An Initialization for Cumulus Convection in Numerical Weather Prediction Models

LEO J. DONNER*
National Center for Atmospheric Research,** Boulder, Colorado
(Manuscript received 19 March 1987, in final form 31 July 1987)

ABSTRACT

A procedure for initializing parameterizations for cumulus convection in numerical weather prediction models is described. The initialization adjusts the temperature and humidity fields such that a simplified version of the Kuo cumulus parameterization will yield diagnosed convective precipitation and vertical heating profiles, if a specified velocity field can support them. In an unfavorable velocity field, the initialization will yield the closest approach to diagnosed convective precipitation possible. The initialization minimizes changes in the humidity and temperature fields while satisfying constraints imposed by the cumulus parameterization.

Slight adjustments in the temperature field and relatively larger adjustments in the humidity field can modify the large scale from a state which does not support cumulus convection to a state whose convective heating, as parameterized by the simplified version of the Kuo scheme, agrees to the extent possible for an imposed velocity field. Use of more complicated versions of the Kuo cumulus parameterization with the initialized temperature and humidity profiles yields heating rates agreeing reasonably with diagnosed heating. If used in conjunction with an initialization for the velocity field, cumulus initialization may ameliorate problems associated with spinup of physical processes in numerical weather prediction.

1. Introduction

The problem of spinup for physical processes is among the key outstanding issues in numerical weather prediction. Its chief characteristic is the significant departure of the fields of convection and precipitation (and thereby clouds and radiative transfer) from observation and diagnosis during the initial stages of integration of numerical weather prediction models. Miyakoda (1978) found that the precipitation rate calculated, using a Geophysical Fluid Dynamics Laboratory model, increased for the first two and one-half days, mostly due to the spinup of convective precipitation in the tropics. Figure 1 shows the global average rate of convective precipitation (ensemble average over four cases in January and February 1979) in the National Center for Atmospheric Research Community Forecast Model (CCM0B) (described by Williamson, 1983), diabatically initialized with nonlinear normal-mode initialization and incorporating a version of the Kuo cumulus parameterization summarized in Donner (1986). Convective precipitation is seen to increase sharply from an insignificant amount during the first two days of the forecast. The spinup problem extends to nonconvective precipitation as well (Lejenäinen, 1979, 1980), but is not so serious. Severe deficiencies in the calculation of heating rates are implied by these deficiencies in the treatment of clouds and condensation.

Inadequate initial response of the cumulus parameterizations may represent a failure of the parameterization itself or an inability of the large-scale model and its initial analysis to produce dynamic, thermodynamic, or moisture fields capable of supporting convection. Since the time-mean, equilibrium circulations simulated by models used for numerical weather prediction are convectively active, the problem of initializing cumulus convection naturally focuses on the initial structure of the large-scale fields. (The possibility nonetheless remains that parameterization deficiencies explain the cumulus convection spinup. In this view, the model circulation drift toward states which accommodate the cumulus parameterization; such states are not necessarily realistic.) The first attempts to initialize model precipitation fields centered on the moisture field and included such techniques as enhancing gradients in the humidity field, relating the humidity and vertical velocity fields (Atkins, 1970, 1974), and adding moisture where satellites detected clouds (Perkey, 1976). Precipitation increases resulted in these experiments. Krishnamurti et al. (1984) and Krishnamurti (1985) introduced the concept of "physical initialization," an approach in which the moisture (and, in some cases, velocity) fields are adjusted to elicit desired initial behavior from the model physical parameterizations. Krishnamurti et al. (1984) adjusted the velocity and humidity fields such that a precipitation pattern consistent with satellite and rain gage data was

** The National Center for Atmospheric Research is sponsored by the National Science Foundation.

Corresponding author address: Dr. Leo J. Donner, The University of Chicago, Dept. of the Geophysical Sciences, 5734 South Ellis Ave, Chicago, IL 60637.

