An Inconsistency in Vertical Discretization in Some Atmospheric Models

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ABSTRACT

A simple example problem is constructed for which the continuous primitive equations are shown to have a solution. The model is then discretized in the vertical in a manner equivalent to that used in the NMC global spectral model. It is shown that, in general, the discretized version of the problem has no solution. The vertical discretization schemes used in the NMC and many other atmospheric circulation models are therefore inconsistent with the continuous equations. Conditions for design of a vertical differencing scheme that does not have this limitation are suggested.

1. Introduction

Understanding of the atmosphere is limited by sparse or nonexistent data over large regions; some fields, such as latent heating, cannot be measured directly. However, atmospheric simulation models, including general circulation models (GCMs) and weather prediction models, produce what are, in principle, perfect datasets, with all fields known; hence, the understanding of how the models behave is not limited by the lack of data. One recent approach, using this high quality data for a diagnostic understanding of simulation model behavior has been to solve the linearized equations governing the time mean flow, using model output for the basic flow and forcing, and employing the same vertical (Nigam et al., 1986) and horizontal (Navarra, 1985) discretization as the simulation model. The advantage of this approach is that spatial interpolation of input and verification data is eliminated in the diagnostic studies. We have been developing a model linearized about a zonally symmetric basic state for the purpose of diagnosing the maintenance of the stationary waves in monthly or seasonal integrations of the NMC global spectral model. Both models use the vertical staggering of the variables suggested by Lorenz (1960). This note reports on an unexpected and interesting problem that we encountered in attempting to verify the linear model.

Analytic or semianalytic solutions can be constructed for the steady inviscid primitive equations linearized about a resting basic state and forced by specified heat sources and sinks. It was thought that it would be useful to compare the response of the discretized linear model to the "exact" solutions for this situation. However, we found that solutions to the discrete system do not exist, as the governing matrix is singular. A simple example is constructed in section 2 to illustrate the reasons for this problem.

2. The continuous model and solution

The problem to be discussed is that of the steady, thermally forced perturbations to a resting inviscid atmosphere with a flat lower boundary. The perturbations are governed by the primitive equations, and the sigma vertical coordinate (Phillips, 1957) is chosen. The equations are simplified by the use of Cartesian geometry and the midlatitude β-plane approximation. These simplifications and others discussed later are introduced for clarity of presentation and do not affect the conclusions.

The perturbation equations are

- the vorticity equation:

$$f_0 D + \beta v = 0, \tag{1}$$

- the divergence equation:

$$-f_0 \zeta + \beta u = -\nabla^2\left(\phi + \frac{R\bar{T}}{p_0} p_g\right), \tag{2}$$

- the hydrostatic equation:

$$\frac{\partial \phi}{\partial \sigma} = -\frac{RT}{\sigma}, \tag{3}$$

- the mass conservation equation:

$$D + \frac{\partial \dot{\sigma}}{\partial \sigma} = 0, \tag{4}$$

- the thermodynamic equation:

$$\sigma^{\kappa} \dot{\sigma} \frac{\partial}{\partial \sigma}(\bar{T}\sigma^{-\kappa}) = \frac{Q}{c_p}. \tag{5}$$

The coordinates are x in the eastward direction, y in the northward direction, and $\sigma = p/p_s$ in the vertical, with p = pressure, p_s = surface pressure, and the operator $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. The dependent perturbation variables are divergence $D = \nabla_2 \cdot \mathbf{v}$ where ∇_2 is the horizontal divergence operator, relative vorticity $\zeta = k \cdot \nabla \times \mathbf{v}$ where k is the vertical unit vector, vertical σ velocity $\dot{\sigma}$, geopotential ϕ, perturbation surface pressure p_g, and temperature T. The horizontal velocity is $\mathbf{v} = (u, v)$ where u and v are the zonal and meridional components. The perturbation heating, which is specified, is Q. The basic state quantities which enter the equations are temperature $\bar{T} = \bar{T}(\sigma)$ and surface pressure p_0. A constant f_0 is the mean Coriolis parameter and the constant β represents the meridional variation of the Coriolis parameter. The gas constant is R, c_p is specific heat at constant pressure, and $\kappa = R/c_p$.

The boundary conditions on (1)–(5) are

$$\dot{\sigma} = 0 \quad \text{at} \quad \sigma = 0, 1 \tag{6}$$

$$\phi = 0 \quad \text{at} \quad \sigma = 1 \tag{7}$$

and periodicity is imposed in the x direction. The forcing Q and the perturbation variables are taken to have zero zonal mean.

