

## A Local, Minimum Aliasing Method for Use in Nonlinear Numerical Models

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### ABSTRACT

The local spectral method is a minimum aliasing technique for the discretization and numerical integration of prognostic systems consisting of nonlinear partial differential equations. The technique embodies many features of both spectral transform methods and conventional finite difference techniques. The method is derived by applying a digital filtering approximation to a formulation of the nonlinear problem similar to the formulation that leads to the spectral transform method, and shares many of the desirable performance characteristics of that method. In contrast to the spectral transform method, the local spectral method can be implemented on a parallel processing computer system without requiring each processor to have a global knowledge of the values of variables in order to compute spatial derivatives. In addition to the computational virtues of the scheme, the local spectral method should have considerable promise as a high performance scheme for limited area models as appropriate boundary conditions are developed.

### 1. Introduction

At the present time most problems requiring the numerical integration of nonlinear meteorological fluid flows are solved using either finite difference (FD) formulations or spectral representations which are calculated using the spectral transform (ST) method proposed by Orzag (1970) and independently by Machenhauer and Rasmussen (1972). A general disadvantage of using FD methods for the solution of nonlinear systems is that the computation of quadratic and higher order product terms generate fields which have spatial wavenumbers that are too high to be represented on the model grid. When left untreated, this phenomenon (usually referred to as aliasing) will often result in the uncontrolled growth of high spatial frequency perturbations on the simulation variables (Phillips 1959). Although this unstable growth can generally be controlled by applying numerical dissipation or spatial low-pass filters to the model fields, the solutions will still be contaminated by those nonlinear products which are aliased onto lower frequencies and the other effects of the additional dissipation which generally are not desirable. In this paper we will show that this aliasing error is unavoidable for a general form encompassing all conventional FD schemes unless measures are taken that limit the model resolution for a given grid spacing although conservative schemes such as those proposed by Arakawa (1966) can avoid

the unstable growth by controlling the cumulative effects of the error.

Fourier methods such as the ST method are not subject to this fundamental error because they include the concept of an explicit spectral truncation of those terms which cannot be represented in terms of the model basis functions. Any components of the calculated time tendency that are at higher spatial wavenumbers than the model resolution are simply dropped and are not included in the further time integration.

In the following section we will derive a general operator notation for describing all FD techniques and a somewhat more general formulation which includes the ST method. A new technique, the local spectral (LS) method, will be developed, which shares the same general algorithmic architecture as the ST scheme while limiting the number of points on the spatial grid that can influence the solution at the next time step for a particular point. The LS technique will then be developed in a more heuristic fashion based on a digital signal processing argument which reduces the problem of designing a LS scheme of order  $M$  to the design of a digital low-pass filter of order  $2M + 1$ . It will be shown that a LS system can be generated which has an arbitrarily small aliasing error corresponding to the stop-band leakage of the low-pass filter.

Although the original motivation for developing the LS procedure was to limit the communications overhead when fluid computations were performed on a 2-D lattice organization parallel computer, the LS method does offer significant advantages over FD methods when used on conventional computers. While we will not discuss limited domain boundary condi-

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tions in this paper, we believe that the imposition of reasonable boundary conditions on the solution could provide a highly competitive method for regional scale simulations.

**2. General forms for the discretization of nonlinear problems**

In this section we will consider several general forms for the numerical integration of nonlinear partial differential systems. Since in most fluid calculations the error resulting from the time discretization of the problem is minimal, we will choose to limit our discussion to methods used for spatial discretization.

We have found the nonlinear advection/linear diffusion equation (1) to be a good prototype problem for the evaluation of schemes that will be used for nonlinear problems:

$$\frac{\partial u}{\partial t} = -u \frac{\partial u}{\partial x} + \nu \frac{\partial^2 u}{\partial x^2}. \tag{1}$$

Here we have confined our attention to one spatial dimension and a system state which is completely described by  $u(x, t)$ , the component of motion in the  $x$  direction. We have included a biharmonic diffusion term represented by  $\nu$ , the kinematic viscosity. This dissipation term will serve as a sink for the high spatial wavenumber power generated by the nonlinear terms in the system and is required to ensure that the asymptotic behavior of the full differential system remains continuous in space. In the absence of spatially scale selective dissipation, the time evolution of (1) will often tend to discontinuous solutions; however, the continuous system will be well behaved for any nonzero value of  $\nu$ .

In order to perform the numerical time integration of (1) from a suitable initial condition we must first discretize the problem, i.e., describe the system in terms of a finite number of parameters. In addition, this system must have suitable schemes for evaluating the first and second derivatives of  $u$  and then for combining all of the terms into a value for the time tendency of each discrete state variable. Once the time tendency has been computed one would then use a suitable discrete time differencing scheme to complete the state variable update.

The most commonly used spatial differencing technique is the finite difference method, which is based on the representation of  $u(x, t)$  as  $u_j(t)$  and samples of  $u$  taken at the discrete points defined by  $x = j\Delta x$ , where  $\Delta x$  is a grid spacing that is considered to be constant. Using this discrete form for  $u$  we can write a generalized finite difference version of (1), which is given as (2). A block diagram of the processing architecture for this scheme appears as Fig. 1.

$$\frac{du_j}{dt} = -u_j u_{xj} + \nu u_{xxj} \tag{2a}$$

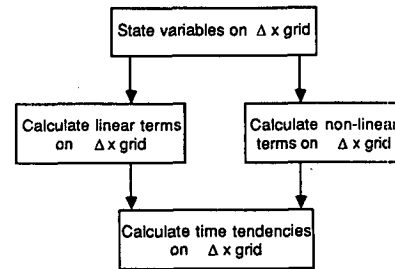


FIG. 1. Block diagram of finite difference method computation architecture.

$$u_{xj} = \sum_{k=-R}^R DX_k u_{j+k} \tag{2b}$$

$$u_{xxj} = \sum_{k=-R}^R DXX_k u_{j+k}. \tag{2c}$$

Here  $u_{xj}$  and  $u_{xxj}$  are finite difference approximations to the first and second spatial derivatives of  $u$  at grid-point  $j$ , which are generated through the convolution sums (2b) and (2c). In (2b) and (2c) the coefficients  $DX_k$  and  $DXX_k$  are chosen by one of several popular techniques such as truncating the Taylor series expansion of the function  $u(x)$ . It is well known that the coefficients  $DX_k$  and  $DXX_k$  can be chosen so that (2b) and (2c) will provide consistent, order  $2R$  approximations to the derivatives as  $\Delta x$  approaches 0, as long as the continuous solution for  $u$  can be differentiated at least  $2R$  times.