© 1988 American Meteorological Society
2. Initialization procedure

The cumulus initialization adjusts the temperature and humidity fields such that precipitation and the vertical distribution of convective heating are consistent with values observed or diagnosed, e.g., from satellite data or rain gages. Given a diagnosis of heating (and, by integration, rainfall)\(^1\), the initialization procedure minimizes, in a least-squares sense, the changes in the uninitialized temperature and humidity profiles, subject to constraints imposed by a simplified version of the Kuo (1974) cumulus parameterization. The humidity field is initialized to yield the rainfall rate, and the temperature field is initialized to yield the pressure dependence of the convective heating. It is assumed that the diagnosed heating profile or other information permits identification of the pressures at the base and top of the convectively active layer.

\(q_c = q_{\text{sat}}(T_c),\)

where \(T_c\) gives the cloud temperature, \(T_{\text{co}}\) the cloud virtual temperature, \(L\) the latent heat of vaporization, \(q_c\) the cloud specific humidity, \(R_g\) the gas constant for dry air, \(g\) the gravitational constant, \(c_p\) the specific heat of air, \(e_s\) the saturation vapor pressure at \(T_c\), \(p\) pressure, \(R_e\) the gas constant for water vapor, and \(q_{\text{sat}}\) refers to saturation specific humidity. Note that the cloud temperature depends on the large-scale environment only through its cloud-base temperature, which is assumed to be the boundary-layer temperature cooled dry adiabatically to the lifting condensation level.

Convective heating \(Q_T\) of the large scale is taken as

\[ Q_T = \frac{L (1 - b) g M_T (T_c - \bar{T})}{c_p \int_{P_b}^{P_s} (T_c - \bar{T}) dp}, \]

where

\[ g M_t = -\int_0^{P_s} \nabla \cdot (\bar{\nabla} \bar{q}) dp + \tilde{\rho}_0 g C_B |\bar{\nabla}| D_w (\bar{q}_s - \bar{q}_0). \]

\(^1\) Rain gage data provides only the vertical integral of heating. The cumulus initialization can be used for precipitation only by initializing just the humidity field. If divergence can be adequately initialized, the vertical distribution of heating could also be inferred diagnostically as a residual from the thermodynamic equation.
Here, \( \bar{T}, \bar{V}, \) and \( \bar{q} \) refer to the large-scale temperature, velocity, and specific humidity, respectively; \( gM_t \) gives the large-scale moisture convergence and surface evaporation, and \( (1 - b) \) denotes the fraction of the large-scale moisture convergence which precipitates. The subscripts \( g, b, t \) and \( c \) indicate ground, boundary layer, convective base, convective top, and cloud, respectively. Atmospheric density \( \bar{p} \), drag coefficient \( C_D \), and surface moisture availability factor \( D_w \) enter the bulk-aerodynamic calculation for surface evaporation.

Equation (3) is the form of the Kuo parameterization used by Donner et al. (1982) and Donner (1986) with heating by cumulus-induced, eddy entropy-flux convergence neglected. For calculations involving the vertical integral of (3) (e.g., precipitation), the flux convergence terms are not important. For calculations involving the vertical profile of \( Q_T \), the results of Donner et al. (1982) imply that neglecting the eddy fluxes is more problematic. A cumulus initialization, using an approximation to the Kuo parameterization including eddy fluxes, could be developed by applying principles similar to those discussed in the following sections to a suitable version of the Kuo parameterization.

b. Humidity adjustment

The vertical integral of (3) is proportional to the rainfall rate and gives the column integral of cumulus thermal forcing:

\[
\int_{p_1}^{p_0} Q_T dp = \frac{L}{0.6 \bar{p} \bar{e} c_p} \left( \int_{0.4 \bar{p}}^{p_0} \bar{q} \frac{dp}{\bar{q}_{sat}} \right)
\times \left[ -\int_0^{p_0} \bar{V} \cdot (\bar{V} \bar{q}) dp + \bar{p}_0 g C_D |\bar{V}_0| D_w (\bar{q}_g - \bar{q}_0) \right].
\] (4)

