NOTES AND CORRESPONDENCE

On the Formation of Potential-Vorticity Anomalies in Upper-Level Jet-Front Systems

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22 January 1990 and 17 March 1990

ABSTRACT

We review and discuss a difference in interpretation of the role of turbulence in modifying the potential-vorticity distribution in the vicinity of upper-level jet–front systems. In the late 1970s, M. A. Shapiro presented observational evidence that turbulent mixing of heat can result in a positive anomaly of the Ertel potential vorticity on the cyclonic-shear side of upper-level jets near the level of maximum wind. E. F. Danielsen and collaborators disputed this evidence and the accompanying interpretation. They argued that the turbulent mixing of potential vorticity can be described in terms of downgradient diffusion, in the same sense as for a passive chemical tracer. Accordingly, turbulent mixing cannot produce anomalies from initially smooth distributions of potential vorticity. In our view, this dispute stems from differences in the averaging procedures used to analyze turbulent flows, which lead to fundamentally different definitions of potential vorticity. Shapiro defined potential vorticity as the scalar product of the averaged absolute vorticity and the averaged potential-temperature gradient, whereas Danielsen et al. defined it, in their analytical framework, as the average of the scalar product of these quantities. We conclude that the positive anomaly of potential vorticity identified by Shapiro is plausible if one accepts the definition of potential vorticity used in his studies. Moreover, we believe Shapiro’s alternative to be the only practical option when working with observed or simulated data. Even if Danielsen’s alternative could be adopted in practice, we suggest that its utility as a tracer is problematic in view of the questionable validity of the downgradient diffusion of potential vorticity.

1. Introduction

The contemporary focus on potential vorticity as a fundamental dynamical variable (Hoskins et al. 1985) motivates us to reexamine a standing controversy regarding the effects of turbulence on potential vorticity in the vicinity of upper-level jet–front systems. In a series of case studies based on the analysis of aircraft measurements together with conventional radiosonde data, M. A. Shapiro (1974, 1976, 1978) identified a positive potential-vorticity anomaly on the cyclonic-shear side of upper-level jets near the level of maximum wind. He argued that this anomaly is generated by the vertical (and downgradient) flux of heat associated with turbulence due to Kelvin-Helmholtz instability in zones of large vertical wind shear above and below the jets. E. F. Danielsen and collaborators (Danielsen and Hipskind 1980; Danielsen et al. 1987) disagreed with this interpretation. They argued that the direct effects of diabatic and frictional processes on potential vorticity are sufficiently small over the period of interest for potential vorticity to be conserved. An implication of their view is that potential vorticity is no different from any other conserved tracer; thus, the potential-vorticity field observed on the cyclonic-shear side of upper-level jets should resemble the smoothly varying distribution of ozone mixing ratio found there, which is not the case in the studies by Shapiro.

We argue that the foregoing difference in interpretation stems from the disparate averaging procedures used in the two sets of investigations. If potential vorticity is defined as the scalar product of the averaged absolute vorticity and the averaged potential-temperature gradient, then the equation governing its evolu-
tion contains eddy fluxes of heat and momentum, which can produce anomalies in potential vorticity such as those suggested by Shapiro’s analysis. On the other hand, if potential vorticity is taken to be the average of the scalar product of the absolute vorticity and potential-temperature gradient, then its governing equation yields only an eddy flux of potential vorticity. If this eddy flux is modeled on the assumption of downgradient mixing, then Shapiro’s anomalies are not possible.

The crux of the dispute thus shifts to which of the two definitions of potential vorticity is more revealing physically or more appropriate in practice. In any observational or numerical study of turbulent flow, averaging (e.g., ensemble, grid-volume, spatial or temporal filtering) is unavoidable; one typically considers averaged fields of velocity and temperature. From these quantities and their respective governing equations one then may derive a potential vorticity and the equation governing its evolution. If potential vorticity is useful in elucidating the dynamics of the flow governed by the averaged equations, then the potential vorticity formed from the averaged equations is the appropriate quantity. This is Shapiro’s position.

