \\ 0 & i\sigma_k & 0 \\ 0 & 0 & -i\sigma_k \end{pmatrix}. \tag{8}$$

Substituting this relation (8) into the relation (7), we then identify the modal components of order k in the two sides of the relation (7) (it is licit since the definition of the modal components is unique), and we obtain

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 & & (0) \\ & i\sigma_k & \\ (0) & & -i\sigma_k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix}. \tag{9}$$

This modal component system can be readily written as

$$\frac{dx_1}{dt} = N_1, \quad \frac{dx_2}{dt} + i\sigma_k x_2 = N_2, \quad \frac{dx_3}{dt} - i\sigma_k x_3 = N_3.$$

b. Linear normal-mode initialization

The linear normal-mode initialization consists in maintaining the Rossby component at its initial value ($x_1 = x_1^{(0)}$) and cancelling the contribution of gravity components (Williamson, 1976) ($x_2 = x_3 = 0$). By using the preceding definitions of the modal components, the first relation can be written

$$(f^2\phi - \bar{\phi}f\zeta, S_k) = (f^2\phi^{(0)} - \bar{\phi}f\zeta^{(0)}, S_k)$$

where the exponent (0) refers to the initial time. Due to the expansion property of the S_k family and since the Coriolis parameter f is not identically equal to zero, it is equivalent to

$$f\phi - \bar{\phi}\,\zeta = f\phi^{(0)} - \bar{\phi}\,\zeta^{(0)}, \tag{10}$$

i.e., maintenance of the quasi-geostrophic potential vorticity at its initial value, independently of any relation about the gravity components. The cancellation of the gravity components gives, by using their definition and the same expansion property of the S_k family,

$$D = 0, \quad f\zeta = m^2\nabla^2\phi, \tag{11}$$

i.e., this relation defines a Rossby mode, as is natural.

In this case, we obtain directly the linear normal mode initialization in the physical space with the relations (10) and (11).

Clearly, the linear normal-mode initialization is equivalent to the linear balance equation (without β-terms) with the constraint expressed by (10). In order to solve the system of Eqs. (10) and (11), we eliminate ζ and get the Helmholtz equation

$$\mathscr{H}\phi' = f^2\phi'^{(0)} - f\bar{\phi}\,\zeta^{(0)}$$

for the deviated geopotential ϕ' as defined in section 2a, with the boundary condition $\phi' = 0$ on $\mathscr{C}$. Returning to the undeviated geopotential ϕ the Helmholtz equation becomes

$$\mathscr{H}\phi = f^2\phi^{(0)} - \bar{\phi}f\zeta^{(0)}$$

with the boundary conditions given by the boundary values of the known geopotential $\phi^{(0)}$ at the initial time.

c. Nonlinear normal mode initialization

Machenhauer's algorithm of nonlinear normal mode initialization consists of maintaining the Rossby component at its initial value ($x_1 = x_1^{(0)}$) and writing the stationarity of gravity components: $dx_2/dt = dx_3/dt$

= 0 in the full nonlinear system (9). This can be reached by the explicit iterative scheme:

$$\left.\begin{aligned} x_1^{(q+1)} &= x_1^{(q)} \\ x_2^{(q+1)} &= N_2(x_1^{(q)}, x_2^{(q)}, x_3^{(q)})/i\sigma_k \\ x_3^{(q+1)} &= N_3(x_1^{(q)}, x_2^{(q)}, x_3^{(q)})/-i\sigma_k \end{aligned}\right\}.$$

We now transform these equations by expressing the nonlinear terms N_2 and N_3 in terms of the time tendencies of the modal components x_2 and x_3. This is the way in which the nonlinear contributions are usually computed in the initialization packages that make use of a forward step of the model run. By using the notation: $\delta x_i^{(q)} = x_i^{(q+1)} - x_i^{(q)}$ we get the following iterative scheme:

$$\begin{pmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{pmatrix}^{(q)} = \begin{matrix} 0 & & (0) \\ & \dfrac{1}{i\sigma_k} & \\ (0) & & \dfrac{-1}{i\sigma_k} \end{matrix} \; \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^{(q)}.$$