The only temperature dependence in (4) is through \( \bar{q}_{sat} \). Since temperature adjustments for cumulus initialization should be fairly small (Observational and analysis errors in temperature should be reasonably small), the humidity field may be modified independently of the temperature field to yield a desired rainfall rate (or, equivalently, vertically integrated convective heating). The boundary-layer specific humidity \( \bar{q}_0 \) is adjusted first by requiring it to take the value of the saturation specific humidity at cloud base, the pressure of which is inferred from the observed or diagnosed heating profile. The quantity \( C_1 = \bar{p}_0 g C_D |\bar{V}_0| D_w (\bar{q}_g - \bar{q}_0) \) is constant for the remainder of the humidity adjustment. Decompose, \( \bar{q} \) as \( q^{(0)} + \Delta \bar{q} \), where \( q^{(0)} \) is the uninitialized specific humidity and \( \Delta \bar{q} \) is the humidity adjustment. By assumption \( \int_{p_1}^{p_0} Q_T dp \) is known. Seek \( \Delta \bar{q} \) satisfying

\[
\int_{p_1}^{p_0} Q_T dp = \frac{L}{0.6 \bar{p} \bar{e} c_p} \left( \int_{0.4 \bar{p}}^{p_0} \frac{q^{(0)} + \Delta \bar{q}}{\bar{q}_{sat}} dp \right)
\times \left[ -\int_0^{p_0} \bar{V} \cdot (\bar{V} (q^{(0)} + \Delta \bar{q})) dp + \bar{p}_0 g C_D |\bar{V}_0| D_w (\bar{q}_g - \bar{q}_0) \right].
\] (5)

For the one-dimensional calculations in this paper, let \( \bar{V} \cdot \nabla (\Delta \bar{q}) = 0 \). The resulting (5) does not have a unique solution.

Obviously, \( 0 \leq \bar{q}(p) \leq q_{sat}(p) \) and \( \bar{q}_0 = q_{sat}(p_1) \). These bounds imply a maximum value \( F_c \) of \( \int_{p_1}^{p_0} Q_T dp \) for specified \( \bar{V} \) and \( \nabla q^{(0)} \) fields. This maximum value should be calculated and checked against the diagnosed \( \int_{p_1}^{p_0} Q_T dp \). If the diagnosed value exceeds \( F_c \), adjustment of the humidity and temperature fields only cannot yield the diagnosed convective heating and rainfall. If the velocity field has been analyzed reasonably, this problem should not occur.

If \( F_c \geq \int_{p_1}^{p_0} Q_T dp \) (the usual case), then a solution to (5), subject to the positivity and saturation constraints described above, is sought which minimizes \( Q = \int_{p_1}^{p_0} (\Delta \bar{q})^2 dp \). The solutions to the optimization problems involving \( F_c \) and \( Q \) are discussed in the Appendix.

In application to a numerical weather prediction model, \( \bar{V} \cdot \nabla (\Delta \bar{q}) \) would not necessarily vanish. If the convection is isolated (i.e., a single grid point is subject to cumulus parameterization while those surrounding it are not), \( \bar{V} \cdot \nabla (\Delta \bar{q}) \) will be small, and the one-dimensional procedure can be followed as a reasonable approximation. (The smallness of \( \bar{V} \cdot \nabla (\Delta \bar{q}) \) for isolated convection is evident if a centered, finite difference is taken for \( \Delta \bar{q} \). Since \( \Delta \bar{q} = 0 \) at all but the grid point undergoing convection, the change in \( \bar{V} \cdot \nabla (\Delta \bar{q}) \) from one side of the centered difference just cancels that from the other). Isolated convection frequently occurs in coarse-resolution numerical weather prediction models. If convection is not isolated (i.e., several adjacent grid points are simultaneously subject to cumulus parameterization), the term \( \bar{V} \cdot \nabla (\Delta \bar{q}) \) should be retained in (5), and a solution which satisfies (5) while minimizing \( \int_{p_1}^{p_0} (\Delta \bar{q})^2 dp da \), where \( Ac \) is the convectively active area, is sought. Note that this case will allow for changes in the humidity gradient occurring as soundings are adjusted. Optimization techniques similar to those for the one-dimensional case would be used. Alternatively, since variations in thermodynamic states across a convectively active area are not likely to be large, the soundings could be averaged and the one-dimensional approach used directly.

