Non-Normal Mode Initialization: Formulation and Application to Inclusion of the β-Terms in the Linearization

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ABSTRACT

The non-normal mode initialization, i.e., an initialization scheme which does not require an explicit computation of the eigenmodes of the linearized equations, is reviewed. The formulation of such a scheme is given in abstract form, in the case of the Machenhauer initialization scheme as well as in the case of higher-order schemes. The particular case of a stationary Rossby mode is examined in detail. In this case, the separation between slow modes and fast gravity modes is explicitly given, and it is conjectured that the formulation of non-normal mode initialization can be given only in such a case. An application to the shallow-water equations, which includes the main β-terms in the linearization is given as a result of the preceding formulation. Such a scheme extends the previous scheme proposed by Bourke and McGregor.

1. Introduction

Nonlinear normal mode initialization is now considered an efficient technique for the initialization of primitive equation models. The formulation developed by Machenhauer (1977) and Baer (1977) has been applied successfully to global models (Daley 1979) or to limited-area models (Brière 1982). This initialization successfully eliminates unwanted gravity wave oscillations of these models.

Nonlinear normal mode initialization was first developed for numerical models spanning the hemispheric or global domain. For a limited-area model, the problem of finding the normal modes of the linearized set of the model equations is not separable due to the variation of the Coriolis factor f (and of the map scale factor m in the case of a conformal projection). This problem is further complicated by the problem of boundary conditions. So it is necessary to make drastic assumptions, i.e., to use constant values of f and m in the linearized equations to deduce the normal modes in a cost efficient way.

Due to these problems, Bourke and McGregor (1983) have suggested a procedure which does not require explicit identification of the normal modes but gives rise to a filtering scheme entirely expressed in physical space. This technique has been previously introduced by Ballish (1980) as "non-normal mode initialization," and was deduced on the basis of the bounded derivative method. In a previous paper (Juvanon du Vachat 1986), we deduced the Bourke McGregor (BMG) scheme as a rigorous consequence of the Machenhauer (1977) algorithm applied to the formulation of normal modes for the continuous eigenproblem. The strategy to do this deduction is not fully general, but rather linked to the particular linearization chosen by BMG. This linearized set of equations indeed includes the variation of f but not the β-terms involving the derivatives of f.

The question arises of how to formulate an initialization scheme in the physical space without this restriction—or to know what other terms involving the derivatives of f can be included in such a physical-space initialization. It is the purpose of this paper to address this question. What linearizations are "admissible" to formulate a non-normal mode initialization, i.e., an initialization scheme which does not require an explicit computation of the eigenmodes of the linearized equations? The technique that will be used is the same as used by Juvanon du Vachat (1986), i.e., to consider the continuous eigenproblem in order to have continuous rather than discrete formulations. We restrict our discussion to the shallow water equations, which embody the essentials of the normal mode approach for primitive equation models. But the approach developed in this paper can be equally applied to a limited-area model or to a global or hemispheric model.

This paper is organized as follows. In section 2, the nonlinear initialization is formulated in abstract form for the Machenhauer scheme (2b), with special reference to the case of a stationary Rossby mode (2c), and extended to the case of a higher-order scheme (2d). These formulations are briefly applied to the BMG linearization (section 3). Then the extension to a linear-

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ization including the β-terms is considered (section 4). First, the existence of orthonormal modes in this case is demonstrated as a consequence of the energy conserving linearization (4b). Then a non-normal mode initialization scheme, with a linearization including the most β-terms is formulated (4b). The general case, with all the β-terms, is briefly discussed (4c). After some additional remarks (section 5), the summary and conclusion are given (section 6).

2. The nonlinear initialization in abstract form

a. The normal modes of the abstract system

We consider the linearized shallow-water equations written with a vector state **E**, depending on the horizontal coordinates and the time t as follows:

$$\frac{\partial \mathbf{E}}{\partial t} + \mathcal{M}(\mathbf{E}) = 0. \qquad (1)$$

We assume that for each t, the function **E** in terms of the horizontal coordinates belongs to the functional space $\mathcal{E}$ (which includes the boundary conditions if necessary). The matrix operator $\mathcal{M}$ includes the chosen linearization, so it only contains spatial operators in terms of the horizontal coordinates. Here **E** is a three-component vector function, for example,

$$\mathbf{E} = \begin{pmatrix} D \\ \zeta \\ \phi \end{pmatrix}$$

with the divergence D, the vorticity ζ, and the geopotential ϕ defined on the domain of interest $\mathcal{D}$, with appropriate boundary conditions. We assume that the matrix operator $\mathcal{M}$ has the following eigenmodes (i.e., a vector **E** different from zero belonging to $\mathcal{E}$ such that $\mathcal{M}(\mathbf{E}) = \lambda\mathbf{E}$): a denumerable family of slow modes $\mathbf{R}_k$ (**R** like Rossby), with k integer; and two denumerable families of gravity modes $\mathbf{G}_k$ and $\mathbf{G}'_k$, with k an integer. We assume three families since we have a three-component problem. Due to the hyperbolic nature of the system (1) we have $\lambda = i\sigma$, with σ real, so that for such a mode the temporal dependency is $e^{-i\sigma t}$, and σ appears as an eigenfrequency. This point will be further rigorously deduced from the properties of the operator $\mathcal{M}$ in section 4. So we have the following relations:

$$\left.\begin{aligned} \mathcal{M}(\mathbf{R}_k) &= i\sigma_{1k}\mathbf{R}_k \\ \mathcal{M}(\mathbf{G}_k) &= i\sigma_{2k}\mathbf{G}_k \\ \mathcal{M}(\mathbf{G}'_k) &= i\sigma_{3k}\mathbf{G}'_k \end{aligned}\right\}. \qquad (2)$$

We assume that there is a frequency separation between the slow modes and the gravity modes:

$$|\sigma_{1k}| \ll |\sigma_{2k}|,\ |\sigma_{3k}| \quad \text{for arbitrary } k,$$

with the possibility of having $\sigma_{1k} = 0$ for arbitrary k. Finally we assume that every vector state **E** belonging to $\mathcal{E}$ can be generated with these modes, i.e. can be decomposed in a unique way along the relation

$$\mathbf{E} = \sum_k x_{1k}\mathbf{R}_k + \sum_k x_{2k}\mathbf{G}_k + \sum_k x_{3k}\mathbf{G}'_k \qquad (3)$$

with three suitable sequences of complex coefficients x_{1k}, x_{2k}, x_{3k} which are the slow modal components (x_{1k}) and the gravity modal components (x_{2k}, x_{3k}). The definition of these modal components can be given by using an orthogonal projection if the modes are orthonormal for a given scalar product, or by using another decomposition technique as done in Juvanon du Vachat (1986).

b. The Machenhauer initialization scheme

In order to implement nonlinear initialization we consider the complete nonlinear equations written in the form

$$\frac{\partial \mathbf{E}}{\partial t} + \mathcal{M}(\mathbf{E}) = \mathbf{N}. \qquad (4)$$

The three-component vector function **N** includes the nonlinear terms but also the linear terms not taken into account in the linearization defined by $\mathcal{M}$.