Danielsen et al. are interested in potential vorticity as a tracer. They develop the governing equation for potential vorticity from the unaveraged equations for velocity and temperature. They then proceed to average the governing equation for the resulting potential vorticity, and complete the derivation by neglecting molecular effects. This procedure yields an equation governing the evolution of the average of the scalar product of the absolute vorticity and potential-temperature gradient; this equation contains eddy fluxes of potential vorticity, which are modeled in terms of a downgradient-mixing hypothesis. In our opinion, there are two difficulties with this position; one practical, the other physical. First, averaged potential vorticity (in the sense advocated by Danielsen et al.) cannot be deduced from contemporary observing/analysis systems, because these yield averaged velocity and temperature rather than point values of gradients of these respective quantities. Second, even if it were possible to measure the averaged potential vorticity, the validity of the Danielsen et al. position hinges on whether this potential vorticity diffuses downgradient. We present supporting and opposing arguments.

In the next section, we review the mechanism proposed by Shapiro for the formation of potential-vorticity anomalies in upper-level jet–front systems, and show that it is consistent with several integral theorems governing potential-vorticity evolution not considered explicitly in his original papers. In section 3, we summarize the objections to this mechanism raised by Danielsen et al. and reconcile the difference in interpretation between them and Shapiro. In section 4, we recapitulate our conclusions and point toward future research necessary to address their implications. An Appendix is included to illustrate the effect of averaging on the conservation of a hypothetical quadratic quantity resembling potential vorticity. Arguments for and against downgradient diffusion of potential vorticity are presented in the context of this hypothetical quantity.

2. Evidence for the turbulent generation of potential-vorticity anomalies in upper-level jet–front systems

In a series of case studies of the structure of upper-level jet–front systems based on aircraft measurements and conventional upper-air data, Shapiro (1974, 1976, 1978) documented the existence of a positive anomaly of potential vorticity on the cyclonic-shear side of upper-level jets, near the level of maximum wind. An example of one of these analyses is reproduced here, showing vertical cross sections of wind speed and potential temperature defining an upper-level jet–front system (Fig. 1), along with the associated distributions of potential vorticity, ozone concentration, and Richardson number (Fig. 2). The potential vorticity shown in Fig. 2 is the hydrostatic form of the Ertel potential vorticity, which, in pressure coordinates, takes the form

$$P = -g_0 \zeta_\phi \cdot \nabla \theta \cdot \nabla \theta,$$  \hspace{1cm} (1a)

where $\zeta_\phi = \nabla \times V_h + f k$, $\nabla \theta = \nabla \rho + k \partial \rho / \partial \rho$, and other symbols have their usual meaning. Under those circumstances where potential temperatures increase monotonically with altitude it is useful to transform (1a) to isentropic coordinates:

$$P = -g \zeta_\phi \frac{\partial \theta}{\partial p}.$$  \hspace{1cm} (1b)

The distribution of potential vorticity in Fig. 2a may be contrasted with the corresponding ozone pattern in Fig. 2b. If potential vorticity were conserved following the three-dimensional air motion, its distribution should match that of the ozone much more closely, since the latter is assumed to act as a tracer of air motions in the lower stratosphere on the time scale of the development of upper-level fronts. The downward-directed “tongue” in the ozone pattern in Fig. 2b results from the descent of stratospheric air in conjunction with upper-level frontogenesis.

Following Staley (1960), Shapiro (1976, 1978) discussed the development of the potential-vorticity anomaly in terms of the equation:

$$\frac{dP}{dt} = -g \zeta_\phi \frac{\partial \theta}{\partial p} + g_0 \frac{\partial \theta}{\partial p} \left[ k \cdot \left( \nabla \theta \times \nabla \theta \right) \right]$$

$$- g \frac{\partial \theta}{\partial p} \left[ k \cdot (\nabla \theta \times F_h) \right],$$  \hspace{1cm} (2)
where \( \frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{V}_h \cdot \nabla \), and \( \mathbf{F}_a \) and \( \mathbf{\hat{\theta}} \) represent frictional and diabatic effects in the prognostic equations for horizontal wind and potential temperature, respectively. In Shapiro’s application of (2), involving smoothed analyses, the dependent variables, \( \theta \), \( \xi_\alpha \), and \( \mathbf{V}_h \), represent Reynolds averages, and the quantities \( \mathbf{F}_a \) and \( \mathbf{\hat{\theta}} \) include (minus) the divergences of the Reynolds stress and Reynolds heat flux, respectively. Consistent with the derivation of (2) from the Reynolds-averaged equations for momentum and heat in Shapiro’s application, the quantity \( P \) in (2) is the scalar product of the averaged absolute vorticity and the averaged potential-temperature gradient, not the average of the product of these quantities. This distinction is expressed as