This expresses the iterative initialization scheme in a canonical form in terms of the time derivatives of the variables. The word "canonical" is used since the matrix appearing in this equation is systematically related to the matrix of the modal equations by the relation:

$$\begin{matrix} 0 & & (0) \\ & \dfrac{1}{i\sigma_k} & \\ (0) & & \dfrac{-1}{i\sigma_k} \end{matrix} = -\frac{1}{\sigma_k^2}\begin{pmatrix} 0 & & (0) \\ & i\sigma_k & \\ (0) & & -i\sigma_k \end{pmatrix}.$$

So we can write the iterative scheme:

$$\begin{pmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{pmatrix}^{(q)} = -\frac{1}{\sigma_k^2}\begin{pmatrix} 0 & & (0) \\ & i\sigma_k & \\ (0) & & -i\sigma_k \end{pmatrix}\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^{(q)}.$$

We now express this scheme back in the physical space. This can be done in a straightforward way by multiplying the preceding equality by $\mathbf{Q}_k S_k$ and, with a little algebra using the relation (8), it leads to the relation:

$$Q_k\begin{pmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{pmatrix}^{(q)} S_k = -\frac{\mathbf{MQ}_k}{\sigma_k^2}\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^{(q)} S_k.$$

After multiplication by $-\sigma_k^2$ and summation on the index k we get the relation

$$\sum_k -\sigma_k^2\mathbf{Q}_k\begin{pmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{pmatrix}^{(q)} S_k = \mathbf{M}\frac{\partial}{\partial t}\begin{pmatrix} D \\ f\zeta \\ \phi \end{pmatrix}^{(q)}. \quad (12)$$

By expansion the first row of the left-hand side of (12) can be written as

$$\sum_k -\sigma_k^2(-i\sigma_k\delta x_2^{(q)} + i\sigma_k\delta x_3^{(q)})S_k.$$

By using the relation (3) for the eigenfunctions S_k:

$$\sum_k (-i\sigma_k\delta x_2^{(q)} + i\sigma_k\delta x_3^{(q)})\mathscr{H}S_k$$

and due to the fact that the quantities inside the parentheses are independent of (x, y), we can get the operator $\mathscr{H}$ out of the symbol Σ

$$\mathscr{H}\sum_k (-i\sigma_k\delta x_2^{(q)} + i\sigma_k\delta x_3^{(q)})S_k,$$

which can be written $\mathscr{H}\delta D^{(q)}$.

The same transformations can be applied to the third row of the left-hand side of (12) and lead to $\mathscr{H}\delta\phi^{(q)}$. The first row and the third row of the relation (12) then become

$$\mathscr{H}\delta D^{(q)} = -f\frac{\partial\zeta^{(q)}}{\partial t} + m^2\nabla^2\frac{\partial\phi^{(q)}}{\partial t} \quad (13)$$

$$\mathscr{H}\delta\phi^{(q)} = \bar{\phi}\frac{\partial D^{(q)}}{\partial t}. \quad (14)$$

For the second row of the left-hand side of (12), the coefficients are dependent on (x, y) and we cannot get the operator $\mathscr{H}$ out of the symbol Σ.

However, by using the definition of the Rossby modal component, the relation $x_1^{(q+1)} = x_1^{(q)}$ can be written:

$$(f^2\phi^{(q+1)} - \bar{\phi}f\zeta^{(q+1)}, S_k) = (f^2\phi^{(q)} - \bar{\phi}f\zeta^{(q)}, S_k).$$

As already noted in linear initialization (section 3b), it is equivalent to the physical space relation:

$$f\phi^{(q+1)} - \bar{\phi}\,\zeta^{(q+1)} = f\phi^{(q)} - \bar{\phi}\,\zeta^{(q)},$$

also written as

$$\bar{\phi}\,\delta\zeta^{(q)} - f\delta\phi^{(q)} = 0. \quad (15)$$

It means the conservation of quasi-geostrophic potential vorticity at each step.