The nonlinear system (4) can be transformed by a straightforward identification of the modal components by using the decomposition (3) assumed to be unique, together with the relation (2) into

$$\left.\begin{aligned} \frac{dx_{1k}}{dt} + i\sigma_{1k}x_{1k} &= N_{1k} \\ \frac{dx_{2k}}{dt} + i\sigma_{2k}x_{2k} &= N_{2k} \\ \frac{dx_{3k}}{dt} + i\sigma_{3k}x_{3k} &= N_{3k} \end{aligned}\right\} \qquad (5)$$

where the nonlinear modal components N_{1k}, N_{2k}, N_{3k} are defined in terms of **N** by the relation [analog of (3)]

$$\mathbf{N} = \sum_k N_{1k}\mathbf{R}_k + \sum_k N_{2k}\mathbf{G}_k + \sum_k N_{3k}\mathbf{G}'_k. \qquad (6)$$

Machenhauer's algorithm of nonlinear normal mode initialization consists of maintaining the slow components at their initial values ($x_{1k} = x_{1k}^{(0)}$) and writing the stationarity of gravity components, $dx_{2k}/dt = dx_{3k}/dt = 0$, in the full nonlinear system (5). This can be reached by the explicit iterative scheme:

$$\left.\begin{aligned} x_{1k}^{(q+1)} &= x_{1k}^{(q)} \\ x_{2k}^{(q+1)} &= N_{2k}^{(q)}/i\sigma_{2k} \\ x_{3k}^{(q+1)} &= N_{3k}^{(q)}/i\sigma_{3k} \end{aligned}\right\}. \qquad (7)$$

By using the notation $\delta x_{ik} = x_{ik}^{(q+1)} - x_{ik}^{(q)}$ and transforming the nonlinear terms N_{2k}, N_{3k} in terms of the time tendencies of the modal components dx_{2k}/dt, dx_{3k}/dt along the equations (5), we get the following iterative scheme

$$\delta x_{1k} = 0 \tag{8a}$$

$$\delta x_{2k} = (1/i\sigma_{2k})(dx_{2k}/dt) \tag{8b}$$

$$\delta x_{3k} = (1/i\sigma_{3k})(dx_{3k}/dt). \tag{8c}$$

The strategy for formulating non-normal mode initialization will then be to express this scheme back in the physical space, i.e., in terms of **E** and $\mathcal{M}$ and of other ingredients that will appear necessary.

At first (8a) can be written, by summing over all the slow components, as

$$(\delta \mathbf{E})_R = \sum_k \delta x_{ik} \mathbf{R}_k = 0 \tag{9}$$

where the notation $(\delta \mathbf{E})_R$ means the restriction of the vector $\delta \mathbf{E}$ to its slow components, and must be understood as an infinite sum over all the slow components.

By synthetizing for a vector $\delta \mathbf{E}$ the relations (8a), (8b) and (8c), we get

$$\delta \mathbf{E} = \sum_k \left(\frac{1}{i\sigma_{2k}} \frac{dx_{2k}}{dt} \mathbf{G}_k + \frac{1}{i\sigma_{3k}} \frac{dx_{3k}}{dt} \mathbf{G}'_k \right).$$

Applying the operator $\mathcal{M}$ to this equality and using the property (2) we eliminate the unknown quantities $1/i\sigma_{2k}$, $1/i\sigma_{3k}$ and get:

$$\mathcal{M}(\delta \mathbf{E}) = \sum_k \left(\frac{dx_{2k}}{dt} \mathbf{G}_k + \frac{dx_{3k}}{dt} \mathbf{G}'_k \right) = \left(\frac{\partial \mathbf{E}}{\partial t} \right)_G . \tag{10}$$

The notation $(\partial \mathbf{E}/\partial t)_G$, i.e., the vector $\partial \mathbf{E}/\partial t$ restricted to its gravity components, must be understood as an infinite sum over all the gravity components. An equivalent relation has also been derived by Temperton (1988) (cf. his Eq. (3.5)) but here it is expressed more directly in terms of sums over gravity modes, rather than by matrix manipulation. This formulation can be equally written as

$$\mathcal{M}(\delta \mathbf{E}) = \frac{\partial \mathbf{E}}{\partial t} - \left(\frac{\partial \mathbf{E}}{\partial t} \right)_R \tag{11}$$

with the equivalent notation $(\partial \mathbf{E}/\partial t)_R$ for the restriction of the same vector $\partial \mathbf{E}/\partial t$ to its slow components.

Moreover, since the decomposition (3) is unique, a consequence of the relation (10) is

$$[\mathcal{M}(\delta \mathbf{E})]_R = 0$$

which is equivalent, if the operator $\mathcal{M}$ is nonsingular, to the relation (9). So, clearly, for a nonsingular operator $\mathcal{M}$, the relations (8a, b, c) are equivalent to the relation (10) or (11). Therefore, it is only necessary to characterize the global expressions $(\mathbf{E})_G$ or $(\mathbf{E})_R$ without the details of the component along each gravity mode or along each slow mode.

c. Particular case of a stationary mode

We examine in more details the case when the slow mode is stationary, since it has been met with the Brière (1982) or BMG linearization. This means that $\sigma_{1k} = 0$ for arbitrary k, and that the operator $\mathcal{M}$ is singular and its kernel is generated by the slow modes. In this case the relation (10) does not imply the relation (9). In other words, the relation (10) written with a singular operator $\mathcal{M}$ give a nonunique solution unless the relation (9) is satisfied. In this case we have to write simultaneously the two relations (9) and (10) [or (11)].

An additional relation can also be deduced in the particular case when the slow modes are stationary. The condition of stationarity of the gravity modal components $dx_{2k}/dt = dx_{3k}/dt = 0$ can be written with the global expressions as

$$\left(\frac{\partial \mathbf{E}}{\partial t} \right)_G = 0$$

or equivalently as

$$\frac{\partial \mathbf{E}}{\partial t} = \left(\frac{\partial \mathbf{E}}{\partial t} \right)_R$$

which is equivalent to $\mathcal{M}(\partial \mathbf{E}/\partial t) = 0$, since the kernel of the operator $\mathcal{M}$ is generated by the slow modes. This last relation can be read as

$$\frac{\partial}{\partial t} \mathcal{M}(\mathbf{E}) = 0. \tag{12}$$

So the condition of stationarity of gravity modal components can be deduced by simply taking the time derivative of the equations defining the slow modes. It is a new result which has not been identified in Temperton (1988) and it will be used in section 3.

d. Higher-order initialization schemes

Machenhauer's initialization algorithm is usually considered a first-order initialization scheme obtained with the hypothesis of the constant nonlinear gravity modal components. If they are assumed to be linear in terms of time, Baer and Tribbia (1977) and recently Tribbia (1984) have proposed another scheme which is practically equivalent to the stationarity of the second-order time derivatives for the gravity modal components: $d^2x_G/dt^2 = 0$. Our development now follows closely Machenhauer (1982). Considering the time-rate equation of a gravity mode

$$\frac{dx_G}{dt} + i\sigma_G x_G = N_G$$

we deduce the second-order equation:

$$\frac{d^2x_G}{dt^2} + i\sigma_G \frac{dx_G}{dt} = \frac{dN_G}{dt}$$

or equivalently

$$\frac{d^2x_G}{dt^2} + i\sigma_G(N_G - i\sigma_G x_G) = \frac{dN_G}{dt}$$

so that the condition $d^2x_G/dt^2 = 0$ can be written as

$$x_G = -\frac{1}{(i\sigma_G)^2}\left[\frac{dN_G}{dt} - i\sigma_G N_G\right]$$

and computing the nonlinear terms N_G and dN_G/dt in terms of the tendencies dx_G/dt and d^2x_G/dt^2, we get

$$\delta x_G = -\frac{1}{(i\sigma_G)^2}\frac{d^2x_G}{dt^2}.$$

Thus, the iterative initialization scheme can be written as

$$\delta x_{1k} = 0,$$

$$\delta x_{2k} = -\frac{1}{(i\sigma_{2k})^2}\frac{d^2x_{2k}}{dt^2},$$

$$\delta x_{3k} = -\frac{1}{(i\sigma_{3k})^2}\frac{d^2x_{3k}}{dt^2},$$

from which we deduce, in the physical space,

$$\mathcal{M}^2(\delta \mathbf{E}) = -\left(\frac{\partial^2\mathbf{E}}{\partial t^2}\right)_G$$

with the same particular case of stationary Rossby mode when the condition $(\delta \mathbf{E})_R = 0$ must be kept. And in this case, the second-order balance condition $d^2x_G/dt^2 = 0$ can be written as

$$\mathcal{M}\left(\frac{\partial^2\mathbf{E}}{\partial t^2}\right) = \frac{\partial^2}{\partial t^2}\mathcal{M}(\mathbf{E}) = 0. \qquad (13)$$

Thus it is obtained by simply taking the second time derivative of the equations defining the slow modes. These formulations can be easily generalized at an arbitrary order n, and become, respectively,

$$\mathcal{M}^n(\delta \mathbf{E}) = (-1)^{n-1}\left(\frac{\partial^n\mathbf{E}}{\partial t^n}\right)_G$$

and $\mathcal{M}(\partial^n\mathbf{E}/\partial t^n) = (\partial^n/\partial t^n)\mathcal{M}(\mathbf{E}) = 0$ for a stationary Rossby mode.