c. Temperature adjustment

Although the humidity adjustment provides the vertical integral of convective heating (or the closest possible approach for a given velocity field), it is also generally desirable to partition the heating in the vertical. This can be achieved by adjusting the temperature
profile to satisfy the integral equation (3) once the humidity has been initialized. Equation (3) involves strictly temperature for a specified humidity profile, except for the slight dependence (for small temperature adjustment) of $b$ on temperature through the saturation specific humidity.

Since $c_p \int_p^0 Q_T dp = L(1 - b)gM_t$, there exist solutions to (3) having the form

$$T_c(T_0^{(0)} + \Delta T_0) - T^{(0)} - \Delta T = \alpha Q_T,$$

(6)

where $\alpha$ is independent of pressure. If $T_c$ is given by (1), $T_c(T_0^{(0)} + \Delta T_0)$, i.e., the cloud temperature depends on the large scale only through the boundary-layer temperature. [Eq. (1) neglects entrainment.] The temperature initialization adjusts the uninitialized profile such that (3) (and thus (6)) are satisfied:

$$T_c(T_0^{(0)} + \Delta T_0) - T^{(0)} - \Delta T = \alpha Q_T.$$

(7)

Noting that $\Delta T = \Delta T(\alpha, \Delta T_0)$, values for $\alpha$ and $\Delta T_0$ are selected to minimize

$$\tau = \int_{T_0}^{T_c} (\Delta T)^2 dp,$$

(8)

subject to bounds on $\Delta T_0$ representing its uncertainty and ensuring that a parcel lifted from the surface will be buoyant at the lifting condensation level. Sufficient conditions on $\alpha$ and $\Delta T_0$ for a discretized $\Delta T(p)$ to minimize $\tau$ are analogous to those for $Q$, discussed in the Appendix, and an identical numerical procedure can be used.

3. Application to one-dimensional profiles

To assess the nature and magnitude of changes in the temperature and humidity profiles required for cumulus initialization, the procedure described in section 2 was applied to some sample profiles. Large-scale divergence (Fig. 2a) and heating rates (Fig. 2b) for a 24-h period centered on 0000 UTC, 27 January 1979 at 11°S, 128°W were calculated as in Kasahara et al. (1986). Corresponding profiles for $Q^{(0)}$ (Fig. 3a), $T^{(0)}$ (Fig. 4a), and $\nabla \cdot Q^{(0)}$ were extracted from a diabatic initialization of CCM0B (Errico, 1983) using Rasch’s (1985) iteration scheme. (The heating rates used in the initialization were those of Kasahara et al. The four largest vertical scales were initialized using six iterations and a 48-h frequency cutoff.) These temperature and humidity profiles are not capable of sustaining any convection when subjected to the cumulus parameterization referenced in Donner (1986). (For the purposes of this paper, such profiles are referred to as “uninitialized,” although actually so with respect to cumulus initialization only.) Note the existence of convergence in the lower troposphere with divergence aloft. Figure 3b shows the changes in the humidity profile required to sustain the diagnosed cumulus heating. The specific humidity increases throughout the troposphere. The increase in boundary-layer specific humidity is necessary to provide that parcels rising from near the surface be saturated upon reaching the lifting condensation level. Because convergence exists throughout the lower troposphere, increased humidities there result in increased moisture convergence.