At this point it is instructive to consider the behavior of (2) in the context of a particularly simple form for  $u(x)$ . If we take  $u$  to be a pure sinusoid in  $x$  on a periodic domain, with  $u = a \sin(kt)$  and then recognize that the convolution sums (2b) and (2c) are simply the application of different digital filters to  $u$ , one can see that  $u_{xj}$  and  $u_{xxj}$  can be written in the form (3a) and (3b):

$$u_{xj} = A_1(k) \sin[kx + \phi_1(k)] \tag{3a}$$

$$u_{xxj} = A_2(k) \sin[kx + \phi_2(k)]. \tag{3b}$$

Here  $A_1(k)$  and  $A_2(k)$  are the amplitude responses of the filters whose impulse responses are  $DX_k$  and  $DXX_k$ , respectively, while  $\phi_1(k)$  and  $\phi_2(k)$  are the associated phase responses. For an ideal scheme  $A_1(k) = k$ ,  $A_2(k) = k^2$  and  $\phi_1(k) = \pi/2$ ,  $\phi_2(k) = \pi$ . We see that for a typical scheme where  $\phi_1(k) = \pi/2$ , the nonlinear term in (2a) will take the form  $-aA_1 \cos(kx) \times \sin(kx)$ , which by trigonometric identity is equal to  $-\frac{1}{2}aA_1 \sin(2kx)$ . The aliasing error arises when  $2k > \pi/\Delta x$  and the time tendency cannot be represented on the  $\Delta x$  grid. In this event the tendency will be aliased onto a new wavenumber  $k' = 2\pi/\Delta x - 2k$ .

As mentioned before, the most serious consequence of this error is the tendency for finite difference systems to unstably accumulate power near the maximum re-

solved wavenumber of  $\pi/\Delta x$ . It can be shown (Orzag 1971) that if one continuously removes all power from the solution at wavenumbers above  $2\pi/3\Delta x$ , the aliasing can be avoided; however, this both increases the complexity of the scheme to near that of our proposed LS method and limits the maximum resolution to less than the representation capability of the grid.

A different approach to avoiding the aliasing problem is embodied in the spectral transform method. Here the discretization is accomplished by approximating  $u(x)$  with a continuous function which is specified by a finite number of spectral coefficients. On a periodic domain these spectral coefficients are naturally discrete and give rise to the following simple form (4),

$$u(x, t) = \sum_{m=0}^{N_s} a_m \cos(k_m x) + b_m \sin(k_m x) \quad (4a)$$

$$k_m = 2\pi \frac{m}{L} \quad (4b)$$

where  $N_s$  is the spectral truncation order and  $L$  is the length of the domain. We will usually take  $b_0 = 0$  and  $b_{N_s} = 0$  so that the system has  $2N_s$  degrees of freedom. Note that we can still choose to represent the system state in the physical  $x$  dimension using the invertible discrete Fourier transform (DFT) to generate  $2N_s$  real, evenly spaced samples of  $u(x)$  and we can regard the effective spatial resolution,  $\Delta x$ , of this scheme as being  $L/2N_s$ . Since (4) gives a continuous representation for  $u(x)$  we can exactly compute the derivatives of this representation and calculate the nonlinear terms. The final step in the method is to truncate the resultant time tendencies to wavenumber  $N_s$  and perform the time integration.

If we were to approach the problem of directly substituting (4) into (1) we would get an expression for the nonlinear terms involving convolution sums of the spectral coefficients that is extremely inefficient to compute for reasonable truncations. It has been shown (Machenhauer and Rasmussen 1972; Orzag 1970) that these terms can be calculated much more efficiently using the fast Fourier transform (FFT) and inverse transform algorithms to interpolate  $u(x)$  onto a grid with  $4N_s$  points evenly spaced by  $\Delta x/2$  on which the nonlinear terms are computed, then transform the tendencies to an order  $2N_s$  spectral representation, and, finally, truncate the spectral time tendencies by ignoring all of the spectral terms with wavenumbers higher than  $N_s$ . In fact, when numerical calculations are made with this method the nonlinear terms are usually computed on a  $3N_s$  point grid since, following Orzag (1971), those terms which are aliased on the  $3N_s$  point grid fall onto wavenumbers that are outside the truncation and thus do not affect the solution. This same argument also applies to the LS method, which could use a fine grid with a spacing of  $2\Delta x/3$ . For notational simplicity we have chosen to present the methods using the  $\Delta x/2$  grid spacing.

In order to put the ST scheme in a form similar to that of the finite difference method we will write it in a somewhat unconventional way where the state variables are represented in physical space using the transform technique mentioned above (5):

$$\frac{du_j}{dt} = \text{LIN}_j + \text{NLIN}_j, \quad j = 0, 2N_s \quad (5a)$$

$$\text{LIN}_j = \nu \text{DFT}_{2N_s}^{-1}[-k_m^2 \text{DFT}_{2N_s}(u_j)] \quad (5b)$$

$$\text{NLIN}_j = \text{DFT}_{2N_s}^{-1}[\text{ADV}_m],$$

ignoring  $\text{ADV}_m$  for  $m > N_s$

$$\text{ADV}_m = \text{DFT}_{4N_s}[-u_{2j2} \text{DX}u_{2j2}], \quad m = 0, 2N_s \quad (5d)$$

$$u_{2j2} = \text{DFT}_{4N_s}^{-1}[\text{DFT}_{2N_s}[u_j]], \quad j2 = 1, 4N_s \quad (5e)$$

$$\text{DX}u_{2j2} = \text{DFT}_{4N_s}^{-1}[-iK_m \text{DFT}_{2N_s}(u_j)],$$

$j2 = 1, 4N_s. \quad (5f)$

Here the notation  $\text{DFT}_N$  denotes a DFT on  $N$  spatial points, and  $\text{DFT}_N^{-1}$  its inverse operator.  $\text{LIN}_j$  and  $\text{NLIN}_j$  are the linear diffusion and nonlinear advection terms, respectively, and  $\text{ADV}_m$  is the spectral representation of the nonlinear term. In (5e) and (5f) the inverse transforms should be evaluated using 0 for the amplitudes of terms where  $m > N_s$ . The transforms are used to interpolate  $u$ , “ $u_2$ ,” and the derivative of  $u$ , “ $\text{DX}u_2$ ,” onto the  $\Delta x/2$  grid where the nonlinear terms are calculated before transforming the tendencies back to the original  $\Delta x$  grid representation, while ignoring terms with wavenumbers above  $N_s$ . This differs somewhat from the normal way of writing ST methods in that ordinarily one performs the time update on the spectral coefficients, thus eliminating one set of the transforms. However, since transformation between the spectral representation and the  $\Delta x$  grid representation is an invertible linear operator it is a completely equivalent procedure. A block diagram of the method architecture is included as Fig. 2.