\[
-g \mathbf{\bar{a}}^\beta \cdot \nabla \mathbf{\bar{\theta}} = -g \mathbf{\bar{a}}^\beta \cdot \nabla \mathbf{\hat{\theta}} - g \mathbf{\bar{a}}^\beta \cdot \nabla \mathbf{\bar{\theta}},
\]

where the overbar indicates a Reynolds average.\(^3\) We have used hydrostatic pressure coordinates in (3), since the transformation to isentropic coordinates may be problematic in turbulent regions.\(^5\) Only the first term on the right of (3) may be readily transformed to the isentropic-coordinate form of \( P \) (1b), since this term is composed of Reynolds-averaged quantities.

Shapiro (1976, 1978) applied (2) in the zone of strong cyclonic shear at the level of maximum wind, eliminating the need to consider the middle term on the right. In addition, he neglected frictional processes on the basis of a scale analysis and focused attention on the role of vertical gradients of turbulent heat flux, \( \mathbf{\theta} = g \mathbf{\hat{\rho}} w' \mathbf{\hat{\theta}} / \partial \mathbf{\hat{p}} \), where \( \mathbf{\hat{\rho}} \) is air density and \( w \) is the vertical velocity. Assuming downgradient mixing of form of the Ertel potential vorticity. Such a distinction does not arise in the case of the quasigeostrophic potential vorticity (e.g., Hoskins et al. 1985, p. 911), since its form is linear, i.e., the sum of the geostrophic absolute vorticity and a deviation static stability.

\(^3\) It is tacitly assumed that the hydrostatic approximation is satisfied well enough in turbulent regions of upper-level jet-front systems to justify formulating (3) in pressure coordinates. If one were to consider (3) from a quantitative rather than conceptual standpoint, Boussinesq height coordinates would be more appropriate.
heat, the eddy flux of potential temperature should be large and negative (downward) in regions of turbulent mixing, which are assumed to correspond to minima in Richardson number (Fig. 2c). Comparing Figs. 1 and 2c confirms that the Richardson number is minimized in the zones of large vertical wind shear above and below the jet core. The vertical distribution of heating due to the convergence of turbulent heat flux contributes respectively toward warming and cooling just above and below the level of maximum wind speed on the cyclonic shear side of the jet core. Further discussion of this mechanism is found in the original references by Shapiro and in the review by Keyser and Shapiro (1986, pp. 461-466).

The formation of the positive potential-vorticity anomaly may be thought of as follows: The vertical distribution of turbulent heating tends to inhibit the vertical spreading of the isentropes immediately above and below the level of maximum wind (which would be required if potential vorticity were conserved) during the frontogenetical scale contraction of the cyclonic-shear zone due to convergent, ageostrophic flow in the
cross-front plane. Reference to the flux form of the isentropic continuity equation,
\[
\frac{\partial}{\partial t} \left( -\frac{1}{g} \frac{\partial p}{\partial \theta} \right) = -\nabla_s \cdot \left( -\frac{1}{g} \frac{\partial p}{\partial \theta} \mathbf{V}_s \right) - \frac{\partial}{\partial \theta} \left( -\frac{1}{g} \frac{\partial p}{\partial \theta} \right),
\]
(4)

shows that if the thickness (mass) of a layer bounded by isentropes remains constant in time, then the presence of horizontal mass convergence must be balanced by a positive vertical gradient of diabatic heating. The same mechanism responsible for enhancing potential-vorticity values in the vicinity of the level of maximum wind should also act to reduce them within the shear layers above and below the jet, where the downward eddy heat flux is maximized. [This argument is only suggestive, however, since the assumption of vanishing vertical wind shear in (2) is violated and the relative importance of the friction term is undetermined.] That there must exist negative potential-vorticity anomalies together with the positive anomaly is demonstrated by the following development.