Cumulus initialization decreases the surface temperature very slightly, with larger decreases in the lower troposphere and increases above about 700 mb (Fig. 4). Reducing the large-scale temperature above the boundary layer increases the parameterized cumulus

---

2 Kasahara et al. diagnosed heating rates as a residual from the thermodynamic equation using ECMWF Level IIIb data from the FGGF Special Observing Period (SOP)-1. The diagnosed heating rates were negative immediately above and below the layer in which heating was assumed to be due to cumulus convection. The diagnosis incorporates all diabatic forcing, but the parameterization inversion for cumulus initialization deals only with cumulus convection, so heating rates were set to zero outside of the convectively active layer. No correction for radiative heating was attempted in these preliminary calculations.
heating [cf. Eq. (3)], while the opposite occurs if the large-scale temperature is increased. The diagnosed cumulus heating profile exhibits a peak where the cumulus initialization decreases the large-scale temperature, but is characterized by lower heating rates where the cumulus initialization increases large-scale temperatures.

The cumulus initialization has changed the large scale from a state which could support no cumulus convection to a state whose convective heating (as calculated with the simplified parameterization) reproduces the diagnosed heating exactly. The cumulus initialization requires temperature changes <4 K with relatively larger humidity changes. Latent heat release calculated using a more complicated version of the Kuo parameterization (described in Donner et al., 1982, and Donner, 1986) and the initialized temperature and humidity profiles is also depicted in Fig. 2b. Latent heat release so calculated peaks more sharply in the vertical at a value about twice that of the diagnosed and simplified heating. Considering that the Kuo parameterization was significantly simplified before it was inverted and, comparatively, that no latent heat release occurred without the cumulus initialization, the agreement among heating rates in Fig. 2 is reasonable.

The cumulus initialization uses a specified divergence profile, and the extent to which it is capable of producing a diagnosed convective heating profile depends on whether the divergence is sufficiently favorable. Often, fields initialized dynamically for numerical weather prediction are characterized by very weak divergence fields in the tropics, in contrast to analyses obtained from intensive observational programs, exemplified by those used in the heating rate calculations of Kasahara et al. (1986). To illustrate the implications for cumulus initialization of an unfavorable divergence field, the calculations described above are repeated without using the heating rates and divergence profiles diagnosed as in Kasahara et al. Instead, the divergence profile is that resulting from a diabatic initialization of CCM0B (Fig. 5a), and the heating profile imposed is shown in Fig. 5b. The vertical integral of the heating is 2.5 K Pa s⁻¹, about three times that of the previous

---

Fig. 3. (a) Uninitialized and initialized humidity profiles and (b) changes in humidity due to cumulus initialization, for first cumulus initialization discussed in text.

Fig. 4. (a) Uninitialized and initialized temperature profiles and (b) changes in temperature due to cumulus initialization for first cumulus initialization discussed in text.
calculation, but still modest. The unfavorable character of the divergence profile precludes any humidity profile which can support fully the integrated convective heating. Figure 6 shows the humidity profile which yields the closest approach to the desired convective heating. The cumulus initialization tends to increase the humidity in convergent layers and decrease the humidity in divergent layers, a redistribution of moisture which increases the vertically integrated moisture convergence in an obvious way. The temperature adjustment (Fig. 7) also differs somewhat from the first case. The boundary-layer temperature is increased by cumulus initialization, while above the surface the two cases are qualitatively similar. Comparison of these two cases shows that the details of the temperature and humidity adjustment vary with the uninitialized profiles and heating diagnosis. When the full Kuo parameterization (Donner, 1986) is used, the distribution of latent heat release is qualitatively similar to that of the simplified parameterization.

4. Parameterization dependence of cumulus initialization

The foregoing discussion has employed exclusively a version of the Kuo parameterization. Clearly, the cumulus initialization should be as consistent as possible with the cumulus parameterization used in the numerical weather prediction model of interest. Although not discussed here in detail, initializations for other cumulus parameterizations could be derived.