Note that in this method the various transform and inverse transform procedures are linear operators, which project interpolated values of  $u$  and the derivatives onto the  $\Delta x/2$  grid, or serve as a linear low-pass filtering scheme, which is applied to the resultant time tendencies before the field is returned to the nominal  $\Delta x$  resolution grid. In each case these operators are global in nature, i.e., they use the value of the data field over the entire domain. This property is associated with two of the principal disadvantages of the method. These disadvantages are the difficulties inherent in trying to use the method on a limited rather than periodic domain and the computational inefficiencies associated with the method when using computers with hierarchical memories, or distributed memories as in the case of massively parallel machines.

If we reformulate the interpolation and anti-aliasing functions of (5) in terms of finite-length digital filters

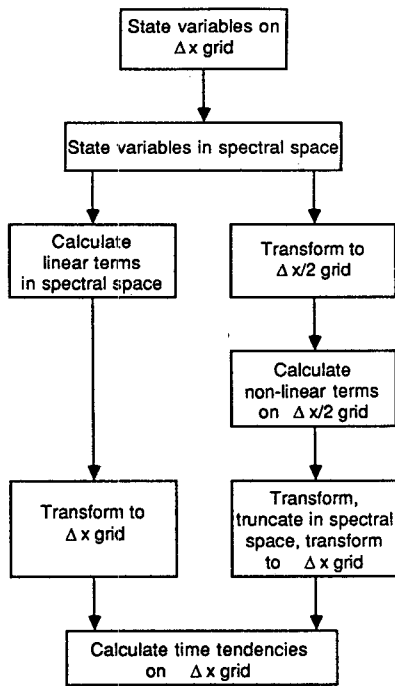


FIG. 2. Block diagram of spectral transform method computation architecture. This figure is written using physical space-time tendencies for consistency with the other methods. Normally the time update is performed in spectral space and the final transform to physical space is omitted.

we will form the system which we have chosen to call the local spectral (LS) method. The system is presented below and a block diagram of the processing appears as Fig. 3.

$$\frac{du_j}{dt} = LIN_j + NLIN_j \quad j = 1, N \quad (6a)$$

$$LIN_j = \nu \sum_{k=-R}^R DXX_k u_{j+k} \quad (6b)$$

$$NLIN_j = \sum_{k=-2R}^{2R} TRUNC_k ADV_{2j+k} \quad (6c)$$

$$ADV_{j2} = -u_{2j2} DX u_{2j2} \quad j2 = 1, 2N \quad (6d)$$

$$u_{2j2} = \sum_{k=-2R}^{2R} INTRP_k u_{(j2+k)/2} \quad (6e)$$

(Ignore terms where  $j2 + k$  is odd)

$$DX u_{2j2} = \sum_{k=-2R}^{2R} INTRPDX_k u_{(j2+k)/2} \quad (6f)$$

(Ignore terms where  $j2 + k$  is odd).

One can see that the form of the LS system is in many ways a hybrid of the FD and ST techniques. It employs the local derivative approximations of the FD method but uses the explicit double density grid and truncation

filters of the ST method. We will now discuss how to design the necessary filter coefficients ( $TRUNC_k$ ,  $INTRP_k$ ,  $INTRPDX_k$ , and  $DXX_k$ ). In section 3 we will present a more heuristic method for the derivation of the LS system and propose a filter design technique which is capable of providing performance equivalent to ST implementations.

### 3. The LS method from a signal processing perspective

Consider a set of values  $u_j$  which are samples of a continuous variable  $u(x)$  taken at a constant sampling interval  $\Delta x$ . If we assume that the continuous function  $u(x)$  is spatially band-limited such that its spatial spectrum is zero at all wavenumbers above  $k = \pi/\Delta x$ , then the well-known Nyquist sampling theorem states that we can uniquely reconstruct  $u(x)$  from the samples. One way in which this can be accomplished is to first construct the continuous time function  $u'(x) = \sum_j u_j \delta(x - j\Delta x)$ , where  $\delta$  is the Dirac delta function, and then low-pass filter  $u'$  as follows:

$$u(x) = \int_{-\infty}^{\infty} h(\tau) u'(x + \tau) d\tau \quad (7)$$

where the frequency response of the filter given by  $h(\tau)$  is unity for all wavenumbers below  $k = \pi/\Delta x$  and zero for all higher wavenumbers. Note that although (7) appears to be an integral it can actually be evaluated for any  $x$  by computing the sum given by (8):

$$u(x) = \sum_{j=-\infty}^{\infty} h(x - j\Delta x) u_j. \quad (8)$$

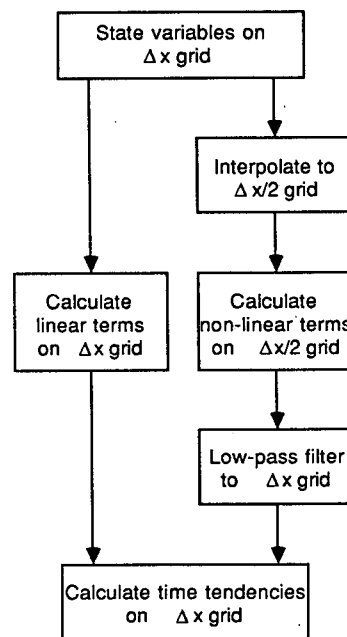


FIG. 3. Block diagram of local spectral method computation architecture.