Use of the transport theorem (e.g., Dutton 1976, pp. 116–117) applied to a material volume in the case of hydrostatic isentropic coordinates leads to
\[
\frac{\partial \langle P \rangle}{\partial t} = M^{-1} \int_{V_s} \nabla^s \cdot (\zeta_s \mathbf{V}_s + F_b \times k) dV_s,
\]
(5)

where the brackets indicate an average with respect to mass, \(\nabla^s = \nabla + k \partial / \partial \theta\), \(\zeta_s = \nabla^s \times \nabla^s + f k\), and \(dV_s = dx dy d\theta\). Derivation of (5) is facilitated by rewriting the right-hand side of (2) in flux form:
\[
\frac{dP}{dt} = -g \frac{\partial}{\partial \theta} \nabla^s \cdot (\zeta_s \mathbf{V}_s + F_b \times k),
\]
(6)

and noting that a mass element in isentropic coordinates is expressed as \(dM = (-g^{-1} \partial p / \partial \theta) dV_s\). Application of the divergence theorem (e.g., Dutton 1976, pp. 109–112) to (5) leads to the inference that, for those material volumes where diabatic heating on the boundaries and the component of frictional force tangential to the boundaries respectively vanish, the mass-averaged potential vorticity is invariant. Consequently, if (5) is applied to a volume sufficiently large to enclose both the upper-level jet–front system and the associated turbulent regions within the interior of the volume, then the formation of the positive anomaly in the vicinity of the level of maximum wind must be compensated by the formation of negative anomalies (or anomaly) to satisfy the invariance of \(\langle P \rangle\). Although the analysis in Fig. 2a apparently does not reflect this constraint, careful visual inspection of the results of numerical experiments by Gidel and Shapiro (1979), designed to reproduce the generation of the positive potential-vorticity anomaly at the level of maximum wind, suggests the concomitant formation of negative anomalies above and below the positive one. (Compare the distributions in their Fig. 9 for the cases with and without parameterized turbulent mixing.) It is also straightforward to show that these results are consistent with the conclusions of Haynes and McIntyre (1987), which are based on a specific application of (5) to control volumes bounded above and below by isentropes.

3. Objections to the turbulent generation mechanism and our reconciliation of the difference in interpretation

Starting from the unaveraged equations of motion and assuming that molecular processes can be neglected, Danielsen et al. (1980, 1987) form the equation for potential vorticity, thus obtaining
\[
\frac{\partial}{\partial t} + \mathbf{V}^P \cdot \nabla P = 0,
\]
(7)

where in this instance \(P\) refers to the potential vorticity of a particular realization and \(\mathbf{V}^P = \mathbf{V}_b + \mathbf{a}\). (Note that the use of hydrostatic pressure coordinates allows one to take advantage of the nondivergence of the averaged and deviation velocity fields.) Danielsen et al. (1980, 1987) proceed to take the average of (7) (corresponding in their papers to the spatial average over a grid volume but alternatively viewed as an ensemble average), and conclude that
\[
\frac{\partial \langle \mathbf{P} \rangle}{\partial t} + \langle \mathbf{V} \rangle \cdot \nabla \langle \mathbf{P} \rangle = -\nabla^s \cdot (\mathbf{V}^P \langle P \rangle).
\]
(8)

It is clear that \(\langle \mathbf{P} \rangle\) in (8) is given by the full expression in (3) and that it evolves following the mean motion only in response to the term on the right. Danielsen et al. hypothesize from the identical forms of (8) and its counterpart applicable to the Reynolds-averaged tracer concentration, \(\bar{\chi}\), in the absence of sources or sinks, that if the averaged potential-vorticity and averaged ozone distributions are highly correlated initially then they should remain so since they are transported and mixed by the same motions. This is the justification for the argument that the potential-vorticity distribution in Fig. 2a should match closely the corresponding ozone distribution in Fig. 2b. Indeed this is the case for the potential-vorticity and ozone profiles shown in Danielsen et al. (1987, Fig. 7).

The foregoing reasoning regarding the correlation of averaged potential vorticity and ozone is valid only if the eddy-flux term in the governing equation for \(\langle \mathbf{P} \rangle\) may be represented in a similar manner as that in the governing equation for \(\bar{\chi}\) (as assumed in Danielsen et al. 1987, p. 2105). In the Appendix, we cite evidence for and against representing the eddy flux of potential vorticity in terms of downstream mixing, which is standard practice for a passive tracer. If such an approach

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4 This is in essence the theorem of Truesdell (1951), recent applications of which are found in Hoskins et al. (1985, p. 931), Thorpe and Emanuel (1985) and Thorpe and Rotunno (1989).