Let us remember that the three relations (13), (14), (15) are true for a deviated geopotential ϕ'. Due to its definition (section 2a) we have the following relations

$$\delta\phi^{(q)} = \delta\phi'^{(q)}$$

$$\nabla^2\phi = \nabla^2\phi'$$

so that

$$\nabla^2\frac{\partial\phi'}{\partial t} = \frac{\partial}{\partial t}\nabla^2\phi' = \frac{\partial}{\partial t}\nabla^2\phi = \nabla^2\frac{\partial\phi}{\partial t}.$$

From them we conclude that the three relations (13), (14), (15) are equally true for the undeviated geopotential ϕ, and the iterative scheme they constitute does not explicitly involve the function ϕ_1, introduced at the beginning of (section 2), in order to fit the boundary conditions.

In the case of a nonhomogeneous boundary condition for D held constant in time, we have to transform the relations (13), (14), (15) written for a deviated divergence $D' = D - D_1$. Due to the fact that in this case D_1 is independent of time, we have $\partial D'/\partial t = \partial D/\partial t$ and we also have $\delta D'^{(q)} = \delta D^{(q)}$, so we get the result that the scheme expressed by (13), (14) and (15) can also be written for undeviated divergence D and does not explicitly involve the function D_1.

We also notice that the Helmholtz equations (13) and (14) need a boundary condition for $\delta D^{(q)}$ and $\delta\phi^{(q)}$, which naturally appears as $\delta D^{(q)} = \delta\phi^{(q)} = 0$.

In fact, these three relations, (13), (14) (with the preceding boundary conditions) and (15), are exactly the filtering scheme B proposed by Bourke and McGregor (1983) on a more intuitive basis. Indeed, the relationship between the Machenhauer scheme and this filtering condition B has already been shown by Brière (see the conclusion of section 2b in his paper) and by Bourke and McGregor (cf. appendix of their paper). These demonstrations, however, only hold in the case of an f-plane linearization and by assuming a horizontal spectral expansion (sinusoidal for Brière). In the present paper we enlarge this relationship by showing that it also holds in the general case of f variable and m variable linearization. Indeed, it is a formal derivation and there is no discussion of the convergence of these schemes. The usefulness of these schemes for initializing limited-area models has been demonstrated largely by Brière and Bourke and McGregor.

Moreover, the condition expressing the stationarity of the gravity components

$$\frac{dx_2}{dt} = \frac{dx_3}{dt} = 0$$

can be written following their definitions (section 3a)

$$\frac{d}{dt}(D, S_k) = 0, \quad \frac{d}{dt}(f\zeta - m^2\nabla^2\phi, S_k) = 0$$

or, equivalently,

$$\left(\frac{\partial D}{\partial t}, S_k\right) = 0, \quad \left[\frac{\partial}{\partial t}(f\zeta - m^2\nabla^2\phi), S_k\right] = 0.$$

By using the expansion property of the S_k family, these relations are equivalent to the physical space relations:

$$\frac{\partial D}{\partial t} = 0 \quad \frac{\partial}{\partial t}(f\zeta - m^2\nabla^2\phi) = 0.$$

A similar transformation into the physical space has already been done for the conservation of the Rossby modal component x_1. We can consider these formulations of x_1, x_2, x_3 efficient tools for transforming any filtering condition for the modal components directly into a physical space formulation. For example, stationarity of the second derivatives of the gravity components immediately gives

$$\frac{\partial^2 D}{\partial t^2} = \frac{\partial^2(f\zeta - m^2\nabla^2\phi)}{\partial t^2} = 0.$$

4. Additional remarks

a. *Other definitions of divergence and vorticity*

It may seem natural to define divergence and vorticity in the following way:

$$\tilde{D} = \frac{\partial}{\partial x}\left(\frac{u}{m}\right) + \frac{\partial}{\partial y}\left(\frac{v}{m}\right)$$

$$\tilde{\zeta} = \frac{\partial}{\partial x}\left(\frac{v}{m}\right) - \frac{\partial}{\partial y}\left(\frac{u}{m}\right)$$

without taking into account the coefficient $m^2(x, y)$ in their definition. In this case, the linearized system will be

$$\left.\begin{aligned} \frac{\partial \tilde{D}}{\partial t} - f\tilde{\zeta} + \nabla^2\phi = 0 \\ \frac{\partial \tilde{\zeta}}{\partial t} + f\tilde{D} = 0 \\ \frac{\partial \phi}{\partial t} + \bar{\phi}m^2\tilde{D} = 0 \end{aligned}\right\}. \qquad (16)$$

