

## A Multigrid Solver for the Helmholtz Equation on a Semiregular Grid on the Sphere

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### ABSTRACT

A multigrid finite-difference solver is developed for the Helmholtz equation on the sphere. The finite-difference grid resolution is constant in the latitudinal direction and variable in the longitudinal direction so as to keep the physical gridpoint spacing approximately uniform over the sphere. The cpu time per grid point required to reduce the residual by a given amount is independent of grid resolution. The discretization error is slightly worse than second order as a result of the variable grid spacing. The method should be applicable to general elliptic equations on the sphere and should be useful for problems where uniform grid spacing is disadvantageous.

### 1. Introduction

Geophysical problems often require the solution of elliptic equations in spherical coordinates, such as the Helmholtz equation,

$$\nabla^2 \psi - \lambda \psi = f(\phi, \theta) \quad (1)$$

where  $\phi$  is longitude and  $\theta$  is latitude. Many different numerical methods have been devised for solving (1) approximately, including spectral, finite-difference, and finite-element techniques. All numerical methods convert the continuous problem (1) into a discrete set of  $N$  algebraic equations in  $N$  unknowns. Two critical measures of the efficiency of a numerical method are the number of computations required per unknown and the amount of memory needed to carry out the computations.

Methods for solving the discrete problem can be broadly divided into direct and iterative solvers. Spectral methods are usually direct solvers, while finite-difference and finite-element methods can be either direct or iterative. Direct solvers tend to be fast, requiring  $O[N \log(N)]$  operations, but typically needing  $O(N^{3/2})$  memory. Classical iterative solvers, such as Gauss-Seidel iteration or successive overrelaxation (SOR) are compact, requiring only  $O(N)$  storage, but are usually slow, needing  $O(N^{3/2})$  or  $O(N^2)$  operations to converge to a solution.

Multigrid solvers are a new class of iterative solvers for elliptic problems introduced by Brandt (1977). They have proven to be optimally efficient in terms of both computation and memory, requiring  $O(N)$  operations and  $O(N)$  storage, for a wide variety of prob-

lems, including problems for which direct solvers are inefficient. The advantages of multigrid solvers hold even for small problems, not merely asymptotically. As a result, multigrid methods are as fast or faster than comparable spectral or finite-difference direct solvers for a given  $N$  and require much less memory for large problems.

Multigrid solvers that can be applied to elliptic problems in spherical coordinates have been developed recently. The MUDPACK package developed by Adams (1989) at NCAR is a general-purpose elliptic solver that can be applied on a regular latitude-longitude grid on the sphere. Because the latitude-dependent coefficients in the Laplacian can slow the convergence of the iteration at different latitudes, Barros (1990) optimized the convergence of the multigrid iteration on the sphere by dividing the grid into subdomains (high and low latitudes) and using the optimum relaxation method in each domain. Both these methods require regularly spaced latitude-longitude grids, which can cause difficulties near the pole in some cases as a result of the convergence of the meridians.

An alternative to using different relaxation methods in different latitude zones is to vary the grid point spacing so as to keep the coefficients of the relaxation operator approximately constant. This paper presents a multigrid finite-difference method for solving the Helmholtz equation in spherical coordinates that uses a semiregular grid. In the method presented here, a simple pointwise relaxation method is used everywhere, but the grid resolution is adapted locally to the Laplacian operator. The method was developed for use in a two-dimensional, time-dependent energy balance climate model, where an elliptic problem had to be solved at each time step. The method may have uses in hyperbolic problems with elliptic components, where avoiding the convergence of the grid points near the

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poles may reduce the CFL time-stepping constraints or the need for filtering near the poles. It may also be useful for such data analysis problems as the blending of observations from different sources (Oort and Rasmusson 1971; Reynolds 1988), which require the solution of Poisson-type problems with data on equal-area rather than regular grids.

## 2. Methods

### a. Background

The multigrid technique was introduced by Brandt (1977) following earlier work in the Russian literature (see Hackbusch 1985; Stuben and Trottenberg 1982 for reviews of the history and theory of the method). Briggs (1987) has provided an introductory tutorial. A review of multigrid methods and applications to meteorology can be found in Fulton et al. (1986). They list a number of available multigrid software packages.