We have shown with very weak hypotheses that the Machenhauer scheme can be put in a very general form only involving the matrix operator $\mathcal{M}$ of the linearization, the time tendency of the vector state $\partial\mathbf{E}/\partial t$, and a general characterization of the Rossby modes or of the fast gravity modes. In the particular case of a stationary Rossby mode, the condition of conservation of the Rossby component is not a simple consequence of this relation and must be written separately. In addition, in this case the stationarity of the gravity modal components is equivalent to the balance condition $\partial/\partial t[\mathcal{M}(\mathbf{E})] = 0$; it is obtained by simply taking the time derivative of the equations defining the slow modes. Similarly, a n-order initialization scheme (Tribbia 1984) can be put in a general form involving the matrix operator $\mathcal{M}$ and the n-order time derivative $\partial^n\mathbf{E}/\partial t^n$ and a general characterization of the Rossby modes or of the fast gravity modes. The same remark holds for the case of a stationary Rossby mode and the balance condition is then written as $\partial^n/\partial t^n[\mathcal{M}(\mathbf{E})] = 0$.

3. Application with the BMG linearization

We consider the BMG linearization of the shallow water model written with a mean free height $\bar{\phi}$ as

$$\left.\begin{aligned} \frac{\partial D}{\partial t} - f\zeta + \nabla^2\phi = 0 \\ \frac{\partial \zeta}{\partial t} + fD = 0 \\ \frac{\partial \phi}{\partial t} + \bar{\phi}D = 0 \end{aligned}\right\} \qquad (14)$$

with the operator

$$\mathcal{M} = \begin{pmatrix} 0 & -f & \nabla^2 \\ f & 0 & 0 \\ \bar{\phi} & 0 & 0 \end{pmatrix}.$$

It is a singular operation with a stationary mode defined by

$$f\zeta - \nabla^2\phi = 0, \quad D = 0. \qquad (15)$$

As indicated in section 2c, the stationarity of gravity modal components can be immediately written as [cf. Eq. (12)]

$$\frac{\partial}{\partial t}(f\zeta - \nabla^2\phi) = \frac{\partial D}{\partial t} = 0$$

which is the Eq. (9) in the BMG paper.

To write the Machenhauer iterative scheme from the relation (11), we impose that the vector

$$\mathcal{M}(\delta \mathbf{E}) - \frac{\partial \mathbf{E}}{\partial t} = \begin{pmatrix} D_1 \\ \zeta_1 \\ \phi_1 \end{pmatrix}$$

belongs to the stationary manifold, i.e., that D_1, ζ_1, ϕ_1 satisfy the equations (15). After rearrangement, we get the two relations

$$f\delta\zeta - \nabla^2\delta\phi = -\frac{\partial D}{\partial t} \qquad (16)$$

$$(f^2 - \bar{\phi}\nabla^2)\delta D = f\frac{\partial \zeta}{\partial t} - \nabla^2\frac{\partial \phi}{\partial t} \qquad (17)$$

which are (14) and (16), respectively, in the BMG paper.

The problem remains of how to express $(\delta \mathbf{E})_R = 0$ or equivalently, that $\delta \mathbf{E}$ belongs to the gravity manifold. It is deduced rigorously in Juvanon du Vachat (1986) by explicitly determining the gravity modes of the system, and then by giving an explicit determination of a Rossby modal component x_1. That gives the relation:

$$\bar{\phi}\delta\zeta - f\,\delta\phi = 0 \tag{18}$$

which is also derived by a different way in Temperton (1988) ("property 2").

This relation which means the conservation of quasi-geostrophic potential vorticity has been previously derived by Brière (1982) and BMG but in the particular case of constant f. It must be noticed that it was not identified in the Ballish (1980) non-normal mode initialization proposal. Clearly, with (16), (17) and (18), we get scheme B proposed by BMG.

The second-order balance condition (13) is simply written as

$$\frac{\partial^2 D}{\partial t^2} = \frac{\partial^2(f\zeta - \nabla^2\phi)}{\partial t^2} = 0.$$

On the contrary, BMG discussed in their paper the condition $\partial D/\partial t = \partial^2 D/\partial t^2 = 0$ proposed by Ballish (1980). This condition clearly cannot be justified on the basis of the preceding filtering conditions as it is impossible to write only $\partial D/\partial t = 0$ (or $\partial^2 D/\partial t^2 = 0$) without writing

$$\frac{\partial}{\partial t}(f\zeta - \nabla^2\phi) = 0,$$

or

$$\frac{\partial^2}{\partial t^2}(f\zeta - \nabla^2\phi) = 0,$$

if we want to be coherent with the BMG linearization (14).

However, the technique used in (Juvanon du Vachat 1986) appears as an "oblique expansion technique" since the geopotential components of the gravity modes are mutually orthogonal, but not the different 3-component gravity modes as a whole. Similarly, an arbitrary gravity mode is not orthogonal to an arbitrary Rossby mode as is the case in a normal mode problem. Also, the inclusion of the β-terms in the linearization is not possible with this technique, which seems restricted to the particular linearization chosen by BMG.

4. Inclusion of the β-terms in the linearization

We now address the question of how to work with the β-terms in the linearization. It is not clear how important the inclusion may be of these terms in the linearization. However, Ballish (1979) showed in the context of a one-dimensional model that the omission of these terms from the derivation of the normal modes leads to larger oscillations than if they are included. Moreover, in order to have a more complete technique at our disposal, we now exhibit all the Hilbert properties that come from a normal mode problem: orthogonal modes for a given scalar product.

a. *Orthonormal modes*

We now attack the problem of the existence of such modes, i.e., those modes that have a temporal dependency as $e^{i\sigma t}$. Their existence is a consequence of a quadratic invariant for the linearization, as will be shown later. So, we now consider the general system (1) written for the divergence, vorticity and geopotential as

$$\left.\begin{aligned} \frac{\partial}{\partial t}\nabla^2\chi - \nabla\cdot(f\nabla\psi) + \mathbf{k}\times\nabla\chi\cdot\nabla f + \nabla^2\phi = 0 \\ \frac{\partial}{\partial t}\nabla^2\psi + \nabla\cdot(f\nabla\chi) + \mathbf{k}\times\nabla\psi\cdot\nabla f = 0 \\ \frac{\partial\phi}{\partial t} + \bar{\phi}\nabla^2\chi = 0 \end{aligned}\right\}. \tag{19}$$

We refer to the Appendix for notations and this formulation of the shallow water equations. In this case, the linearized operator includes all the Coriolis effects. It is well known that the whole nonlinear shallow water equations conserve the energy, i.e., the quantity

$$\int_{\mathcal{D}} [\phi(|\nabla\chi|^2 + |\nabla\psi|^2) + |\phi|^2]d\omega$$

where $d\omega$ is the surface differential element.