With the same usage of $\eta = f\zeta$ the matrix operator will be

$$\begin{pmatrix} 0 & -1 & \nabla^2 \\ f^2 & & \\ \bar{\phi}m^2 & (0) & \end{pmatrix}$$

and some factors will be changed in the definition of the normal modes, but there will be no change concerning the operator $\mathscr{H}$ defining the geopotential of gravity modes. As a conclusion we note that linear or nonlinear normal mode initialization will be written in terms of physical variables in a way that can be achieved by making the changes $D = m^2\tilde{D}$, $\zeta = m^2\tilde{\zeta}$ directly in the initialization scheme itself.

b. *On the (u, v) fields*

These formulations are only written in terms of divergence and vorticity. If we need the (u, v) fields, it is necessary to use the classical relations involving the velocity potential χ and the streamfunction ψ, i.e.,

$$D = m^2\nabla^2\chi, \quad \zeta = m^2\nabla^2\psi$$

$$u = m\left(\frac{\partial\chi}{\partial x} - \frac{\partial\psi}{\partial y}\right), \quad v = m\left(\frac{\partial\chi}{\partial y} + \frac{\partial\psi}{\partial x}\right).$$

In order to solve these Poisson equations for ψ and χ, knowing D and ζ, it is necessary to prescribe some boundary conditions for ψ and χ. This problem occurs when constructing the normal modes as well as when using the filtering scheme. This can be done by using Dirichlet or Neumann boundary conditions, as those discussed by Bourke and McGregor (1983), or even by using nonhomogeneous or nonlinear boundary conditions for the filtering scheme. It appears that we are not restricted to the homogeneous Dirichlet boundary conditions used for ψ and χ in Brière's method.

c. Relation to Brière's scheme

First of all, we verify that the preceding filtering scheme (13), (14), (15) restricted to the particular case $f = \bar{f} =$ constant, $m = \bar{m} =$ constant is equivalent to Brière's scheme written in the physical space in terms of divergence and vorticity. We transform in the physical space the Eqs. (5), (6), (7) in Brière (1982), written for the Fourier components of ψ, χ, ϕ with the following values:

$$\phi_0 = \bar{\phi}\bar{m}^2$$

$$\sigma_{kl}{}^2 = \phi_0 \alpha_{kl}{}^2 + \bar{f}^2,$$

since he used the linearized system (16) with $f = \bar{f}$, $m = \bar{m}$. We then get the filtering scheme:

$$(\bar{f} - \bar{\phi}\bar{m}^2\nabla^2)\delta\tilde{D}^{(q)} = -\bar{f}\frac{\partial\tilde{\zeta}^{(q)}}{\partial t} + \nabla^2\frac{\partial\phi^{(q)}}{\partial t} \tag{17}$$

$$(\bar{f}^2 - \bar{\phi}\bar{m}^2\nabla^2)\delta\phi^{(q)} = \bar{\phi}\bar{m}^2\frac{\partial\tilde{D}^{(q)}}{\partial t} \tag{18}$$

$$\bar{\phi}\bar{m}^2\delta\tilde{\zeta}^{(q)} - \bar{f}\,\delta\phi^{(q)} = 0.$$

This equivalence implies that even if Brière's presentation concerning the boundary conditions seems restrictive, his scheme can potentially be applied with more general boundary conditions such as those chosen in sections 2a and 5b.