For example, the equations governing moist adiabatic adjustment (Manabe et al., 1965) are

$$\frac{\partial}{\partial p} \theta(T^{(0)} + \Delta T + \delta T, q^{(0)} + \Delta q + \delta q, p) = 0,$$

(9)

$$q^{(0)} + \Delta q + \delta q = q_{sat}(T^{(0)} + \Delta T + \delta T, p)$$

(10)

$$\int_{p_0}^{p} (C_p \delta T + L \delta q) dp = 0.$$  

(11)
large-scale environment. Since these properties enter into the calculation of quasi-equilibrium of the cloud work function, which closes the parameterization and permits an evaluation of large-scale forcing by cumulus convection, a relationship between environmental properties and diagnosed heating exists upon which an inversion for cumulus initialization could be based. It is also possible to construct cumulus initializations for versions of the Kuo parameterization using other formulations of the moisture partitioning parameter (e.g., Krishnamurti et al., 1983).

5. Conclusion

A cumulus initialization procedure has been developed which inverts a simplified version of the Kuo (1974) cumulus parameterization, yielding adjustments to the temperature and humidity fields such that the cumulus parameterization will produce a diagnosed heating profile, including its vertical structure, for a specified velocity field. If the divergence field is very unfavorable to cumulus convection, it may not be possible to obtain the diagnosed precipitation or vertical integral of convective heating; the cumulus initialization then yields the closest approach possible to the diagnosed precipitation.

Since the divergence field plays a crucial role in cumulus initialization, an initialization for the velocity field (e.g., Julian, 1984) should be used in conjunction with an initialization for temperature and humidity. It may also be desirable to initialize the humidity in non-convective areas, perhaps using a principle such as Krishnamurti et al.'s (1984) radiative-advective balance. Since cumulus initialization produces temperature changes, it also has implications for dynamic initialization based on balanced states, a matter requiring further study; in principle, cumulus initialization should not interfere with dynamic initialization if the latter can be constructed to incorporate an appropriate balance involving diabatic heating. In practice, temperature changes associated with nonlinear normal-mode initialization are not large (Rasch, 1985). Imposing a normal-mode initialization following a cumulus initialization would not be likely, therefore, to remove the conditional instability produced by the cumulus initialization in many cases, and a few iterations between the cumulus and normal-mode initializations would likely converge quickly to eliminate any exceptions. More research is clearly required on this problem, however.

Unlike Krishnamurti et al. (1984), this cumulus initialization does not guarantee that total precipitable water remain invariant. It has not been assumed that the uninitialized humidity analysis should have the property that its total precipitable water is accurate. However, the constraint that total precipitable water remain invariant under cumulus initialization could be added to the humidity initialization, if desired.
Other versions of the Kuo parameterization than the simplified form given by (3) could also be inverted, such as an inversion incorporating a simplified treatment of cumulus eddy-flux convergence. It would also be possible to construct an inversion including diagnosed moistening, in addition to heating.

By initializing both model physical processes and velocity fields, many of the spinup problems in numerical weather prediction may be ameliorated. The application of these initializations to models for numerical weather prediction should provide crucial insights into the development from initial states of simulated large-scale flows.

Acknowledgments. This research was motivated by problems in initialization for numerical weather prediction under study by A. Kasahara, whose active interest in the work has been appreciated. A. Mizzi calculated the heating rates used in the first example discussed in section 3. Discussions with J. Tribbia, P. Rasch and R. Errico have also been helpful in this research. Partial support for this research has been provided through the National Oceanic and Atmospheric Administration under P.O. NA85AAG02575.

APPENDIX

Optimization Problems in Cumulus Initialization

The problems of maximizing $F_c$ and minimizing $Q$, defined in section 2b, represent optimization subject to bounds and (for $Q$) a nonlinear constraint. In practice, a vector $q$, defined in section 2b, whose components are values of $\Delta q$ at discrete pressure levels, is sought. The vector $q^*$ gives the minimum value of $(-F_c)$ or $Q$ if:

1) $q^*$ is a feasible point relative to the bounds and (for $Q$) satisfies (5).