But the question addressed here is to know what linearizations can conserve the linearized energy i.e., the quantity EL:

$$\mathrm{EL} = \int_{\mathcal{D}} [\bar{\phi}(|\nabla\chi|^2 + |\nabla\psi|^2) + |\phi|^2]d\omega.$$

This conservation of the energy is examined in more details in the Appendix. It follows that if we only consider the Coriolis effects, the terms $\nabla\cdot(f\nabla\psi)$ and $\nabla\cdot(f\nabla\chi)$ must be kept as a whole and simultaneously but each of the terms $\mathbf{k}\times\nabla\chi\cdot\nabla f$ and $\mathbf{k}\times\nabla\psi\cdot\nabla f$ may or may not be kept, each one independently. These last terms have no contribution to the conservation of the energy. A correct specification of the boundary conditions is also required to have conservation of the energy. As shown in the Appendix, we can choose the following boundary conditions:

$$\psi = \chi = \phi = 0 \tag{BC1}$$

or

$$\left.\begin{aligned} \psi &= 0 \\ \frac{\partial \chi}{\partial n} &= 0 \\ f\nabla\psi\cdot\mathbf{n} &= \nabla\phi\cdot\mathbf{n} \end{aligned}\right\} \qquad \text{(BC2)}$$

where $\mathbf{n}$ is the outward pointing normal vector at the boundary.

As a consequence, the BMG linearization given by the system (14) where only the terms $f\nabla^2\psi$ and $f\nabla^2\chi$ are retained does not conserve the energy. We refer to subsection 5c for quadratic invariant conservation and orthogonality relationships in this particular case.

We now establish the result that normal modes exist with such an energy conserving linearization. We define the scalar product over

$$\mathscr{E} = \left\{ \mathbf{E} = \begin{pmatrix} \chi \\ \psi \\ \phi \end{pmatrix} \right\} :$$

$$\langle \mathbf{E}_1, \mathbf{E}_2 \rangle = \int_{\mathscr{D}} [\bar{\phi}(\nabla\chi_1\cdot\nabla\chi_2^* + \nabla\psi_1\cdot\nabla\psi_2^*) + \phi_1\phi_2^*]d\omega \qquad (20)$$

where $*$ means complete conjugate.

It is clearly associated with the energy since $\langle \mathbf{E}, \mathbf{E} \rangle = \mathrm{EL}$. As can be seen in the Appendix, the conservation of the energy can be written in a more general sense as

$$\left\langle \frac{\partial \mathbf{E}_1}{\partial t}, \mathbf{E}_2 \right\rangle + \left\langle \mathbf{E}_1, \frac{\partial \mathbf{E}_2}{\partial t} \right\rangle = 0. \qquad (21)$$

If $\mathscr{L}$ denotes the linearized operator for

$$\frac{\partial}{\partial t}\begin{pmatrix} \chi \\ \psi \\ \phi \end{pmatrix},$$

i.e., the system (19) is written equivalently as

$$\frac{\partial \mathbf{E}}{\partial t} + \mathscr{L}(\mathbf{E}) = 0,$$

the relation (21) can be read

$$\langle \mathscr{L}\mathbf{E}_1, \mathbf{E}_2 \rangle + \langle \mathbf{E}_1, \mathscr{L}\mathbf{E}_2 \rangle = 0. \qquad (22)$$

Clearly (22) means that $\mathscr{L}$ is a "skew-adjoint" operator, corresponding to a skew-hermitian matrix in a finite dimension space. From this we deduce that the eigenvalue λ for an eigenmode $\mathbf{V}$ is such that

$$0 = \langle \mathbf{V}, \mathscr{L}\mathbf{V} \rangle + \langle \mathscr{L}\mathbf{V}, \mathbf{V} \rangle = (\lambda^* + \lambda)\langle \mathbf{V}, \mathbf{V} \rangle$$

and since $\mathbf{V}$ is not equal to zero, $\lambda + \lambda^* = 0$, so that $\lambda = i\sigma$ with a real number σ.

With two eigenmodes $\mathbf{V}_1(\lambda_1)$, $\mathbf{V}_2(\lambda_2)$ we have

$$0 = \langle \mathbf{V}_1, \mathscr{L}\mathbf{V}_2 \rangle + \langle \mathscr{L}\mathbf{V}_1, \mathbf{V}_2 \rangle = (\lambda_2^* + \lambda_1)\langle \mathbf{V}_1, \mathbf{V}_2 \rangle$$

so that if $\lambda_1 + \lambda_2^* \neq 0$, we get $\langle \mathbf{V}_1, \mathbf{V}_2 \rangle = 0$, i.e., $\mathbf{V}_1$ and $\mathbf{V}_2$ are orthogonal for the scalar product (20).

The existence of such orthogonal modes with an eigenvalue $\lambda = i\sigma$ is a consequence of the property (22) for the operator $\mathscr{L}$. If we write $\mathscr{L} = i\mathscr{L}'$, the relation (22) is equivalent to saying that the operator $\mathscr{L}'$ is self-adjoint. So the normal modes exist as the eigenfunctions of the first-order differential system (19), with a self-adjoint operator and their family is complete in the Hilbert space $\mathscr{E}$ with the scalar product (20) (Courant and Hilbert 1953). This new structure gives the possibility of writing orthogonality relationships between different modes, especially between Rossby and gravity modes.

b. Machenhauer initialization in the case of a stationary Rossby mode

We look for a linearization including the maximum β-terms but with a stationary Rossby mode. We consider the problem of the continuous eigenmodes defined as

$$\left.\begin{aligned} -\nabla\cdot(f\nabla\psi) + \mathbf{k}\times\nabla\chi\cdot\nabla f + \nabla^2\phi &= \lambda\nabla^2\chi \\ \nabla\cdot(f\nabla\chi) + \mathbf{k}\times\nabla\psi\cdot\nabla f &= \lambda\nabla^2\psi \\ \bar{\phi}\nabla^2\chi &= \lambda\phi \end{aligned}\right\}. \qquad (23)$$

The particular case $\lambda = 0$ gives $\nabla^2\chi = 0$ from (23c) and putting $\chi = 0$ in (23b) we obtain the necessary condition:

$$\mathbf{k}\times\nabla\psi\cdot\nabla f = 0$$

for the existence of such a stationary Rossby mode. It is then clearly defined by (23a) as the classical linear balance equation:

$$\left.\begin{aligned} \chi &= 0 \\ \nabla\cdot(f\nabla\psi) &= \nabla^2\phi \end{aligned}\right\}. \qquad (24)$$

From the general relation (12) we get the stationarity of gravity components as:

$$\left.\begin{aligned} \frac{\partial \chi}{\partial t} &= 0 \\ \frac{\partial}{\partial t}[\nabla\cdot(f\nabla\psi) - \nabla^2\phi] &= 0 \end{aligned}\right\}.$$

The conservation of the Rossby component can be obtained by imposing that $\delta\mathbf{E}$ is a pure gravity mode or, equivalently, that $\delta\mathbf{E}$ which is orthogonal to an arbitrary Rossby mode $\mathbf{V}_R$ defined by (24):

$$\langle \delta\mathbf{E}, \mathbf{V}_R \rangle = 0.$$

This relation can be written by using the scalar product (20) as

$$\int_{\mathscr{D}} [\bar{\phi}(\nabla\delta\psi\cdot\nabla\psi_R^*) + \delta\phi\phi_R^*]d\omega = 0$$

for arbitrary ψ_R, ϕ_R satisfying (24). If we define δx

$= \nabla^{-2}\delta\phi$, we have, using the Gauss theorem, together with (24) (dropping the surface differential element $d\omega$):

$$\int \delta\phi\phi_R^* = \int (\nabla^2\delta x)\phi_R^* = \int \delta x\nabla^2\phi_R^*$$

$$= \int \delta x\nabla\cdot(f\nabla\psi_R^*) = -\int f\nabla\psi_R^*\cdot\nabla\delta x.$$

Thus the relation $\langle \delta\mathbf{E}, \mathbf{V}_R\rangle = 0$ can be written as

$$0 = \int (\bar\phi\nabla\delta\psi - f\nabla\delta x)\cdot\nabla\psi_R^*$$

$$= \int [\nabla\cdot(f\nabla\delta x) - \bar\phi\nabla^2\delta\psi]\psi_R^*$$

for an arbitrary ψ_R^*. We notice that the boundary contributions to the integrals vanish if we use the following boundary conditions: $\delta x = 0$ together with (BC1), or $(\partial/\partial n)(\delta x) = 0$ together with (BC2). That entirely defines the operator ∇^{-2}.