One further simplification that is introduced is to take $\partial/\partial y = 0$ and $Q = \hat{Q}_0(\sigma)\exp(imx)$. The perturbation variables are taken to have a similar form, with $D(x, \sigma) = \hat{D}(\sigma)\exp(imx)$ and so on. Then

$$D = \frac{\partial^2 \chi}{\partial x^2}, \quad \hat{D} = -m^2 \hat{\chi}$$

$$\zeta = \frac{\partial^2 \psi}{\partial x^2}, \quad \hat{\zeta} = -m^2 \hat{\psi}$$

$$\hat{v} = im\hat{\psi}$$

$$\hat{u} = im\hat{\chi} \tag{8}$$

where χ is the velocity potential and ψ is the streamfunction. The system (1)–(7), using (8), becomes

$$-m^2 f_0 \hat{\chi} + im\beta\hat{\psi} = 0, \tag{9}$$

$$m^2 f_0 \hat{\psi} + im\beta\hat{\chi} = m^2\hat{\phi} + m^2 \frac{R\bar{T}}{p_0}\hat{p}_g, \tag{10}$$

$$\frac{\partial\hat{\phi}}{\partial\sigma} = -\frac{R\hat{T}}{\sigma}, \tag{11}$$

$$-m^2\hat{\chi} + \frac{\partial\hat{\dot{\sigma}}}{\partial\sigma} = 0, \tag{12}$$

$$\sigma^\kappa \hat{\dot{\sigma}} \frac{\partial}{\partial\sigma}(\bar{T}\sigma^{-\kappa}) = \frac{\hat{Q}_0}{c_p}, \tag{13}$$

$$\hat{\dot{\sigma}}(0) = \hat{\dot{\sigma}}(1) = 0, \tag{14}$$

$$\hat{\phi}(1) = 0. \tag{15}$$

The solution to (9)–(15) is simple to find schematically. The vertical velocity is found from (13) by

$$\hat{\dot{\sigma}} = \frac{\hat{Q}_0}{c_p \sigma^\kappa}\left[\frac{\partial}{\partial\sigma}(\bar{T}\sigma^{-\kappa})\right]^{-1}. \tag{16}$$

The specified heating is constrained by the boundary condition (14) at $\sigma = 1$ to satisfy $\hat{Q}_0(1) = 0$. As $\sigma \to 0$, (16) and (14) imply that $\hat{Q}_0$ increases less rapidly than $1/\sigma$ for finite $\bar{T}$. Then $\hat{\chi}$ is found by substituting the solution for $\hat{\dot{\sigma}}$ in (12), giving

$$\hat{\chi} = \frac{1}{m^2}\frac{\partial\hat{\dot{\sigma}}}{\partial\sigma}. \tag{17}$$

The streamfunction is then found by substituting $\hat{\chi}$ from (17) into (9), so that

$$\hat{\psi} = -\frac{imf_0}{\beta}\hat{\chi}. \tag{18}$$

The streamfunction and velocity potential solutions are used in the divergence equation, (10), to give the equation for $\hat{\phi}$ and $\hat{p}_g$:

$$\hat{\phi} + \frac{R\bar{T}}{p_0}\hat{p}_g = g(\sigma) \tag{19}$$

where

$$g(\sigma) = f_0\hat{\psi} + i(\beta/m)\hat{\chi}.$$

The surface pressure is found from (19) by using the boundary condition (15):

$$\hat{p}_g = \frac{p_0}{R\bar{T}(1)}g(1), \tag{20}$$

and $\hat{\phi}(\sigma)$ is then determined from (19) and (20) by

$$\hat{\phi} = g(\sigma) - \frac{R\bar{T}}{p_0}\hat{p}_g. \tag{21}$$

Finally, the temperature is found from the hydrostatic equation, using $\hat{\phi}$ from (21):

$$\hat{T} = -\frac{\sigma}{R}\frac{\partial\hat{\phi}}{\partial\sigma}. \tag{22}$$

3. The discrete model and analysis

The vertical differencing scheme used by the NMC global spectral model, as described by Sela (1980), is applied to the model equations. The conclusions drawn using this scheme, however, apply to those of most other GCMs and weather prediction models. The arrangement of the discrete variables for $K + 1$ equally spaced $\dot{\sigma}$ levels is shown in Fig. 1. The level spacing is taken to be a constant, Δ. The discretized versions of (9)–(15) are

- vorticity:

$$-m^2 f_0 \hat{\chi}_k + im\beta\hat{\psi}_k = 0, \quad k = 1, \cdots, K, \tag{23}$$

- divergence:

$$m^2 f_0 \hat{\psi}_k + im\beta\hat{\chi}_k = m^2\hat{\phi}_k + m^2\frac{R\bar{T}_k}{p_0}\hat{p}_g, \quad k = 1, \cdots, K, \tag{24}$$