Relaxation methods can be viewed as smoothers that iteratively reduce the errors in the approximate solution to (1). Relaxation with the standard 5-point finite-difference star, which is frequently used to solve elliptic problems (Fig. 1a), efficiently removes the smallest-scale components of the error, reducing them by a factor of 0.25 in each relaxation sweep on a uniform isotropic grid. The slow global convergence of classic iterative methods is a result of the slow damping of the

large-scale components of the error. Multigrid methods improve the convergence of the iteration by smoothing the error components of different scales on a hierarchy of different resolution grids. Large-scale errors can be effectively damped on coarse grids while small-scale errors are damped on fine grids. With proper communication between the grids, the multigrid method efficiently dampens errors at all scales from that of the entire domain to that of the finest grid spacing.

In outline, the multigrid method works as follows. Given an elliptic problem such as (1), we seek the solution to the equation

$$Lu = f \quad (2)$$

where  $L$  is the discrete linear operator resulting from applying a finite-difference approximation to (1);  $u$  is the exact solution to the discrete problem; and  $f$  is an arbitrary forcing. Given an approximation to  $u$  designated  $v$ , the defect  $d$  is defined by

$$u = v + d. \quad (3)$$

The residual  $r$ , defined by

$$r = f - Lv, \quad (4)$$

measures the degree to which  $v$  fails to satisfy the linear operator locally. Substituting for  $v$  in (4) from (3) and using the linearity of  $L$  yields

$$Ld = r. \quad (5)$$

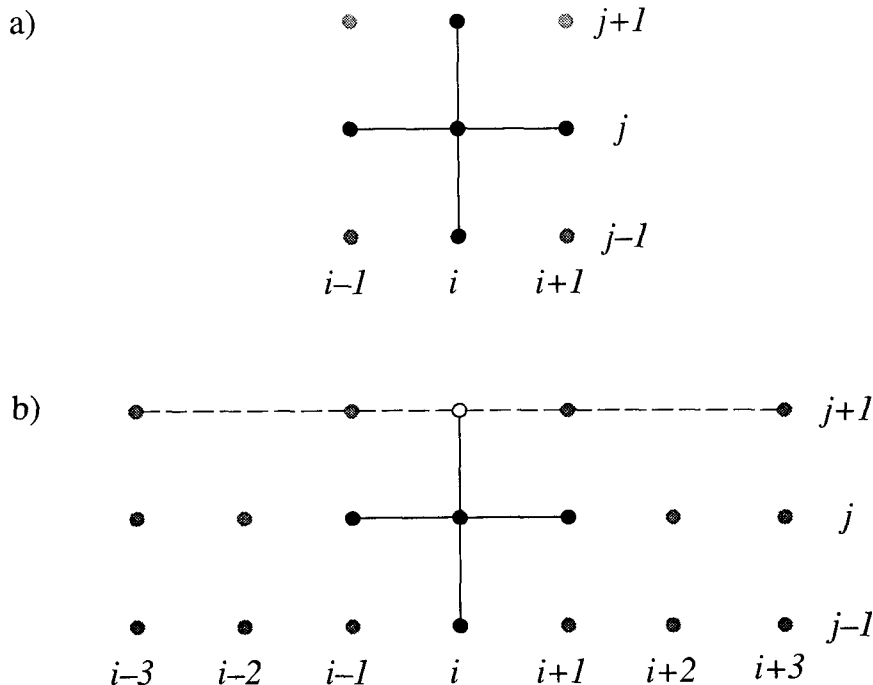


FIG. 1. Standard 5-point finite-difference approximation used for regular portions of the grid (a), and the finite-difference operator used at internal grid boundaries (b). The dashed line connects the points used to interpolate the value at the open circle.

The defect  $\mathbf{d}$  thus satisfies the same equation as the solution  $\mathbf{u}$ , but with the residual  $\mathbf{r}$  as the forcing. Solving for  $\mathbf{d}$  by using (5) would allow  $\mathbf{u}$  to be calculated from (3); unfortunately, (5) is as difficult to solve as (1), so this formal manipulation does not appear to have gained anything.