The last relation is equivalent to the relation:

$$\nabla\cdot[f\nabla(\nabla^{-2}\delta\phi)] = \bar\phi\nabla^2\delta\psi \tag{25}$$

which thus expresses the conservation of the Rossby component.

By expanding the left-hand side of Eq. (25) we get

$$f\delta\phi + \nabla f\cdot\nabla\nabla^{-2}\delta\phi = \bar\phi\nabla^2\delta\psi = \bar\phi\delta\zeta.$$

Thus, it is analogous to Eq. (18) modified through the presence of the β-terms.

The Machenhauer iterative scheme is deduced with the general expression (11) by writing that the vector

$$\mathcal{M}(\delta\mathbf{E}) - \frac{\partial\mathbf{E}}{\partial t} = \begin{pmatrix}\nabla^2\chi_1\\ \nabla^2\psi_1\\ \phi_1\end{pmatrix}$$

belongs to the Rossby manifold, i.e., that χ_1, ψ_1, ϕ_1 satisfy (24). We obtain the two relations

$$-\mathscr{H}(\delta\psi) + \mathbf{k}\times\nabla\delta\chi\cdot\nabla f + \nabla^2\delta\phi - \frac{\partial\nabla^2}{\partial t}\chi = 0 \tag{26}$$

$$\mathscr{H}\nabla_d^{-2}\left[\mathscr{H}(\delta\chi) - \frac{\partial}{\partial t}\nabla^2\psi\right] = \nabla^2\left[\bar\phi\nabla^2\delta\chi - \frac{\partial\phi}{\partial t}\right] \tag{27}$$

with the operator $\mathscr{H} \equiv \nabla\cdot(f\nabla\ )$.

The computation of ψ_1 requires the solution of the equation $\nabla^2\psi_1 =$, with the boundary condition $\psi_1 = 0$ [in the two cases (BC1), (BC2)]. Thus ∇_d^{-2} is the inverse of the operator ∇^2 subject to this homogeneous Dirichlet boundary condition. Similarly, by expanding the left-hand side of the Eqs. (26), (27) we note that they are analogous to the equations (16), (17) modified through the presence of the β-terms. To demonstrate that the scheme given by the relations (25), (26), (27) is workable we show that these three relations give a unique vector

$$\begin{pmatrix}\delta\chi\\ \delta\psi\\ \delta\phi\end{pmatrix}.$$

We examine the relation (27) for the unknown function $\delta\chi$, which can be written as

$$\mathcal{G}(\delta\chi) \equiv -\mathscr{H}\nabla_d^{-2}\mathscr{H}(\delta\chi) + \bar\phi\nabla^2\nabla^2\delta\chi$$

$$= \frac{\partial}{\partial t}[\nabla^2\phi - \mathscr{H}(\psi)].$$

By multiplying $\mathcal{G}(\delta\chi)$ by the function $\delta\chi^*$ and integrating over the domain $\mathcal{D}$ we get the identity

$$\int \mathcal{G}(\delta\chi)\delta\chi^* d\omega$$

$$= \int [-\mathscr{H}\nabla_d^{-2}\mathscr{H}(\delta\chi)\delta\chi^* + \bar\phi(\nabla^2\nabla^2\delta\chi)\delta\chi^*]d\omega.$$

The first term of the integral can be written as

$$\int -\mathscr{H}[\nabla_d^{-2}\mathscr{H}((\delta\chi)]\delta\chi^* d\omega$$

$$= -\int \nabla_d^{-2}\mathscr{H}(\delta\chi)\mathscr{H}(\delta\chi^*)d\omega$$

$$= -\int \nabla_d^{-2}\mathscr{H}(\delta\chi)\nabla^2\nabla_d^{-2}\mathscr{H}(\delta\chi^*)d\omega$$

$$= \int |\nabla[\nabla_d^{-2}\mathscr{H}(\delta\chi)]|^2 d\omega$$

by using the divergence theorem with vanishing boundary contribution since we have at the boundary $\delta\chi = \delta\chi^* = 0$ (BC1) or $\partial/\partial n(\delta\chi) = \partial/\partial n(\delta\chi^*) = 0$ (BC2). The second term of the integral can be written as

$$\bar\phi\int [\nabla^2(\nabla^2\delta\chi)]\delta\chi^* d\omega = \bar\phi\int (\nabla^2\delta\chi)(\nabla^2\delta\chi^*)d\omega$$

with the supplementary boundary condition $\nabla^2\delta\chi = 0$ or $\partial/\partial n(\nabla^2\delta\chi) = 0$. These supplementary boundary conditions appear plausible in the cases (BC1) and (BC2), respectively. This is a consequence of the fourth order operator $\mathcal{G}$ due to the presence of the term $\nabla^4\delta\chi$. Then we get the result that

$$\int \mathcal{G}(\delta\chi)\delta\chi^* d\omega = \int |\nabla[\nabla_d^{-2}\mathscr{H}(\delta\chi)]|^2 d\omega$$

$$+ \bar\phi\int |\nabla^2\delta\chi|^2 d\omega$$

is a positive-definite quantity. Therefore it is a well-posed elliptic problem (cf. Lax-Milgram lemma in (Ciarlet 1978)), from which we deduce that the relation (27) has a unique solution $\delta\chi$ under the preceding boundary conditions. Since $\delta\chi$ is a known function the other unknown functions are then $\delta\psi$, $\delta\phi$ defined by the relation (26) written in the following form:

$$-\nabla\cdot(f\nabla\delta\psi)+\nabla^2\delta\phi=g \tag{28}$$

and the relation (25). Considering the homogeneous problem with $g = 0$, the relation (28) expresses that the vector

$$\begin{pmatrix}0\\ \delta\psi\\ \delta\phi\end{pmatrix}$$

belongs to the stationary manifold and the relation (25) expresses that it is orthogonal to the same manifold. Thus this vector is equal to zero. It follows that (25), (26) have a unique solution for $\delta\psi$, $\delta\phi$.

The extension to the n-order initialization is straightforward since such a scheme uses the n-iterate of the linearized operator, and it similarly gives a well-posed problem.

As a conclusion, we've shown that in the case of a linearization including all the Coriolis terms but $\mathbf{k}\times\nabla\psi\cdot\nabla f$, the slow mode is still stationary, and the Machenhauer initialization scheme is expressed by the three relations (25), (26), and (27) which give a unique solution. It has been shown that these relations are an extension of the BMG scheme through the inclusion of β-terms. During the development of this work, we learned that Temperton (1988) has formulated and tested a similar scheme including the same Coriolis terms and appearing as an extension of the BMG scheme. Thus, the feasibility of such a scheme has been clearly demonstrated not only for Machenhauer initialization but also for a second-order initialization scheme.

c. *Initialization in the case of a nonstationary Rossby mode*

We address first the question, Is it possible to formulate a non-normal mode initialization in this case? The necessary and sufficient condition to have a nonstationary Rossby mode is to keep the term $\mathbf{k}\times\nabla\psi\cdot\nabla f$ in the linearization.