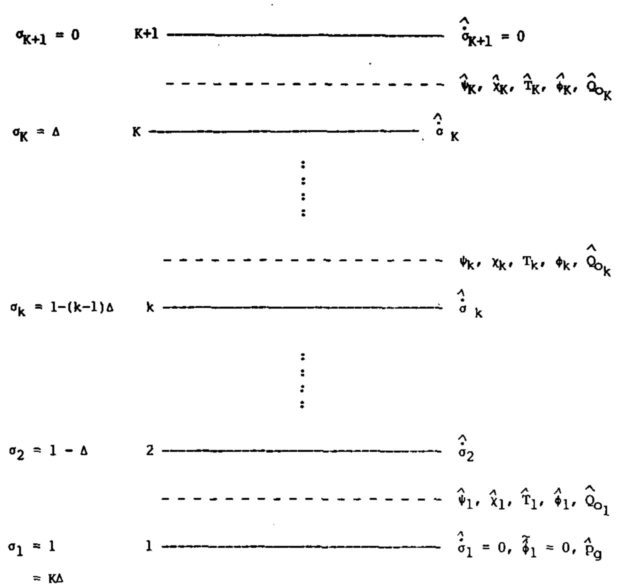

FIG. 1. Vertical structure of the model, equivalent to that used in the NMC global spectral model. (Notation is defined in the text.)

• hydrostatic:

$$\hat{\phi}_k-\hat{\phi}_{k-1}+\frac{c_p}{2}\left[\left(1-\frac{\pi_{k-1}}{\pi_k}\right)\hat{T}_k+\left(\frac{\pi_k}{\pi_{k-1}}-1\right)\hat{T}_{k-1}\right]=0,$$
$$k=2,\cdots,K \quad (25a)$$

$$R\Delta\sum_{k=1}^{K}\hat{T}_k=\Delta\sum_{k=1}^{K}\hat{\phi}_k-\tilde{\phi}_1, \quad (25b)$$

• mass conservation:

$$-m^2\hat{\chi}_k-\frac{\hat{\sigma}_{k+1}-\hat{\sigma}_k}{\Delta}=0, \quad k=1,\cdots,K \quad (26)$$

• thermodynamic:

$$-\frac{1}{2\Delta}\left[\hat{\sigma}_{k+1}\left(\frac{\pi_k}{\pi_{k+1}}\bar{T}_{k+1}-\bar{T}_k\right)+\hat{\sigma}_k\right.$$
$$\left.\times\left(\bar{T}_k-\frac{\pi_k}{\pi_{k-1}}\bar{T}_{k-1}\right)\right]=\frac{\hat{Q}_{0k}}{c_p}, \quad k=1,\cdots,K \quad (27)$$

$$\hat{\sigma}_1=\hat{\sigma}_{K+1}=0, \quad (28)$$

$$\tilde{\phi}_1=0. \quad (29)$$

In (25a) and (27), π_k represents σ^κ at the dashed levels and is a function of the values of σ_j. The number of discrete variables is K (for $\hat{\psi}_k$) + K (for $\hat{\chi}_k$) + $K + 1$ (for $\hat{\sigma}_k$) + K (for $\hat{\phi}_k$) + K (for $\hat{T}_k$) + 1 (for $\tilde{\phi}_1$) + 1 (for $\hat{p}_g$) = $5K + 3$. The number of equations is K [for (23)] + K [for (24)] + K [for (25)] + K [for (26)] + K [for (27)] + 2 [for (28)] + 1 [for (29)] = $5K + 3$. The number of variables equals the number of equations, and therefore the solution to the discrete problem may be found by inverting a square matrix. However, it turns out that this matrix is singular.

The solution of the discrete system is attempted by Gaussian elimination, following the procedure outlined for the continuous system. First, the values $\hat{\sigma}_k$ are determined from the thermodynamic equation (27), and the boundary conditions on $\hat{\sigma}_1$ and $\hat{\sigma}_{K+1}$, (28). These are K equations from (27) and 2 from (28), or $K + 2$ equations, for $K + 1$ values of $\hat{\sigma}_k$. Therefore, the system is overdetermined, so that for a solution to exist only $K - 1$ of the K values of the forcing $\hat{Q}_{0k}$ may be arbitrarily chosen.

A constraint on the vertical distribution on the heating is implicit in the system. In the two-level system ($K = 2$), this constraint is found to be

$$\hat{Q}_{02}=\frac{\pi_2}{\pi_1}\hat{Q}_{01}. \quad (30)$$

In the multilevel case, taking $\sigma^\kappa\partial(\bar{T}\sigma^\kappa)/\partial\sigma$ to be constant gives the constraint

$$\sum_{k=1}^{K}(-1)^k\hat{Q}_{0k}=0. \quad (31)$$

In this constant stability case, the constraints on the continuous solution heating are $\hat{Q}_0(1) = \hat{Q}_0(0) = 0$.