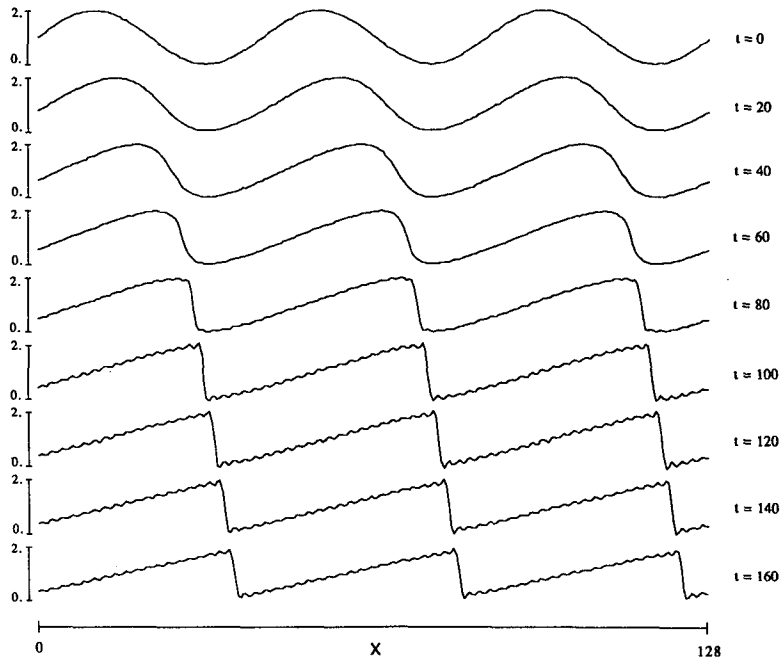


FIG. 4. Nonlinear advection equation solution using ST method;  $\nu = 0.12$ .

This expression defines a technique for interpolating  $u(x)$  to any point in space under the assumption of its band limited nature. For periodic domains the infinite sum can be evaluated and the result is equivalent to the procedure used by the ST technique in interpolating  $u$  to the  $\Delta x/2$  spacing grid. To derive the LS system

we will consider the approximation to (8) which involves designing a filter  $h(\tau)$  such that  $h(\tau) = 0$  for all  $\tau$  greater than some finite limit  $T = R\Delta x$ . This filter design problem is a central topic in digital signal processing and many approaches to its solution can be found in standard texts such as Oppenheim and Schaffer

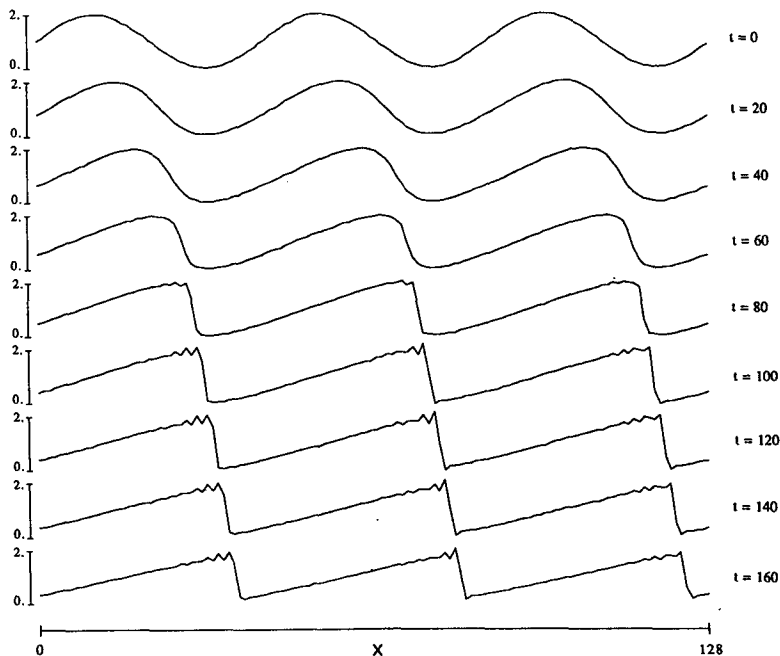


FIG. 5. As in Fig. 4 but using LS method.

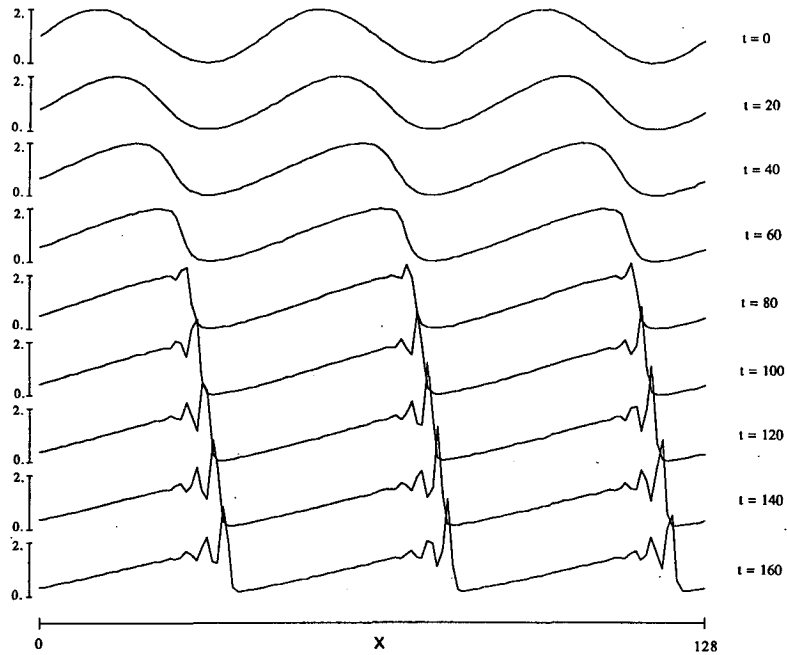


FIG. 6. Nonlinear advection equation solution using second-order FD method;  $\nu = 0.12$ .