7 Haynes and McIntyre (1990) raise fundamental objections to the plausibility of downstream mixing of potential vorticity, citing inter alia evidence on the role of gravity-wave drag in producing a dipole pattern in the potential vorticity distribution on isentropic surfaces.
is untenable, then we can no longer pursue the line of thinking based on (8). If it is defensible, then we can reconcile the difference in interpretation between Shapiro and Danielsen et al. as follows:

As discussed in the previous section, Shapiro constructs potential vorticity from the Reynolds-averaged equations of motion, with the resulting form given by the first term on the right-hand side of (3). In this approach, Reynolds stresses and heat fluxes arise respectively in the prognostic equations for Reynolds-averaged momentum and heat [cf. (2) or (6)]. These terms may be viewed properly as frictional and diabatic effects, which may render the potential vorticity non-conservative along a trajectory defined by the mean motion. On the other hand, Danielsen et al. average the potential-vorticity equation (7) to obtain (8), where now $\vec{P}$ is the term on the left-hand side of (3). If potential vorticity mixes downgradient, then (8) is a classical advection–diffusion equation for $\vec{P}$. Under these circumstances, no anomaly should form from a smoothly varying distribution of $\vec{P}$. If the distribution of $\vec{P}$ defined by the term on the left-hand side of (3) is smoothly varying at the same time that the distribution of $\vec{P}$ defined by the first term on the right-hand side of (3) exhibits a positive anomaly, then a compensating negative anomaly is required in the distribution of the second term on the right-hand side of (3). Thus, the difference in interpretation between Shapiro and Danielsen et al. reduces to the difference in their respective definitions of $\vec{P}$, which corresponds precisely to the second term on the right-hand side of (3).

It is apparent that there is a choice to be made in the form of the potential vorticity appropriate to a given application, but in practice one is forced to adopt that used by Shapiro [the first term on the right-hand side of (3)] because measurements are made of velocity and temperature, which, following some type of filtering, are taken to represent Reynolds averages. Although the form of potential vorticity discussed by Danielsen et al. [the term on the left-hand side of (3)] may be appealing for transport studies, it is problematic in the respect that simultaneous direct measurements are not made of the absolute vorticity and the potential-temperature gradient, and in view of the debatable validity of the downgradient mixing of potential vorticity. Despite the emphasis by Danielsen et al. (1980, 1987) on using the Reynolds average of the potential vorticity rather than the potential vorticity constructed from Reynolds averages, we contend that their potential vorticity analyses represent the latter alternative, that adopted by Shapiro.

To bring the underlying mathematical issues into sharper focus, we consider in the Appendix a hypothetical quadratic quantity, $C = AB$, where the equations for $A$ and $B$ contain diffusion terms and where $C$ is conserved in the absence of these terms. This hypothetical problem illustrates the role of averaging in yielding a distinction in the behavior of the quantities $\vec{C}$ and $\vec{AB}$. In particular, it shows that the intrinsic nature of the governing equations demands a compensation between $\vec{AB}$ and $\vec{A}^{\perp}B$ of the type described in relation to (3) for potential vorticity. This hypothetical problem is also used to motivate the differing justifications for and against parameterizing eddy fluxes in terms of downgradient diffusion of mean quantities, alluded to previously in relation to (8).

4. Concluding remarks

Our reconciliation of the difference in interpretation between Shapiro and Danielsen et al. concerning the formation of potential-vorticity anomalies in upper-level jet–front systems may be summarized as follows: Shapiro considers potential vorticity to be the scalar product of the Reynolds-averaged absolute vorticity and potential-temperature gradient, whereas, in their analytical framework, Danielsen et al. consider it to be the Reynolds average of the scalar product of the absolute vorticity and the potential-temperature gradient. In Shapiro’s framework, diabatic and frictional effects associated with turbulent-mixing processes (in which heat and momentum may be mixed downgradient) can introduce structural detail to the potential-vorticity field. The behavior of this form of the potential vorticity is fundamentally different from that of a passive tracer in the presence of turbulent mixing, as elucidated by Haynes and McIntyre (1987, 1990). Thus, we conclude that the positive anomaly of potential vorticity identified by Shapiro is plausible if one accepts the definition of potential vorticity used in his studies. On the other hand, in the framework adopted by Danielsen et al., changes in the potential vorticity following the mean motion are in the same sense as for a passive tracer, provided that molecular effects can be neglected and downgradient diffusion of potential vorticity is assumed. Subject to their restrictions, turbulent mixing cannot produce anomalies from initially smooth distributions of potential vorticity.