The constraint on $\hat{Q}_{0k}$ in the two-level case represented by (30) is evidently spurious as $|\hat{Q}_{02}| < |\hat{Q}_{01}|$. The constraint (31) for the multilevel case does not represent the continuous constraints and is also spurious, in particular as $K \to \infty$. The vertical differencing structure of Fig. 1 implicitly introduces this nonphysical constraint on the allowed thermal forcing due to having one less "interior" level for $\hat{\sigma}_k$ than the number of levels at which the thermodynamic equation is applied; as discussed in the Introduction, the vertical differencing scheme is therefore inconsistent. If the $\hat{\sigma}$ levels were the same as the heating levels, no constraint other than the natural constraints analogous to the continuous system would occur.

Suppose now that the imposed heating satisfies the constraint that allows a steady solution to the discretized equations. Thus, $\hat{\sigma}_k$ may be assumed to be known, and a solution is sought to the system (23)–(26) and (29). It may then be verified that the reduced problem is a set of $4K + 1$ equations in $4K + 2$ unknowns, and the complete solution to this system cannot be found without the imposition of an additional (arbitrary) constraint.

Using (26), $\hat{\chi}_k$ may be found and substituted into (23) to determine $\hat{\psi}_k$. There is no problem encountered in completely determining the total velocity field of the solution. Also, (29) determines $\tilde{\phi}_1$. Substitution for $\hat{\psi}_k$ and $\hat{\chi}_k$ in (24) leaves $2K$ equations (24) and (25) in the $2K + 1$ unknowns ϕ_k, T_k, and $\hat{p}_g$, and the solution can proceed no further. In the continuous system the lower boundary condition $\hat{\phi}(1) = 0$ was used in the reduced divergence equation to determine $\hat{p}_g$, while in the discretized system this is not possible, because the discretized divergence equation is not applied at $\sigma = 1$.

If an arbitrary set of values of the forcing $\hat{Q}_{0k}$ is used, then there is no *steady* solution to the finite difference system. However, an unsteady solution to the time dependent version of the linearized discretized system will exist, a result that can be demonstrated using time differencing. So for the situation presented here in which a steady solution to the continuous system exists, the finite difference system can produce only a time varying solution. The vertical differencing scheme then leads to spurious transience in this simplified problem.

4. Discussion

A simple model problem has been constructed to illustrate inconsistencies in the type of vertical differencing scheme used in some atmospheric general circulation and weather prediction models. One of the simplest problems that can be constructed from the continuous equations has no solution in the discretized system.

The difficulties we encountered were due only to the manner in which the variables are arranged in the vertical in the Lorenz scheme, and not to the specific interpolation formulae used in developing the equations. For the example problem previously discussed—the steady inviscid thermally forced primitive equations linearized about a resting basic state—the choice of a continuous, grid, or spectral representation in the horizontal is also irrelevant. Additionally, the singularity in the thermodynamic equation is easily removed by including, for example, thermal diffusion or a nonresting basic state (but not viscous dissipation).

Since the problem for which the difficulty is encountered involves linearization and an extremely simple basic state, it is not obvious that the use of a consistent vertical discretization (with respect to the linear model) would be of importance in the integration of the full primitive equations. The inconsistency found in the application of the discretized model presented here is specific to the case of steady non-zero thermal forcing. No difficulty is encountered in obtaining the normal modes or the solutions with thermal forcing oscillitory in time. The normal modes of the discretized model are used successfully in the elimination of undesired initial transience in the (ad hoc) normal mode initialization procedure. Since the linear solutions in the transient case then have predictive value in terms of the model behavior, it is not inconceivable that the failure of the steady thermally forced discretized linear model is indicative of a serious inconsistency in the full discretized primitive equation model. The effects of this problem would be most evident in the tropical upper tropospheric flow and in the long time mean zonally averaged flow, as in these cases the thermodynamic equation in the form (13) is a good approximation to the full thermodynamic equation.

It is not difficult to design energy conserving vertical differencing schemes which are consistent with the continuous equations for this problem. The simplest cure is to 1) have the levels at which temperature and vertical sigma velocity coincide, and 2) place the lowest layer at which vorticity, divergence, and geopotential are defined at the lower boundary level, $\sigma = 1$. A model which satisfies both conditions is described by Staniforth and Daley (1977). Intercomparison of the various schemes which satisfy these two constraints is a matter for further research, as is the evaluation of the quantitative importance of changing the vertical discretization to one that is consistent with the linearized equations.

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