(1975). One technique for designing the filter is presented in the Appendix. Once we have designed a finite length low-pass filter we have an interpolation scheme for transforming  $u$  to the  $\Delta x/2$  grid. If we use a filter design technique such as the window method presented in the Appendix, where  $h(\tau)$  has an analytic form that

can be differentiated with respect to  $\tau$ , then we can evaluate the derivatives of (8) to get a complete interpolation set (9):

$$u_{2j_2} = \sum_{k=-2R}^{2R} h(k\Delta x/2) u_{(j_2+k)/2}, \quad j_2 = 1, 2N \quad (9a)$$

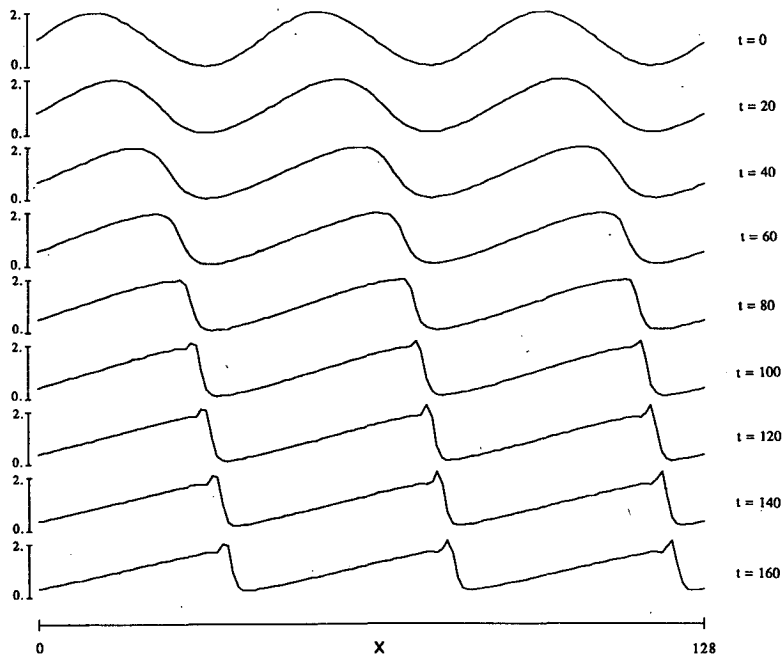


FIG. 7. As in Fig. 6 but for  $\nu = 0.40$ .

$$DXu_{2j_2} = \sum_{k=-2R}^{2R} \frac{dh(k\Delta x/2)}{dx} u_{(j_2+k)/2} \quad (9b)$$

$$DXXu_{2j_2} = \sum_{k=-2R}^{2R} \frac{dh^2(k\Delta x/2)}{d^2x} u_{(j_2+k)/2} \quad (9c)$$

In these sums only terms where  $j_2 + k$  is even are included since only these terms are part of the original integral (7). The interpolation set (9) gives us a way to generate the values of  $u$  and its derivatives on a  $\Delta x/2$  grid with arbitrary accuracy, which is limited only by our filter design skill. In general, the error will decrease without limit as we let the filter length,  $R$ , increase. When designing the filter we will need to choose a value for  $R$  that represents a reasonable compromise between accuracy and computational efficiency. For the rest of this paper we will concentrate on an  $R = 5$  filter, the design of which is detailed in the Appendix. For linear problems this can be considered the analog to a tenth-order finite difference system.

Once the nonlinear terms are computed on the  $\Delta x/2$  grid we will then use a low-pass filter to remove those components having  $k$  greater than  $k = \pi/\Delta x$ , which have been produced during the computation of the nonlinear terms. The form of the anti-aliasing filter which takes a nonlinear product  $q_{2j_2}$  on the  $2N$  grid and samples it back to the original grid is given by (10):

$$q_j = \sum_{k=-2R}^{2R} \frac{h(k\Delta x/2)}{2} q_{2j_2+k} \quad (10)$$

Here the factor of one-half in the impulse response results from renormalizing the filter so that it is appropriate for the  $\Delta x/2$  sample spacing. We will then add these dealiased tendencies to the linear terms, which are computed on the coarse grid to get the final time tendencies. The whole system can finally be assembled to yield the LS system presented as (6) at the end of section 3. To date we have only considered systems where the same filter is used for the initial interpolation and the final low-pass filtering. However, for some applications where complete freedom from aliasing is a strong requirement one may wish to consider using a filter with a somewhat lower wavenumber cutoff for the final filter.

#### 4. Nonlinear numerical experiments

In this section we will perform a nonlinear experiment to evaluate the performance of the LS method on an actual time integration of (1), which represents a severe test of the ability of a scheme to resist the accumulation of aliasing error. We will perform an initial value problem integration describing the evolution of the initial condition sine wave given by (11):

$$u(x) = 1 + \sin(kx). \quad (11)$$

For this initial condition and sufficiently small values for the viscosity  $\nu$ , the continuous system will ultimately approach an arbitrarily close approximation to a sawtooth pattern moving at unity phase velocity in the positive  $x$  direction. For fluid problems of this sort the diffusion term is generally included to provide numerical smoothing; we would like to be able to choose as small a value for the diffusion as possible. Note that even though the ST technique is completely free of aliasing error some scale-dependent dissipation is required to prevent the buildup of high spatial frequency power known as spectral blocking. For FD solutions most investigators have found it necessary to include substantially greater amounts of dissipation to prevent aliasing instability.

In order to make the problem as simple as possible we take the size of our resolution element  $\Delta x$  to be 1 and consider a periodic domain where the domain length is 128. We have chosen  $k$  for the initial condition to be  $6\pi/128$  so that there are three complete cycles of the initial sinusoid. We performed the time integration using a second-order Adams-Bashforth scheme with a time step of 0.1; however, we have verified that the solutions are essentially identical when  $\Delta t = 0.05$ .

When performing nonlinear calculations such as this, one cannot generally take the dissipation parameter  $\nu$  to be a free parameter since there are minimum dissipation values required to control the buildup of high frequency power. This buildup can be controlled by either using artificially high values for  $\nu$  or through the use of smoothing filters (Shapiro 1970) with FD methods. For simplicity, in this paper we have chosen to control the high wavenumber components with the diffusion term. It should be noted that better performance with FD schemes can be achieved with other filtering techniques; however, all of these filters require lower cutoff frequencies than the LS filters since they are limited to the original model grid and thus must remove components with wavenumbers above  $2/3\pi$  to prevent aliasing. In Figs. 4 and 5 we present the results of this integration using the ST and LS schemes for a dissipation value of  $\nu = 0.12$ , which we found to be a reasonable choice for both methods. The figures show a generally good solution with a suitable evolution toward the sawtooth wave. In each case there is some tendency to accumulate power near the truncation wavenumber but it is well controlled and accounts for about the same amount of error with each method, although the ST solution tends toward a global distribution of this spurious power while the error in the LS scheme is more local in character.