Thus a formulation of normal modes as well as of initialization can be done in the particular case $f = \bar{f}$ = constant, $m = \bar{m}$ = constant, in terms of divergence and vorticity, which can handle the preceding boundary conditions for ϕ (section 2a) and for ψ and χ (section 5b) (Juvanon du Vachat, 1985). With the supplementary constraints that the domain is a rectangle with sides parallel to the axes and Dirichlet boundary conditions for ψ and χ, we are in the original Brière's scheme where the usage of Fourier series simplifies the solution of the Helmholtz equations (17), (18) with constant coefficients.

d. The discrete case

It is interesting to note that the relations given for the modes in the continuous case also hold for the discrete analog of the system (1). We suppose a regular grid to be set on a rectangular domain for the sake of simplicity, but the results hold for a more general domain. The Δx and Δy are the mesh spacings, and all variables are defined at the same grid point, denoted (i, j). The discrete system can be written as

$$\left.\begin{aligned} \frac{\partial D_{ij}}{\partial t} - f_{ij}\zeta_{ij} + m_{ij}^2\nabla_{ij}^2\phi_{ij} = 0 \\ \frac{\partial\zeta_{ij}}{\partial t} + f_{ij}D_{ij} = 0 \\ \frac{\partial\phi_{ij}}{\partial t} + \bar{\phi}D_{ij} = 0 \end{aligned}\right\}$$

where ∇_{ij}^2 can be chosen as the classical five-point Laplacian

$$\nabla_{ij}^2\alpha_{ij} = (\alpha_{i+1,j} - 2\alpha_{i,j} + \alpha_{i-1,j})/\Delta x^2 + (\alpha_{i,j+1} - 2\alpha_{i,j} + \alpha_{i,j-1})/\Delta y^2$$

and where ϕ always stands for a deviated geopotential. The deviated geopotential of the gravity modes satisfies the relation

$$\mathscr{H}_{ij}\phi_{ij} \equiv (-f_{ij}^2 + \bar{\phi}m_{ij}^2\nabla_{i,j}^2)\phi_{i,j} = -\sigma^2\phi_{ij}$$

with zero boundary condition for some ϕ_{ij}. By using the classical ordering by successive rows (or columns) for the double index (i, j) we define the column Φ whose elements are the ϕ_{ij} so ordered, and we put the problem into the matrix form

$$\mathbf{H}\Phi = -\sigma^2\Phi$$

where the matrix $\mathbf{H}$ is associated with the discrete operator $\mathscr{H}_{ij}$. It is a symmetrical negative definite matrix like the one associated with a Laplacian with zero boundary conditions (even if some f_{ij} are zero, or for a nonrectangular domain). So the eigenfrequencies are easily related to the eigenvalues of this matrix and can be determined explicitly with a standard eigenvalue package. The eigenvectors Φ^k [the discrete analogs of the eigenfunctions $\phi_k(x, y)$] are obtained in the same process. It is of interest to determine these eigenfrequencies to see what is the practical separation between the stationary mode and the gravity modes for a discrete shallow water model involving the variation of f and m in the linearization. Finally, the author can mention that the Machenhauer scheme applied to this discrete normal mode decomposition leads, after tedious algebraic manipulations, to the discrete analog of the Bourke and McGregor scheme. It has not been reported here since the preceding derivation done for the continuous case is simpler to present.

5. The case of a latitude–longitude formulation

The shallow water equations written for a latitude–longitude formulation are investigated by using the same linearized system:

$$\left.\begin{aligned} &\frac{\partial D}{\partial t} - f\zeta + \nabla^2\phi = 0 \\ &\frac{\partial \zeta}{\partial t} + fD = 0 \\ &\frac{\partial \phi}{\partial t} + \bar{\phi}D = 0 \end{aligned}\right\} \tag{19}$$

with the same definition for the deviated geopotential ϕ (as in section 2a). Now the divergence D, the vorticity ζ, and the Laplacian ∇^2 are spherical operators. The Rossby mode is defined by

$$D = 0, \quad f\zeta = \nabla^2\phi$$

for arbitrary ϕ, and the deviated geopotential of the gravity modes are defined as

$$\mathcal{G}\phi = (f^2 - \bar{\phi}\nabla^2)\phi = \lambda^2\phi.$$

If we make explicit the dependency of the Coriolis parameter f and the Laplacian ∇^2 on the longitude φ and the latitude θ, we obtain the relation:

$$\left[4\Omega^2\sin^2\theta - \frac{\bar{\phi}}{a^2\cos^2\theta} \times \left(\frac{\partial^2}{\partial\varphi^2} + \cos\theta\frac{\partial}{\partial\theta}\cos\theta\frac{\partial}{\partial\theta}\right)\right]\phi = \lambda^2\phi$$