The multigrid method, however, takes advantage of the above relationships in the following way. Beginning with an initial guess  $\mathbf{v}$ , a number of iterations are made with a relaxation method. After a few relaxation sweeps the small-scale error is greatly reduced, while the large-scale error is only slightly damped. The residual  $\mathbf{r}$  is calculated by using (4). Because the relaxation has efficiently damped the small-scale errors, (5) can be solved on a coarser grid than (2), where the solution can be found more rapidly, and the defect  $\mathbf{d}$  can then be added to  $\mathbf{v}$  (interpolating where necessary) to give an improved approximation to  $\mathbf{u}$ . Because the defect  $\mathbf{d}$  has not been computed using the full resolution of the fine grid, the corrected solution will not be exact; therefore, the process must be repeated until the residual is sufficiently small. The net result of this two-grid iteration is to accelerate the convergence by efficiently damping the errors of different scales on different grids. The method can be recursively applied to a series of grid levels until a very coarse grid is reached, on which the problem can be solved very rapidly either directly or by relaxation. A multigrid solver therefore consists of four parts: a relaxation method, which should effectively smooth small-scale errors on each grid; a restriction method for transferring the residuals from the fine grid to the coarse grid; an interpolation method (also called prolongation) for adding the coarse-grid correction to the fine-grid solution; and a cycling strategy to organize the relaxation sweeps and intergrid transfers.

### b. Multigrid solver

A finite-difference multigrid solver was developed to solve (1) approximately in two-dimensional spherical coordinates. The components of the solver are described below.

#### 1) FINITE-DIFFERENCE GRIDS

The starting point for the semiregular grid used here is a regular latitude-longitude grid. At each grid level, grid points are located at the points  $(\phi_i, \theta_j)$ , where  $\phi_i = i\delta\phi - \pi$ ;  $i = 0, \dots, m_l$ ; and  $\theta_j = j\delta\theta - \pi/2$ ,  $j = 0, \dots, n_l$ . The grid resolution is  $(m_l + 1$  by  $n_l + 1)$ , where  $m_l = m_o 2^l$  and  $n_l = n_o 2^l$ . The gridpoint spacing is  $\delta\phi = 2\pi/m_l$  by  $\delta\theta = \pi/n_l$ . The coarsest grid ( $l = 0$ ) has dimensions  $m_o + 1$  by  $n_o + 1$ , which for efficiency are chosen to be as small as possible. In all calculations presented here,  $m_o = 4$  and  $n_o = 2$ . Therefore, the longitudinal and latitudinal gridpoint spacing are equal. Grid points are located at both poles, and for programming convenience, a duplicate column of grid points

exists at  $\phi = \pm\pi$ . The finer grids are created by increasing  $l$  until sufficient resolution is achieved. Increasing the grid resolution by a factor of two at each grid level ensures that the coarse grid points coincide directly with fine grid points.

The regular finite-difference grid described above is modified to provide approximately equal gridpoint spacing in physical space while remaining as regular in latitude and longitude as possible. The method chosen is to retain the uniform gridpoint spacing in the latitudinal direction, while varying the spacing in longitude. On a unit sphere the physical distance  $\delta x$  between two grid points separated in longitude by  $\delta\phi$  is  $\cos\theta \delta\phi$ . Moving poleward from the equator, in each latitude row the number of grid points around the latitude circle is decreased by a factor of two whenever  $\delta x$  becomes less than  $\delta\phi/2$ . In the numerical code, this is carried out by assigning a gridpoint skipping factor to each latitude row. The resulting grid is regular between  $+60^\circ$  latitude (in this band  $\cos\phi \geq 0.5$ ), and becomes progressively coarser in longitude poleward of  $60^\circ$ . The resulting grid is shown in both polar and cylindrical projections in Fig. 2.