For contrast we have performed a second order FD integration with the same value for diffusion. The results of this integration are presented as Fig. 6 and clearly show an uncontrolled aliasing error. In fact, this calculation develops an explosive aliasing instability soon after the end of the time period displayed in the figure. We have explored various values for the

diffusion and have found that a value of  $\nu = 0.4$  yields approximately the same amount of high wavenumber noise level as the two other methods do with  $\nu = 0.12$ . The results of this experiment appear as Fig. 7. In order to directly view the effects of aliasing on the solutions, the wavenumber spectra of the solutions corresponding to Figs. 4, 5, and 6 at time  $t = 120$  have been reproduced as Fig. 8. Since the initial condition included only domain wavenumbers 0 and 3 and the system is quadratically nonlinear, the evolution of the system should be limited to those wavenumbers which are a multiple of 3. In Fig. 8c we see that the finite difference calculation has produced significant amplitudes at other wavenumbers due to aliasing. The ST calculation is completely free of aliasing effects and the LS solution contains a very small amount of aliased power in the high frequency wavenumbers due to the finite leakage of the low-pass filter.

**5. Linear properties of the methods**

In the study of numerical techniques much emphasis is usually placed on the linear phase propagation performance of a method. In this section we will show that when we include the effects of enough dissipation to control the solution for the nonlinear problem, phase error effects are generally not dominant. In Fig. 9 we have plotted the linear advection phase speed errors for the second order FD scheme, a standard tenth-order FD scheme and our  $R = 5$  LS scheme. The phase response of the second-order scheme is well known and exhibits significant phase speed retardation at higher wavenumbers. Since the  $R = 5$  LS scheme uses five points on either side of the object point to compute the derivative, it is natural to compare the performance to a tenth-order scheme even though for our particular choice of filter design our set of LS coefficients are formally only second order. As one can see from Fig. 9

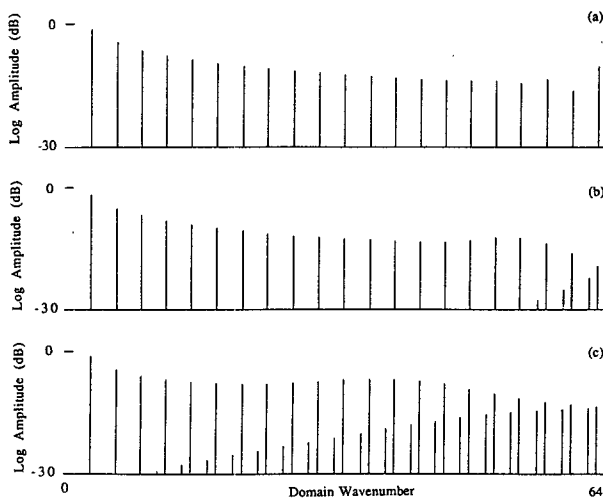


FIG. 8. Wavenumber spectra  $T = 120$  for (a) ST (b) LS and (c) FD methods with  $\nu = 0.12$ . Wavenumber 0 has been omitted.

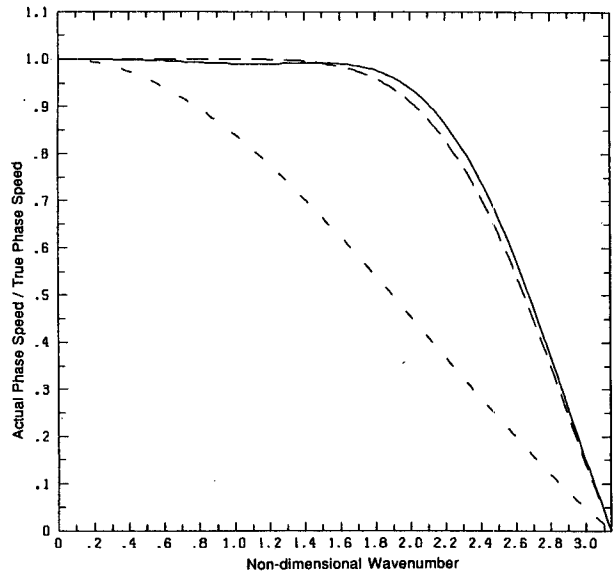


FIG. 9. Linear phase speed errors for second-order FD method (short dashed line), tenth-order FD method (long dashed line), and  $R = 5$  LS method (solid line).

the linear properties of the LS scheme are indeed quite similar to the tenth-order FD method. The phase propagation of the ST method is exact once the initial condition has been truncated to the spectral resolution.

In order to examine the significance of these results we have performed an initial value problem integration for the linear advection-diffusion equation with the initial condition given by (12) and unity advection velocity; the rest of the parameters are identical to those in section 3:

$$u(x) = \exp[-(x^2/25)]. \tag{12}$$

These results appear as Figs. 10-13 and show essentially identical performance for the LS and ST schemes. For the FD case with  $\nu = 0.12$ , the solution exhibits an approximately 3% phase speed retardation of the peak but is certainly an acceptable solution. The  $\nu = 0.4$  FD case shown in Fig. 13 demonstrates the dramatic effect of the added dissipation that was necessary to control the errors produced by the poor handling of the nonlinear terms. In general, for problems of this class we have found the effects of the required dissipation to be much more significant than the phase velocity errors per se.

**6. Summary**

In conclusion, we have presented a technique which we have called the local spectral method. We believe that this is a new approach to the numerical integration of nonlinear problems. The method has been shown to provide similar performance to the well-known spectral transform method and requires about the same level of computational smoothing.