where Ω is the angular rotation and a the radius of the earth. Since the left-hand side operator depends on the longitude φ only by the term $\partial^2/\partial\varphi^2$, separable solutions exist for ϕ of the kind:

$$\phi = \sin(\omega\varphi + x)_x F(\mathcal{O})$$

if the domain is a rectangle of the kind $[\varphi_0, \varphi_1] \times [\theta_0, \theta_1]$, with $\omega = k\pi/(\varphi_1 - \varphi_0)$, and k an integer. The functions $F(\theta)$ are defined by solving the following one-dimensional eigenproblem:

$$\left(4\Omega^2\sin^2\theta + \frac{\bar{\phi}\omega^2}{a^2\cos^2\theta}\right)F - \frac{\bar{\phi}}{a^2\cos\theta}\frac{\partial\cos\theta}{\partial\theta}\frac{\partial F}{\partial\theta} = \lambda^2 F$$

with the boundary conditions $F(\theta_0) = F(\theta_1) = 0$.

This problem has to be solved numerically to find the eigenvalues λ^2 and the functions F for the different values of ω. Since this is only a one-dimensional eigenproblem, it is possible to use such functions $\sin(\omega\varphi + \nu)_x F(\mathcal{O})$ as a basis for practical decomposition. This is a situation similar to the spectral Hough harmonics where a separability appears in terms of $e^{ik\varphi}xF(\theta)$, where the functions $F(\theta)$ are solutions of a one-dimensional eigenproblem. Moreover, as compared to the situation of f = constant in the linearized system (19), there is no additional numerical work to find the normal modes. If we are only interested in nonlinear initialization, we naturally recover the following filtering scheme:

$$\left.\begin{aligned} &\mathcal{G}\delta D^{(q)} = -f\frac{\partial\zeta^{(q)}}{\partial t} + \nabla^2\frac{\partial\phi^{(q)}}{\partial t} \\ &\mathcal{G}\delta\phi^{(q)} = \bar{\phi}\frac{\partial D^{(q)}}{\partial t} \\ &\bar{\phi}\delta\zeta^{(q)} - f\delta\phi^{(q)} = 0 \end{aligned}\right\},$$

and in this case the preceding eigenfunctions can be a useful tool in solving the preceding Helmholtz equations. We have to remember, however, that this occurs for a particular rectangular domain: $[\varphi_0, \varphi_1] \times [\theta_0, \theta_1]$ and by making use of the dependency of the Coriolis parameter on latitude alone. This favorable circumstance would not appear for a pole-rotated model, since the Coriolis parameter would depend on both latitude and longitude, and the problem would no longer be separable.

6. Conclusion

A general formulation of normal modes for limited-area models has been proposed. It does not need the following hypotheses used in Brière's (1982) scheme: (i) constant values for the Coriolis parameter f and for the map scale factor m in the linearization and (ii) a rectangular domain. This formulation is expressed in the physical space, so it can be equally applied to any discretization (for example, finite-element formulation). Due to the choice of the linearization that does not include any derivative of f, we get a stationary Rossby mode. The gravity modes have a geopotential ϕ, which is an eigenfunction of a Helmholtz operator, and vorticity defined by $f\phi/\bar{\phi}$; so their quasi-geostrophic potential vorticity is zero.

These eigenfunctions are not separable for a conformal projection, but if a latitude–longitude grid is used with a rectangular domain they are separable and can be used as a basis for decomposition.

However, by applying Machenhauer's (1977) initialization algorithm to formal expansions along these eigenfunctions and returning to the physical space, we have deduced the vertical mode initialization proposed by Bourke and McGregor (1983) on a more intuitive basis. We have given the formulation of normal modes compatible with Bourke and McGregor's initialization scheme. Moreover, the modal components that have been introduced appear to be a useful tool to give a balance condition with physical variables.

We hope that this physical formulation of normal modes will help future interpretation of limited-area model simulations.