It would appear that for dynamical studies of the two alterations discussed here Shapiro’s is the only practical option for dynamical studies because observing analysis systems yield averages of velocity and temperature rather than point values of the gradients of these respective quantities. Similarly, most numerical models of turbulent flows predict averages of velocity and temperature. Even if the alternative of Danielsen et al. could be adopted in practice, the utility of this form of the potential vorticity as a tracer would be rendered problematic by the uncertain validity of the downgradient diffusion of potential vorticity. Further research on the effects of turbulent mixing on potential vorticity is needed in the context of both definitions. The mathematical properties of potential vorticity indicate that the role of turbulent mixing in its evolution may be quite different compared to conventional chemical tracers. As a start in addressing these issues, we propose that tracking the evolution of an Ertel-like quantity, $\vec{\zeta} \cdot \nabla S$, in simulations of fully developed turbulence in a constant-density flow, where $S$ is a passive scalar satisfying a classical advection–diffusion equation, might prove illuminating.
Acknowledgments. The first author appreciates the support of the Mesoscale and Microscale Meteorology Division and the Advanced Study Program of the National Center for Atmospheric Research for his visit during July 1988, when the collaboration leading to this work was initiated. Additional support was provided by the National Science Foundation through Grant ATM-8721478. Discussions and correspondence with Drs. L. F. Bosart, K. A. Emanuel, D. R. Fitzjarrald, P. H. Haynes, I. M. Held, H. D. Lenschow, M. E. McIntyre, M. A. Shapiro, R. B. Smith, A. J. Thorpe, and J. C. Wyngaard were essential to the evolution and refinement of our ideas on this topic.

APPENDIX

A Study of the Conservation Properties of a Hypothetical Conserved Quadratic Quantity

Consider a hypothetical quantity C that happens to be the product of two other quantities, A and B, for which we know the governing equations. Suppose they take the form

\[
\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) A = \delta A^2 + \nu \frac{\partial^2 A}{\partial x^2},
\]

(A1)

\[
\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) B = -\delta AB + \nu \frac{\partial^2 B}{\partial x^2},
\]

(A2)

where \( U \) is independent of \( x \) and \( t \), but may vary among realizations. These equations then yield

\[
\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) C = \nu \left( \frac{\partial^2 C}{\partial x^2} - 2 \frac{\partial A \partial B}{\partial x} \right),
\]

(A3)

[cf. Eq. (6) of Thorpe and Rotunno (1989)]. Clearly, if \( \nu = 0 \), then \( C \) is conserved following the motion \( U \); if, in addition, \( \delta = 0 \), then \( A \) and \( B \) are also conserved. With \( \delta = 1 \), \( C \) may be thought of as an abstraction of the Ertel potential vorticity, with \( A \) corresponding to a generalized velocity gradient (including vector vorticity) and \( B \) to the vector potential-temperature gradient. To illustrate the effect of averaging on the form of the conservation law for \( C \), we first set \( \nu = 0 \). In developing rationales for representing eddy fluxes in terms of mean quantities, we will return to the full form (A1)–(A2).

Performing Reynolds averaging on \( C \) yields the result analogous to (3) in the text:

\[
\bar{C} = \bar{A}\bar{B} + \bar{A}'\bar{B}',
\]

(A4)

and, since \( C' = C - \bar{C} \),

\[
C' = \bar{A}\bar{B}' + \bar{A}'\bar{B} + A'\bar{B}' - \bar{A}'\bar{B}'.
\]

(A5)

Averaging (A3), we find

\[
\left( \frac{\partial}{\partial t} + \bar{U} \frac{\partial}{\partial x} \right) \bar{C} = -\frac{\partial}{\partial x} \bar{U}'C',
\]

(A6)

If \( C \) is thought of as the potential vorticity, then (A3) and (A6) are the analogues of (7) and (8) in the text.