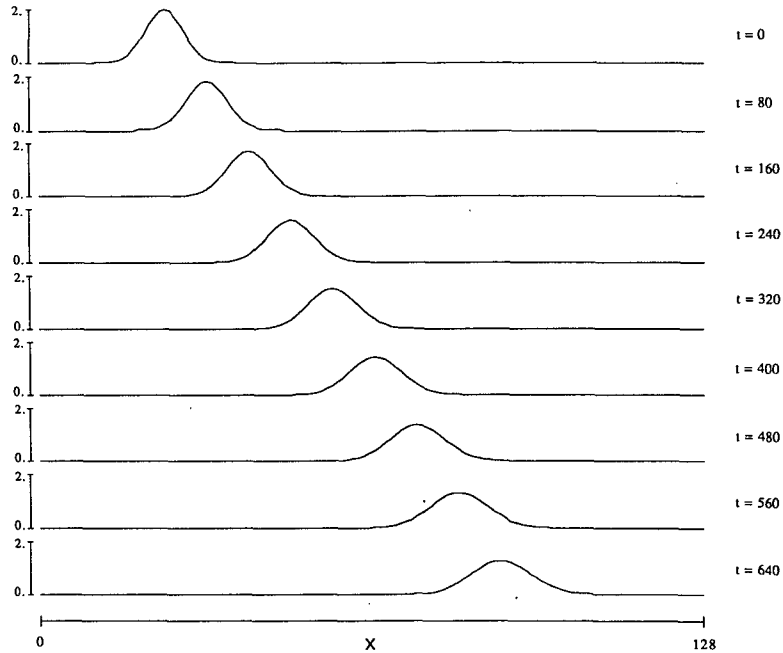


FIG. 10. The ST method linear solution;  $\nu = 0.12$ .

In terms of computational efficiency, the example problems given in this paper can be performed about a factor of 2 faster using the LS formulation than the ST technique when run on a conventional scalar processor using standard FFT library routines for the transforms involved in the ST calculation. In addition,

the LS scheme, unlike the ST method, is well suited to running on computers with distributed processor and memory configurations. This results from the fact that at each time step the LS method requires knowledge of only the state variables that are in some sense "near" the point for which one is computing the time

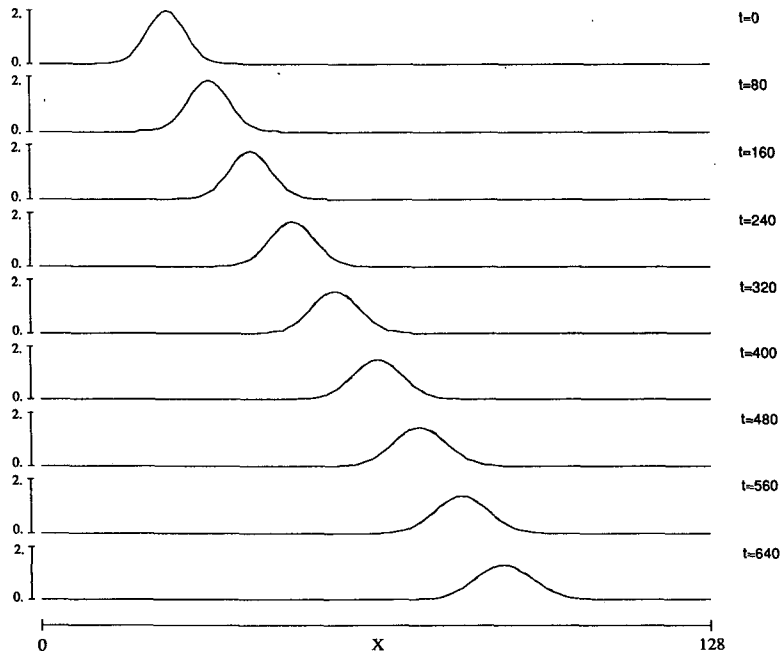


FIG. 11. The LS method linear solution;  $\nu = 0.12$ . Dotted line is ST solution.

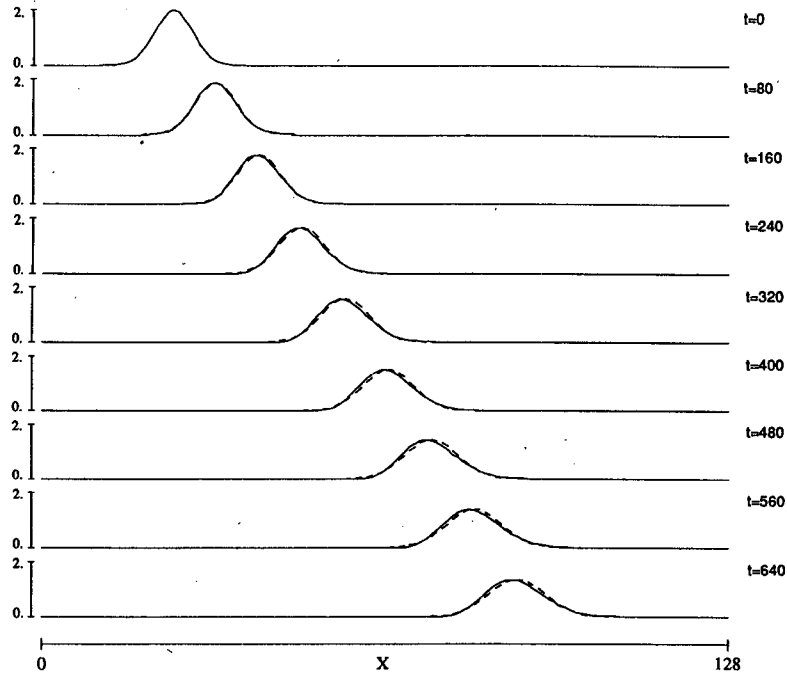


FIG. 12. The FD method linear solution;  $\nu = 0.12$ . Dotted line is ST solution.

tendencies, while the ST calculation requires a global knowledge of the state variables. To be fair we should point out that the LS scheme is not as well suited as the ST scheme for semi-implicit time integrations, since the LS technique does not have the property of diagonalizing the linear system that must be solved for

the implicit part of the time step. We do, however, believe that the LS method will be able to achieve comparable performance to the semi-implicit techniques through the use of explicit "slow sound wave" methods. A discussion of these methods is being prepared.