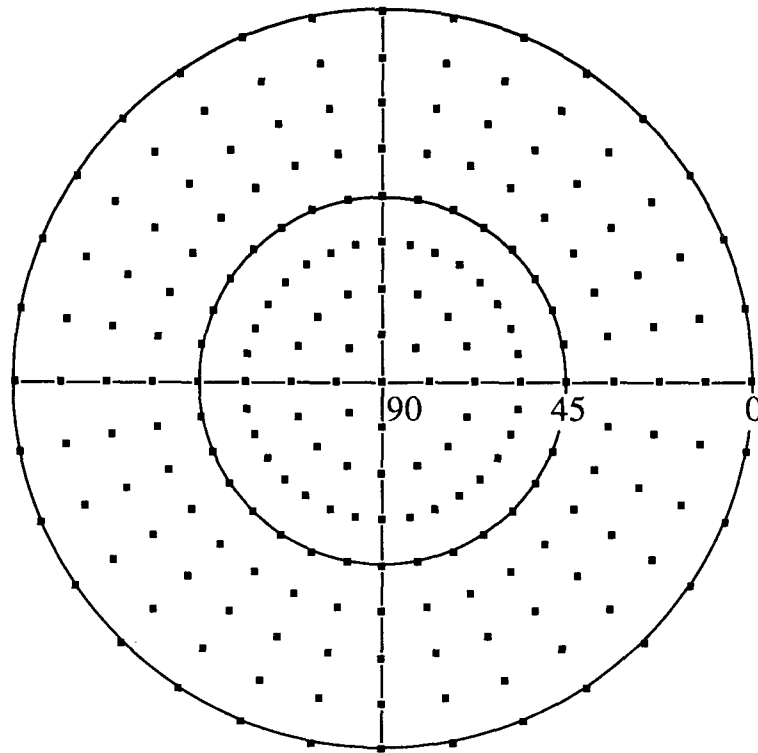
#### 2) RELAXATION

The relaxation operator used on the regular portions of the grid is the simple 5-point star (Fig. 1a). The finite-difference approximation is second order in the gridpoint spacing, i.e.,  $O(\delta\phi^2, \delta\theta^2)$ . At internal boundaries of the grid, the 5-point star cannot be used directly where  $v_{i,j+1}$  is skipped, as in Fig. 1b. In this case, the missing point in the row  $j + 1$  is interpolated using cubic polynomial interpolation in longitude. That is, the skipped point indicated by the open circle in Fig. 1b is interpolated by using  $v_{i-3,j+1}$ ,  $v_{i-1,j+1}$ ,  $v_{i+1,j+1}$ , and  $v_{i+3,j+1}$ . (Linear interpolation in longitude was found to reduce the accuracy of the overall solution too much to be acceptable.) Only a small number of points must be interpolated in this manner.

Because the differential operator is singular at the poles, the 5-point difference star cannot be used there, and an alternative method must be found. Relaxation equations for the values at the poles are obtained by integrating (1) over polar caps of radius  $\delta\theta/2$  and applying the divergence theorem. By using centered finite differences in latitude between the pole point and the surrounding row of points, the integral of the flux across the boundary of the polar cap can be equated to the integral of the forcing over the cap. There are always at least four points in the surrounding row, so the resulting approximation is very similar to the standard 5-point difference star. As Barros (1990) has indicated the finite-difference approximation is second order.

The relaxation method is Gauss-Seidel (immediate replacement of updated values) with red/black ordering of the points. Because of the internal boundaries, a perfect division into red and black points is not possible, but the deviations do not appear to be important.

(a)



(b)