It now remains to form prognostic equations for \( \bar{A}\bar{B} \) and \( \bar{A}'\bar{B}' \). In the case of the former, one suitably combines the Reynolds averages of (A1) and (A2) to obtain

\[
\left( \frac{\partial}{\partial t} + \bar{U} \frac{\partial}{\partial x} \right) \bar{A}\bar{B} = -\left( \bar{B} \frac{\partial}{\partial x} U'A' + \bar{A} \frac{\partial}{\partial x} U'B' \right) + \delta(\bar{B}A'^2 - \bar{A}A'B').
\]

(A7)

Derivation of the corresponding equations for \( \bar{A}'\bar{B}' \) involves developing prognostic equations for \( A' \) and \( B' \) by subtracting the respective equations for \( A \) and \( B \) from (A1) and (A2). Thus,

\[
\left( \frac{\partial}{\partial t} + \bar{U} \frac{\partial}{\partial x} \right) \bar{A}'\bar{B}' = -\left( \bar{B} \frac{\partial}{\partial x} U'A' + \bar{A} \frac{\partial}{\partial x} U'B' \right) - \delta(\bar{B}A'^2 - \bar{A}A'B') - \frac{\partial}{\partial x} (\bar{A}U'B')
\]

\[
+ \bar{B}U'A' + \bar{U}'A' \bar{B}'.
\]

(A8)

Observe that summing (A7) and (A8), along with use of the definition of \( C' \) in (A5), recovers (A6) as required by the decomposition of \( C \) in (A4).

The significance of the mathematical form of (A7) and (A8) is that terms arise of equal magnitude and opposite sign (analogous to 'conversion' terms in energetics diagnostics). These terms derive from the Reynolds-averaging procedure, with a contribution remaining even if the source terms in (A1) and (A2) vanish (\( \delta = 0 \)). These terms allow the formation of compensating anomalies of opposite sign in \( \bar{A}\bar{B} \) and \( \bar{A}'\bar{B}' \) at the same time that \( \bar{C} \) is modified by an eddy-flux divergence, provided that this modification may be modeled in terms of downgradient diffusion of \( \bar{C} \). This supports the plausibility of the formation of anomalies of opposite sign in \( \bar{A}\bar{B} \) and \( \bar{A}'\bar{B}' \) at the same time that \( \bar{C} \) remains smoothly varying, which is precisely the case considered in the text for potential vorticity.

In the interest of completeness, we observe that expansion of the first two members of the flux divergence term on the right side of (A8) and partial cancellation with the first term on the right side of (A8) suggest an alternative form for the contribution to the conversion term arising from advection. The expressions corresponding to (A7) and (A8) are

\[
\left( \frac{\partial}{\partial t} + \bar{U} \frac{\partial}{\partial x} \right) \bar{A}\bar{B} = \left( \bar{U}'A' \frac{\partial \bar{B}}{\partial x} + U'B' \frac{\partial \bar{A}}{\partial x} - \frac{\partial}{\partial x} (\bar{B}U'A' + \bar{A}U'B') \right) + \delta(\bar{B}A'^2 - \bar{A}A'B'),
\]

(A7')

and

\[
\left( \frac{\partial}{\partial t} + \bar{U} \frac{\partial}{\partial x} \right) \bar{A}'\bar{B}' = \left( \bar{U}'A' \frac{\partial \bar{B}}{\partial x} + U'B' \frac{\partial \bar{A}}{\partial x} \right) - \frac{\partial}{\partial x} (\bar{B}A'^2 - \bar{A}A'B'),
\]

(A8')

For clarity of illustration, \( A, B \) and \( C \) are taken to be scalars, but it is understood that the following development can be generalized to pertain to vector and tensor quantities.
where now the conversion term due to advection assumes the conventional form of generic Reynolds stresses multiplying gradients of mean quantities.

A fundamental issue arising in the discussion of (8) concerns justifying the representation of eddy fluxes. There appear to be two schools of thought, to be referred to as the viscous and the inviscid schools, respectively. The viscous school observes that although the direct effect of viscosity is negligible on the mean flow, it acts inexorably to destroy the variance. This fact has been used to justify modeling the eddy flux of a quantity as down its mean gradient (Gill 1982, pp. 83–84). Since the viscous terms in (A1) and (A2) are of diffusive form, whereas their counterpart in (A3) is not, one can only justify parameterizing the eddy fluxes of $A$ and $B$ in terms of downgradient transport. On the other hand, the basic tenet of the inviscid (or mixing-length) school is that fluid properties \ldots adhere to each particle of the fluid during its motion” (Taylor 1921; also refer to Tennekes and Lumley 1972, pp. 51–52). In this framework, one clearly would be on firmest ground in modeling the eddy flux of $C$ as downgradient since it is conserved in the inviscid limit. We feel that reconciliation of these apparently conflicting views is in order.

REFERENCES