Finally we believe that the LS method has a sub-

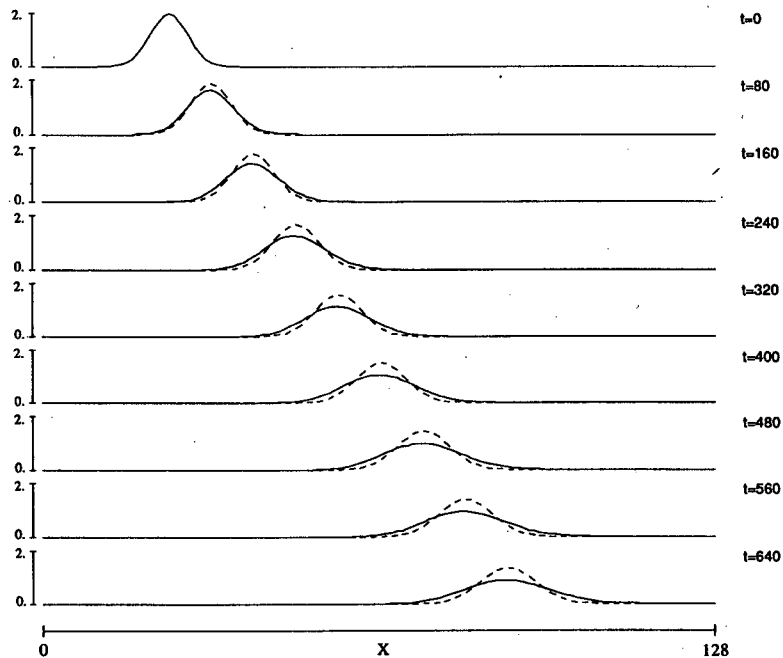


FIG. 13. The FD method linear solution;  $\nu = 0.40$ . Dotted line is ST solution.

stantial amount of promise for use in limited area models; however, we have not yet undertaken a study of the appropriate boundary conditions for such use.

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APPENDIX

One of the steps that needs to be performed during the course of constructing a complete LS method is the actual design of the low-pass filter impulse response,  $h(\tau)$ , which will determine the various scheme coefficients. Ideally this filter would have a frequency response which is unity over the wavenumber interval of 0 to  $\pi/2$  and zero over the interval from  $\pi/2$  to  $\pi$ . It is unfortunately impossible to achieve this result with a filter whose impulse response has finite length; therefore, one must employ a suitable approximation technique.

Filter design is a well-studied problem which has generated many well-known techniques. Many of the common approaches, particularly those based on optimality criteria, are fundamentally discrete in nature and are solved using linear programming methods. For our use we need to generate a filter which is a continuous function of  $\tau$  since we will also need to calculate the derivatives of the impulse response. For the sake of simplicity we have generated our initial filters using a straightforward inverse transform method with a multiplicative window function. Such filters are commonly used and are capable of high performance even though they have no optimality properties. In addition, we have decided to limit ourselves to window functions where the Fourier transform of the window function is a positive definite quantity, and from this set of windows we have taken the one proposed by Papoulis (1973) for use in spectral estimation. Following this procedure we can define  $h(\tau)$  on the interval  $-(R + 0.5)$  to  $R + 0.5$  as follows:

$$h(\tau) = \frac{W(\tau)}{2\omega_c} \int_{-\omega_c}^{\omega_c} \cos \omega \tau d\omega \quad (13)$$

$$W(\tau) = \frac{1}{\pi} \sin \frac{\text{abs}(\pi\tau)}{R + 0.5} + \left[ 1 - \frac{\text{abs}(\pi\tau)}{R + 0.5} \right] \cos \frac{\pi\tau}{R + 0.5} \quad (14)$$

Here  $W(\tau)$  is the Papoulis window function and  $\omega_c$  is the filter cutoff frequency, which we have take to be  $\pi$ . We have chosen to take both INTERP and TRUNC to be samples of  $h(\tau)$  taken at the appropriate sampling

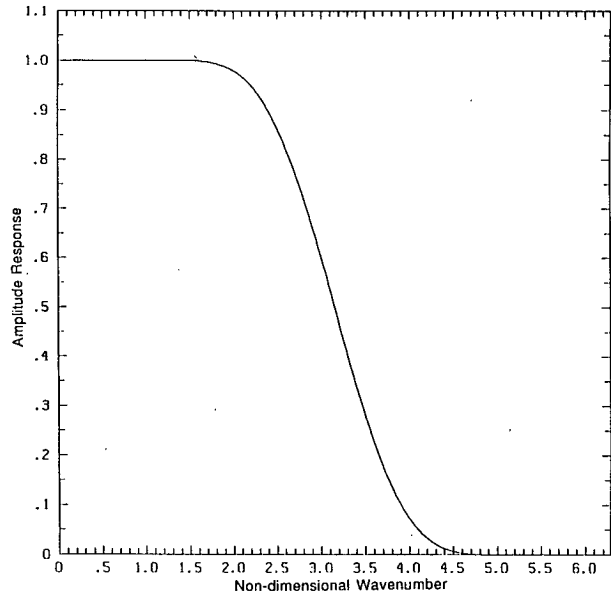


FIG. A1. Frequency response of  $R = 5$  low-pass filter.

intervals and suitably normalized. We have computed DXX and INTERPDX by numerically differentiating  $h(\tau)$  with a sufficiently small  $\Delta x$  at the needed intervals. If more freedom from aliasing than is achieved here is required, the coefficients for TRUNC can be computed with a lower cutoff frequency than INTERP.

A complete description of this process is given in Oppenheim and Schaffer (1975). When this process is performed for the  $R = 5$  case, i.e., we constrain the impulse response to be zero at all lags greater than 5.5, we achieve the frequency response portrayed in Fig. A1.

REFERENCES

Arakawa, A., 1966: Computational design for long-term numerical integrations of the equations of atmospheric motion. *J. Comput. Phys.*, **1**, 119-143.

Machenhauer, B., and E. Rasmussen, 1972: On the integration of the spectral hydrodynamical equation by a transform method. Rep. No. 3, Institut for Teoretisk Meteorologi, Kobenhavns Universitet, Denmark, 44 pp.

Oppenheim, A. V., and R. W. Schaffer, 1975: *Digital Signal Processing*. Prentice-Hall, 585 pp.

Orzag, S. A., 1970: Transform method for the calculation of vector-coupled sums: Application to the spectral form of the vorticity equation. *J. Atmos. Sci.*, **27**, 890-895.

—, 1971: On the elimination of aliasing in finite-difference schemes by filtering high-wavenumber components. *J. Atmos. Sci.*, **28**, 1074.

Papoulis, A., 1973: Minimum-bias windows for high-resolution spectral estimates. *IEEE Trans. Inform. Theory*, **IT-19**, 9-12.

Phillips, N. A., 1959: An example of non-linear computational instability. *The Atmosphere and the Sea in Motion*, Rossby Memorial Volume, Rockefeller Institute Press, 509 pp.

Shapiro, R., 1970: Smoothing, filtering, and boundary effects. *Rev. Geophys. Space Phys.*, **8**, 359-387.