In this case, the problem of the continuous eigenmodes is given by the system (23) with all the terms as well as for Rossby modes as for fast gravity modes. Instead of the preceding case, when $\lambda = 0$ was an eigenvalue of infinite multiplicity, with an eigensubset entirely defined by the explicit relations (24) there is a denumerable family of Rossby modes for different eigenvalues.

To make a frequency separation between Rossby modes and fast gravity modes, we can write a third-order "characteristic equation" for λ. It can be obtained by eliminating ϕ in (23a), defined by (23c) as $\phi = \bar{\phi}(\nabla^2\chi/\lambda)$ and then writing (23a) as

$$\psi=\mathscr{H}^{-1}\left[\mathbf{k}\times\nabla\chi\cdot\nabla f+\bar{\phi}\,\frac{\nabla^2\nabla^2\chi}{\lambda}-\lambda\nabla^2\chi\right]$$

which defines ψ only in terms of χ, which can be put in (20b), giving

$$\begin{aligned}\nabla^2\mathscr{H}^{-1}[\lambda^2\mathbf{k}\times\nabla\chi\cdot\nabla f+\lambda\bar{\phi}\nabla^4\chi-\lambda^3\nabla^2\chi]\\ =\lambda\mathscr{H}(\chi)+\mathbf{k}\times\nabla\mathscr{H}^{-1}[\lambda\mathbf{k}\times\nabla\chi\cdot\nabla f\\ +\bar{\phi}\nabla^4\chi-\lambda^2\nabla^2\chi]\cdot\nabla f.\end{aligned}$$

This is a generalized eigenproblem for the function χ. Thus, it is a third-order equation for λ with three family of solutions: one for Rossby modes and two for gravity modes. Clearly, there is no particular solution $\lambda = 0$ since the term $\mathbf{k}\times\nabla\psi\cdot\nabla f$ has been kept. There is not an obvious separation between the different solutions. To illustrate our affirmation we consider the case of the midlatitude β-plane approximation with double Fourier expansions studied by De Maria and Schubert (1984). In this case we get an ordinary third-order characteristic equation for the frequencies [cf. Eq. (4.20)] and the separation between Rossby and fast gravity frequencies cannot be expressed in an analytical way unless approximations are done.

A way of attacking the problem in its full generality might be to use the technique of Lynch (1985) to do this separation in the Laplace space, or to work with variational constraints or Rayleigh continuous quotients. Another way of considering the problem is to use the formulation given in section 4b for the initialization and to ask the question, What happens for the real Rossby modes defined with the term $\mathbf{k}\times\nabla\psi\cdot\nabla f$? Since this term is taken into account in the nonlinear terms, it is not possible to say, as sometimes argued, that the Rossby modes are partly considered gravity modes and are thus altered in such an initialization process.

Tests will be done in the future to see for what limited-area models the formulation given in section 4b is better than the previous BMG scheme to remove gravity waves oscillations, and if a new formulation, such as the formulation given in section 4c, is necessary.

5. Additional remarks

a. *Coordinate systems and discretization*

These formulations have been written in a general form involving the operators and thus can be applied with any system of coordinates, e.g., Cartesian coordinates with a stereographic projection, or a latitude–longitude system. They can also be equally applied with any discretization. Indeed, BMG applied their scheme to a grid-point model but Ballish (1980) has applied similar scheme to a spectral model and the scheme equivalent to that given in section 4b has been tested by Temperton (1988) for a finite-element discretization.

Concerning the time discretization, it may be interesting to develop an initialization scheme which takes into account the fact that the time tendency of the model run is formulated in a semi-implicit way. In this case, referring to section 2b, and using the fact that the linear terms $i\sigma_{nk}x_{nk}$ (with $n = 2, 3$) in Eqs. (5) are discretized for the forward time step as $i\sigma_{nk}(x_{nk}^{(\Delta t)} + x_{nk}^{(0)})/2$ we get the following iterative scheme for the modal components:

$$\delta x_{1k} = 0$$

$$\delta x_{nk} = \frac{1}{i\sigma_{nk}} \frac{dx_{nk}}{dt} + \frac{\Delta t}{2} \frac{dx_{nk}}{dt} \quad \text{with} \quad n = 2, 3$$

instead of (8a), (8b) and (8c). This new scheme can be easily expressed in the physical space as

$$\mathcal{M}(\delta \mathbf{E}) = \left(\frac{\partial \mathbf{E}}{\partial t}\right)_G + \frac{\Delta t}{2} \mathcal{M}\left(\frac{\partial \mathbf{E}}{\partial t}\right)_G$$

with the time step Δt. Such a scheme has been applied to the BMG linearization and tested in a shallow water model (Juvanon du Vachat 1988). However, it must be noted that such a deduction is only valid if the Coriolis terms introduced in the linearized operator $\mathcal{M}$ are treated implicitly in the semi-implicit scheme, which is not usually the case.

b. Boundary conditions

For a hemispheric or a global model there is no particular problem, and the balance relations written in the Appendix are verified as a consequence of the closed domain. It is the same situation for a limited-area model with cyclic boundary conditions. But in the case of a limited-area model with boundary conditions prescribed by a large-scale model, we must consider some deviated functions χ', ψ', ϕ' such that $\chi' = \psi' = \phi' = 0$ at the boundary (BC1) in order to have a vectorial set $\mathcal{E}$ (Brière 1982). Then the balance relations written in the Appendix are satisfied and, as a consequence, orthogonality relationships are also satisfied for appropriate linearizations. As demonstrated in section 4b, that also leads to consistent boundary conditions for the initialization scheme. Moreover, in the case of a limited-area model surrounded by a solid wall we must use the boundary conditions (BC2), and the same conclusion holds, as demonstrated by Temperton (1988).

c. Orthonormal modes for the BMG linearization

This linearization, given by (14), does not conserve the energy unless f is taken as constant, as can be seen from the balance relations written in the Appendix. Indeed, Brière (1982) has defined orthonormal modes in such an hypothesis.

In the case of variable f however, looking for a quadratic invariant, we can eliminate the f-term by writing $D(\partial D/\partial t) + \zeta(\partial\zeta/\partial t)$; then we eliminate $D\nabla^2\phi$ with the third equation by writing

$$\bar{\phi}\left(D\frac{\partial D}{\partial t} + \zeta\frac{\partial \zeta}{\partial t}\right) - \frac{\partial \phi}{\partial t}\nabla^2\phi = 0.$$

By integrating over the domain and using the divergence theorem we get the following quadratic invariant:

$$\int_D [\bar{\phi}(|D|^2 + |\zeta|^2) + |\nabla\phi|^2] d\omega$$

which is neither the energy nor a physical classical quantity. The associated scalar product is

$$\langle \mathbf{E}_1, \mathbf{E}_2 \rangle = \int \bar{\phi}(D_1 D_2^* + \zeta_1\zeta_2^*) + \nabla\phi_1 - \nabla\phi_2^*$$

for which the different modes are orthogonal. Then the condition of conservation of the Rossby modal component $\delta \mathbf{E}_R = 0$ can be written as the orthogonal set to the Rossby set, i.e., $0 = \langle \delta \mathbf{E}, \mathbf{V}_R \rangle$ for arbitrary $\mathbf{V}_R$ satisfying (15). We directly get the condition

$$\bar{\phi}\delta\zeta = f\delta\phi.$$

Therefore, we have exhibited the scalar product for which the normal modes of the BMG linearization are orthogonal and verified its usefulness.