2) The gradient $g(q^*)$ of $(-F_c)$ with respect to the components of $q$ not on their bounds is zero. For $Q$, this gradient must be a scalar multiple of the gradient (with respect to the components of $q$) of the right-hand side of (5). (The scalar is the Lagrange multiplier $\lambda$.)

3) The Hessian matrix of $(-F_c)$ at $q^*$ is positive definite. For $Q$ at $q^*$, the projected Hessian matrix of the Lagrangian function is positive definite. The Lagrangian function $L$ of $Q$ is

$$L(\lambda, q) = Q(q) + \lambda \int_{P_0}^{P_s} Q_T dp$$

$$- \frac{L}{0.6c_p p_s} \int_{0.4p_s}^{P_s} \left[ \frac{q_0^{(0)} + \Delta q}{q_{sat}} dp \right] \times \left[ \int_{P_0}^{P_s} \nabla \cdot \nabla q_0^{(0)} dp \right.$$

$$+ \int_{P_0}^{P_s} (q_0^{(0)} + \Delta q)(\nabla \cdot \nabla) dp - C_1 \right].$$  \hspace{1cm} (A1)

[The integrals in the term in (6) factored with the Lagrange multiplier would be expressed as finite sums to display explicitly their dependence on $q$.] If $G_L$ is the Hessian matrix of $L(\lambda, q)$ and $q$ has dimension $n$, its projected Hessian matrix is $Z^T G_L Z$, where $Z$ is an $(n \times n - 1)$ matrix whose columns form an orthonormal basis for the null space of the gradient of the right-hand side of (5).

Optimal humidity profiles were evaluated using a descent method (for $-F_c$) and a sequential augmented Lagrangian method solving its minimization subproblems using a modified Newton method (for $Q$) (Gill et al., 1981). NAG algorithms E04LBF and E04VBF (NAG, 1984), respectively, were used in the actual computations.

As noted in the conclusion, moisture conservation is not imposed and need not necessarily be imposed. If moisture conservation were desired, however, the additional constraint $\int_{P_0}^{P_s} \Delta q dp = 0$ would supplement (5). The optimization problem would be solved as before, except that to (A1) would be added a term consisting of the product of the moisture conservation integral and another Lagrange multiplier.

The minimization problem for the objective function $\tau$, defined in section 2c, is analogous to that for $Q$. The integral to be minimized is given by (8) with $0 \leq \Delta T_0 \leq \Delta T_{err}$, where $\Delta T_{err}$ is the uncertainty in boundary-layer temperature. A constraint ensuring that a parcel lifted from the surface will be buoyant at the lifting condensation level is

$$c[\Delta T_0, \Delta T(p_b)]$$

$$= (T_0^{(0)} + \Delta T_0)(\frac{p_b}{p_0})^K - T^{(0)}(p_b) - \Delta T(p_b) \geq 0,$$  \hspace{1cm} (A2)

where $K$ is the ratio of the gas constant to the specific heat for dry air. Introducing a slack variable $x$, (A2) can be written

$$c[\Delta T_0, \Delta T(p_b)] - x = 0,$$  \hspace{1cm} (A3a)

$$x \geq 0.$$  \hspace{1cm} (A3b)

The Lagrangian function for $\tau$ is

$$L(\lambda, T, x) = \tau(T) + \lambda \{ c[\Delta T_0, \Delta T(p_b)] - x \}.$$  \hspace{1cm} (A4)

Here $T$ denotes a vector containing values of $\Delta T(p)$ at discrete pressures. The solutions for $T$ in section 3 were obtained by following a procedure similar to that used for $q^*$, using NAG routine E04WAF (NAG, 1984). Sufficient conditions for $T^*$ to minimize $\tau$ are analogous to those for $q^*$, except that the inequality in (A2), as opposed to the equality in (5), requires, in addition, that the slack variable $x$ satisfy the condition (A3b).

REFERENCES