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APPENDIX

Eigenfunctions and Eigenvalues

On the problem for eigenfunctions and eigenvalues:

$$\mathscr{H}\phi \equiv [-f^2(x,y) + m^2(x,y)\nabla^2]\phi = -\mu\phi \qquad \text{(A1)}$$

with the boundary condition

$$\phi = 0 \qquad \text{(A2)}$$

or $\partial\phi/\partial n = 0$.

Let us recall that $f(x, y)$ and $m(x, y)$ are known functions of the Cartesian coordinates (x, y) and $\bar{\phi}$ is a given positive constant. At first, we consider the case of the Dirichlet boundary condition (A2). We put the relation (A1) under the generalized Sturm–Liouville formulation [cf. Courant and Hilbert, 1953, Vol. I, p. 292, Eq. (19)] by dividing it by the quantity $m^2(x, y)$, which never equals zero.

So we have

$$H\phi \equiv \left(\frac{f^2}{m^2} - \bar{\phi}\nabla^2\right)\phi = \frac{\mu\phi}{m^2}. \qquad \text{(A3)}$$

The operator H is self-adjoint for the ordinary scalar product $[\phi_1, \phi_2] = \int_{\mathscr{D}} \phi_1\phi_2 dxdy$ and for the boundary condition (A2).

This can be seen by multiplying the quantity $H\phi$ by an arbitrary function ϕ' satisfying (A2) and by integrating by parts, so we get

$$[H\phi, \phi'] = \int \frac{f^2}{m^2}\phi\phi' dxdy + \bar{\phi}\int \overrightarrow{\text{grad}}\,\phi\, \overrightarrow{\text{grad}}\,\phi' dxdy.$$

The right-hand side of this equality is now a symmetric bilinear form in ϕ and ϕ' denoted $a(\phi, \phi')$. So we obtain

$$[H\phi, \phi'] = a(\phi, \phi') = a(\phi', \phi) = [\phi, H\phi'].$$

Then, by using this property together with (A3), we deduce that the eigenfunctions ϕ_1, ϕ_2 corresponding to different eigenvalues μ_1, μ_2 are orthogonal for the scalar product

$$(\phi_1, \phi_2) = \int_{\mathscr{D}} \phi_1\phi_2 \frac{dxdy}{m^2}.$$

This follows from

$$[H\phi_1, \phi_2] - [\phi_1, H\phi_2] = (\mu_1 - \mu_2)(\phi_1, \phi_2).$$

The eigenfunctions $\phi/m(x, y)$ are orthogonal in the ordinary sense.

Concerning the existence of such eigenfunctions and the expansion theorem for every function satisfying (A2), we refer to Courant and Hilbert (1953, p. 293). We now examine the sign and the existence of a bound α for the lowest eigenvalue. Clearly we have the relation $a(\phi, \phi) = \mu(\phi, \phi)$ and the bilinear form a is positive definite:

$$a(\phi, \phi) > 0 \quad \text{unless} \quad \phi = 0.$$

Moreover, we have $a(\phi, \phi) \geqslant \alpha(\phi, \phi)$ with α positive real for an arbitrary function ϕ satisfying the boundary condition (A2). We can give an explicit value of α by making use of Poincaré's inequality

$$\int_{\mathscr{D}} (\overrightarrow{\text{grad}}\,\phi)^2 dxdy \geqslant c\int_{\mathscr{D}} \phi^2 dxdy$$

valid for ϕ satisfying the Dirichlet boundary condition $\phi = 0$ and for a bounded domain $\mathscr{D}$, with a positive constant c depending only on the domain. We can choose $\alpha = f_0{}^2 + c\bar{\phi}m_0{}^2$ where $f_0{}^2$ and $m_0{}^2$ are the minima of the functions $f^2(x, y)$ and $m^2(x, y)$ on the domain $\mathscr{D}$.

In the case where the equator intersects the domain $f_0 = 0$ there still exists a positive lowest bound for the eigenvalues $\alpha = c\bar{\phi}m_0{}^2$.

All of the preceding results hold for the case of Neumann boundary condition $\partial\phi/\partial n = 0$, except the preceding evaluation of the constant α. Due to its weak interest for application, it is no longer being investigated.

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