6. Summary and conclusions

In this paper the non-normal mode initialization introduced by Ballish (1980) on the basis of the bounded-derivative method has been considered. It is an initialization scheme which does not require an explicit computation of the normal modes of the linearized equations.

Such an initialization process is known to be interesting for limited-area models for which the problem of defining the normal modes is not separable unless drastic assumptions are done (f taken as constant) in the linearized equations. But the interest in such an initialization scheme has also been expressed recently for a global model with variable resolution (Courtier and Geleyn 1987), for which the completion of the normal modes is too expensive due to nonseparability.

The formulation of such a scheme has been given in an abstract form, for the Machenhauer algorithm as well as for a higher-order scheme as developed by Tribbia (1984). This abstract form has been deduced with very weak hypotheses about the modes, which can be considered as "oblique modes." It is a natural generalization of a technique already used in Juvanon du Vachat (1986). It relies on the same intuition as does what Temperton has called implicit normal mode initialization (Temperton 1985 and 1988), since we exhibit some properties of normal modes but without explicitly computing them in the initialization process.

The particular case of a stationary Rossby mode, studied by Brière (1982) and in BMG, has also been examined in detail. In this case, the stationarity of gravity modal components (Machenhauer 1977) can be written in the general form: $\partial/\partial t[\mathcal{M}(\mathbf{E})] = 0$ where

$\mathcal{M}$ is the linearized operator and **E** the vector state. Thus, this condition can be deduced by simply taking the time derivative of the equations defining the slow modes. The extension to the order n (Tribbia 1984) of such a condition naturally is $\partial^n/\partial t^n[\mathcal{M}(\mathbf{E})] = 0$. On theoretical grounds, that extends the relation done by Leith (1980) between quasi-geostrophic theory and nonlinear normal mode initialization which is only valid if f is taken as constant.

Thus, the existence of "orthonormal modes," i.e., normal modes which are orthogonal for a given scalar product, has been demonstrated as a consequence of the conservation of the energy by a given linearization. The usefulness of such orthogonality relationships has been used for formulating a non-normal mode initialization which includes most of β-terms in the linearization and which provides an extension of the BMG scheme. Indeed, it is a formal deduction but during the completion of this work, it appears that Temperton (1988) has formulated and tested a similar scheme using the principle of implicit normal mode initialization. The main difference with our work is that we've used continuous modes rather than matrix manipulation on discrete vectors and also that we've clearly written the conditions $\partial/\partial t[\mathcal{M}(\mathbf{E})] = 0$, $\partial^n/\partial t^n[\mathcal{M}(\mathbf{E})] = 0$ which were not previously identified as a consequence of a stationary Rossby mode. It is conjectured by this paper but not demonstrated that this condition is also necessary to deduce a non-normal mode initialization.

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APPENDIX

Linearization of the Shallow Water Equations and Conservation of the Energy

The shallow water equations may be linearized as follows:

$$\left.\begin{aligned} \frac{\partial \mathbf{V}}{\partial t} + f\mathbf{k}\times\mathbf{V} + \nabla\phi = 0 \\ \frac{\partial \phi}{\partial t} + \bar{\phi}\nabla\cdot\mathbf{V} = 0 \end{aligned}\right\}$$

where **V** is the horizontal vector wind, ϕ the geopotential of the free surface, f the Coriolis parameter, **k** the unit vector in the vertical and $\bar{\phi}$ a mean free height. The time rate equations for the divergence, $D = \nabla\cdot\mathbf{V}$, and the vorticity, $\zeta = \mathbf{k}\cdot\nabla\times\mathbf{V}$, are deduced as

$$\frac{\partial D}{\partial t} + \nabla\cdot(f\mathbf{k}\times\mathbf{V}) + \nabla^2\phi = 0$$

$$\frac{\partial \zeta}{\partial t} + \nabla\cdot(f\mathbf{V}) = 0.$$

These equations can also be written in terms of the velocity potential χ and of the streamfunction ψ, by using the decomposition of the wind along the rotational and divergent part:

$$\mathbf{V} = \nabla\chi + \mathbf{k}\times\nabla\psi$$

together with the relations:

$$D = \nabla\cdot\mathbf{V} = \nabla^2\chi$$

$$\zeta = \mathbf{k}\cdot\nabla\times\mathbf{V} = \nabla^2\psi.$$

We get:

$$\frac{\partial}{\partial t}\nabla^2\chi - \nabla\cdot(f\nabla\psi) + \mathbf{k}\times\nabla\chi\cdot\nabla f + \nabla^2\phi = 0 \quad [\chi]$$

$$\frac{\partial}{\partial t}\nabla^2\psi + \nabla\cdot(f\nabla\chi) + \mathbf{k}\times\nabla\psi\cdot\nabla f = 0 \quad [\psi]$$

$$\frac{\partial\phi}{\partial t} + \bar{\phi}\nabla^2\chi = 0 \quad [\phi]$$

where the brackets $[\chi]$, $[\psi]$, $[\phi]$ denote the right-hand side of the respective equations.

The terms $\nabla\cdot(f\nabla\alpha)$ can be decomposed as $f\nabla^2\alpha + \nabla f\cdot\nabla\alpha$ if necessary and the terms $\mathbf{k}\times\nabla\alpha\cdot\nabla f$ can be written equivalently as $\nabla\cdot(f\mathbf{k}\times\nabla\alpha)$. Considering this linearization of the shallow water equations, we now examine the role of the different terms in the conservation of the linearized energy, i.e., the quantity EL:

$$\mathrm{EL} = \int_{\mathcal{D}} [\bar{\phi}(|\nabla\chi|^2 + |\nabla\psi|^2) + |\phi|^2]d\omega.$$

By computing $d\mathrm{EL}/dt$ and using the divergence theorem (with appropriate boundary conditions) we get

$$\frac{d\mathrm{EL}}{dt} = \int_{\mathcal{D}}\left[-\bar{\phi}\left(\frac{\partial}{\partial t}\nabla^2\chi\chi^* + \chi\frac{\partial}{\partial t}\nabla^2\chi^* + \frac{\partial}{\partial t}\nabla^2\psi\psi^* + \psi\frac{\partial}{\partial t}\nabla^2\psi^*\right) + \frac{\partial\phi}{\partial t}\phi^* + \phi\frac{\partial\phi^*}{\partial t}\right]d\omega$$

where $*$ means complex conjugation.

Then we write the following combination of the preceding equations:

$$-\bar{\phi}([\chi]\chi^* + \chi[\chi]^* + [\psi]\psi^* + \psi[\psi]^*) + [\phi]\phi^* + \phi[\phi]^* = 0.$$

We integrate over the domain and get (dropping henceforth the surface differential element $d\omega$):

$$\frac{d\mathrm{EL}}{dt} = \int \bar{\phi}[-\nabla\cdot(f\nabla\psi)\chi^* + \nabla\cdot(f\mathbf{k}\times\nabla\chi)\chi^* + \chi^*\nabla^2\phi] + \text{c.c.} + \int \bar{\phi}[\nabla\cdot(f\nabla\chi)\psi^* + \nabla\cdot(f\mathbf{k}\times\nabla\psi)\psi^*] + \text{c.c.} - \int \bar{\phi}[\phi^*\nabla^2\chi] + \text{c.c.}$$

with the notation c.c. meaning the complex conjugate of the preceding term. We examine the conservation of the energy of the preceding linearization in the two cases of boundary conditions (BC1) and (BC2).

At first, for the boundary conditions (BC1) $\chi = \psi = \phi = 0$ such a conservation come from the relations

$$\int \chi^*\nabla^2\phi = \int \phi\nabla^2\chi^* \qquad (29)$$

$$\int \nabla\cdot(f\nabla\psi)\chi^* = \int \nabla\cdot(f\nabla\chi^*)\psi \qquad (30)$$

and their complex conjugates, and from the relations

$$\int \nabla\cdot(f\mathbf{k}\times\nabla\alpha)\alpha^* = -\int \nabla\cdot(f\mathbf{k}\times\nabla\alpha^*)\alpha \quad \text{for} \quad \alpha = \chi, \psi. \qquad (31)$$

These balance relations are a consequence of the Gauss divergence theorem with vanishing boundary contributions due to these boundary conditions (BC1).

For the boundary conditions (BC2)

$$\psi = 0,$$
$$\frac{\partial\chi}{\partial n} = 0,$$
$$f\nabla\psi\cdot\mathbf{n} = \nabla\phi\cdot\mathbf{n}$$

the balance relations (31) are still valid but instead of (29), (30), we have the balance relation

$$\int -\nabla\cdot(f\nabla\psi)\chi^* + \int \chi^*\nabla^2\phi = \int -\nabla\cdot(f\nabla\chi^*)\psi + \int \phi\nabla^2\chi^*$$

and its complex conjugate.

The property of the conservation of the energy can be generalized in the following way. We now consider the scalar product:

$$\langle \mathbf{E}_1, \mathbf{E}_2 \rangle = \int [\bar{\phi}(\nabla\chi_1\cdot\nabla\chi_2^* + \nabla\psi_1\cdot\nabla\psi_2^*) + \phi_1\phi_2^*]$$

and write the combination

$$-\bar{\phi}([\chi_1]\chi_2^* + \chi_1[\chi_2]^* + [\psi_1]\psi_2^* + \psi_1[\psi_2]^*) + [\phi_1]\phi_2^* + \phi_1[\phi_2]^* = 0.$$

We can develop the quantity

$$\left\langle \frac{\partial}{\partial t}\mathbf{E}_1, \mathbf{E}_2 \right\rangle + \left\langle \mathbf{E}_1, \frac{\partial}{\partial t}\mathbf{E}_2 \right\rangle$$

as

$$\int \bar{\phi}[-\nabla\cdot(f\nabla\psi_1)\chi_2^* + \nabla\cdot(f\mathbf{k}\times\nabla\chi_1)\chi_2^* + \chi_2^*\nabla^2\phi_1] + [1\leftrightarrow 2]^* + \int \bar{\phi}[\nabla\cdot(f\nabla\chi_1)\psi_2^* + \nabla\cdot(f\mathbf{k}\times\nabla\psi_1)\psi_2^*] + [1\leftrightarrow 2]^* - \int \bar{\phi}[\phi_2^*\nabla^2\chi_1] + [1\leftrightarrow 2]^*$$

where the notation $[1\leftrightarrow 2]^*$ denotes the complex conjugation plus the index exchange $[1\leftrightarrow 2]$, applied to the preceding term.

Similarly, if we have the following relations:

$$\int \chi_2^*\nabla^2\phi_1 = \int \phi_1\nabla^2\chi_2^* \quad \text{and} \quad [1\leftrightarrow 2]^*$$

$$\int \nabla\cdot(f\nabla\psi_1)\chi_2^* = \int \nabla\cdot(f\nabla\chi_2^*)\psi_1 \quad \text{and} \quad [1\leftrightarrow 2]^*$$

for the boundary conditions (BC1),

$$\int -\nabla\cdot(f\nabla\psi_1)\chi_2^* + \int \chi_2^*\nabla^2\phi_1 = \int -\nabla\cdot(f\nabla\chi_2^*)\psi_1 + \int \phi_1\nabla^2\chi_2^* \quad \text{and} \quad [1\leftrightarrow 2]^*$$

for the boundary conditions (BC2), and

$$\int \nabla\cdot(f\mathbf{k}\times\nabla\alpha_1)\alpha_2^* = -\int \nabla\cdot(f\mathbf{k}\times\nabla\alpha_2^*)\alpha_1 \quad \text{for} \quad \alpha = \chi, \psi$$

for the two cases (BC1) and (BC2), then we get

$$0 = \left\langle \frac{\partial\mathbf{E}_1}{\partial t}, \mathbf{E}_2 \right\rangle + \left\langle \mathbf{E}_1, \frac{\partial\mathbf{E}_2}{\partial t} \right\rangle.$$

Particular linearizations which are energy conserving in this more general sense can be found if the preceding balance relations are satisfied.

REFERENCES

Baer, F., 1977: Adjustment of initial conditions required to suppress gravity oscillations in nonlinear flows. *Contrib. Atmos. Phys.*, **50**, 350–366.

——, and J. Tribbia, 1977: On complete filtering of gravity modes through nonlinear initialization. *Mon. Wea. Rev.*, **105**, 1536–1539.

Ballish, B., 1979: Comparison of some nonlinear initialization techniques. *Preprints Fourth Conf. on Numerical Weather Prediction*, Silver Spring, Amer. Meteor. Soc., 9–12.

——, 1980: Initialization, theory and application to the NMC spectral model. Ph.D. thesis, University of Maryland, 151 pp.

Bourke, W., and J. L. McGregor, 1983: A nonlinear vertical mode initialization scheme for a limited area prediction model. *Mon. Wea. Rev.*, **111**, 2285–2297.

Brière, S., 1982: Nonlinear normal mode initialization of a limited area model. *Mon. Wea. Rev.*, **110**, 1166–1186.

Ciarlet, Ph.G., 1978: *The finite-element method for elliptic problems.* North-Holland, 530 pp.

Courant, R., and D. Hilbert, 1953: *Methods of Mathematical Physics.* Vol. I, Wiley-Interscience, 560 pp.

Courtier, Ph., and J. F. Geleyn, 1987: A global model with variable resolution. Application to the shallow-water equations. *Quart. J. Roy. Meteor. Soc.*, in press.

Daley, R., 1979: The application of non-linear normal mode initialization to an operational forecast model. *Atmos. Ocean.*, **17**, 97–124.

De Maria, M., and W. H. Schubert, 1984: Experiments with a spectral tropical cyclone model. *J. Atmos. Sci.*, **41**, 901–924.

Juvanon du Vachat, R., 1986: A general formulation of normal modes for limited-area models. Application to initialization. *Mon. Wea. Rev.*, **114**, 2478–2487.

——, 1988: A general formulation for nonlinear initialization of a numerical weather prediction model. Experiments with a shallow-water limited-area model. *Comp. Meth. App. Mech. Eng.*, in press.

Leith, C. E., 1980: Nonlinear normal mode initialization and quasi-geostrophic theory. *J. Atmos. Sci.*, **37**, 957–968.

Lynch, P., 1985: Initialization using Laplace transforms. *Quart. J. Roy. Meteor. Soc.*, **111**, 243–258.

Machenhauer, B., 1977: On the dynamics of gravity oscillations in a shallow water model, with application to normal mode initialization. *Contrib. Atmos. Phys.*, **50**, 253–271.

——, 1982: Fundamentals of nonlinear normal mode initialization. *Proc. of the Stanstead Seminar on the Interaction between Objective Analysis and Initialization.* (12–16 July 1982) Bishop's University, Lennoxville, Québec, Canada.

Temperton, Cl., 1985: Applications of a new principle for normal mode initialization. *Preprints Seventh Conf. on Numerical Weather Prediction,* Montréal, Amer. Meteor. Soc., 105–107.

——, 1988: Implicit normal mode initialization. *Mon. Wea. Rev.*, **116**, 1013–1031.

Tribbia, J., 1984: A simple scheme for higher-order nonlinear normal mode initialization. *Mon. Wea. Rev.*, **112**, 278–284.