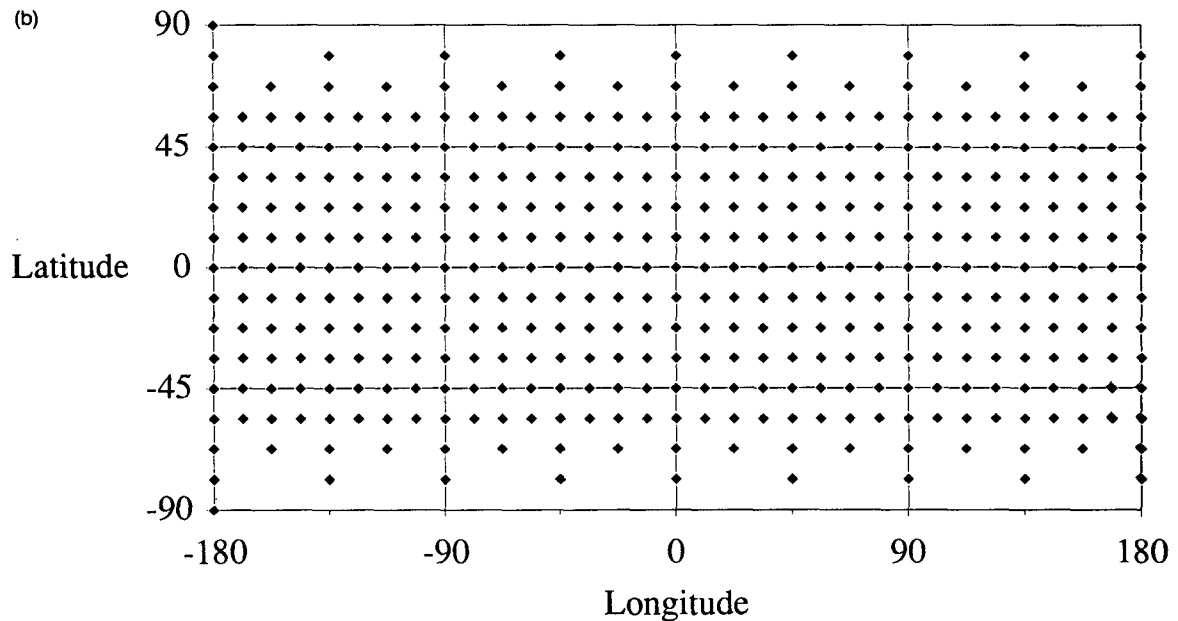


FIG. 2. Typical semiregular grid used shown in polar and cylindrical equidistant projections.

### 3) RESTRICTION OF RESIDUALS

Restriction of the residuals to the coarse grid is carried out with a 9-point weighted operator. At a given coarse grid point, the weighted residual from the nine surrounding fine grid points is computed using the area

of the rectangular cell surrounding each grid point as the weight [proportional to  $(\cos\theta)\delta\phi$ ]. The cell boundaries are halfway between a given grid point and the neighboring grid points in each direction. Because perfect red/black ordering cannot be obtained, and because the interpolation at the internal boundaries is

repeated after the relaxation, the black residuals are not necessarily zero. Therefore, all nine residuals are used in the injection.

4) INTERPOLATION OF THE COARSE-GRID CORRECTION

Interpolation of the defect (correction) to the fine grid is done with bilinear interpolation, interpolating first in longitude and then in latitude. At grid points where the coarse and fine grids coincide this reduces to simple addition.

5) MULTIGRID CYCLING STRATEGY

The final component of the multigrid solver is the iteration strategy; that is, the order in which relaxation sweeps, restriction, and interpolation are made. A great variety of fixed and adaptive schemes are available (Brandt 1977; Hackbusch 1985). After some experimentation, the method of fixed  $V$  cycles was found to be generally fastest in terms of overall time. In the descending part of a  $V$  cycle, relaxation on each grid is immediately followed by transfer of the residuals to the next coarser grid. When the relaxation on the coarsest grid is completed, the correction is added to the next finer grid and a single relaxation sweep is carried out. This process is repeated until the finest grid is reached. The full multigrid method is used, beginning with an initial guess of  $\mathbf{v} = 0$  on the coarsest grid. A series of increasing  $V$  cycles is used to construct the initial guess on the finest grid, whereupon full  $V$  cycles are repeated until convergence is achieved.

The iteration strategy can easily be changed to accommodate different numbers of relaxation sweeps or other variations.

3. Results

a. Convergence of the multigrid iteration

The multigrid code is tested by solving problems for which the analytical solution is known. The eigenfunctions of the Laplacian in spherical coordinates are spherical harmonics  $Y_{l,m}$ , so if the forcing is chosen to be  $[-l(l+1) + \lambda]Y_{l,m}$ , then the solution to (1) is simply  $Y_{l,m}$ . In the examples below, the forcing is chosen to be  $Y_{2,1}$ , which has large spatial scales that single grid Gauss-Seidel relaxation will typically solve inefficiently. Convergence is measured by computing the global rms residual, while the discretization error  $\mathbf{e}$  can be determined from the difference between the analytical and numerical solutions

$$\mathbf{e} = \psi - \mathbf{v}. \tag{6}$$

The convergence of the multigrid iteration is illustrated in Fig. 3 for a finest grid with dimension  $128 \times 64$ . At this resolution there are six grid levels. Work is measured in units of relaxation sweeps on the fine grid, with injection and interpolation counted as one-fourth of a relaxation sweep on the finer of the two grids. The work required to relax a coarse grid is one-fourth the work required on the next finer grid. Total work for a single  $V$  cycle is  $\sim 3$  work units. The rms residual is reduced to less than  $10^{-7}$  in about 30 work units ( $< 20 V$  cycles). The error is reduced to the discretization error in  $\sim 5 V$  cycles ( $\sim 10$  work units).

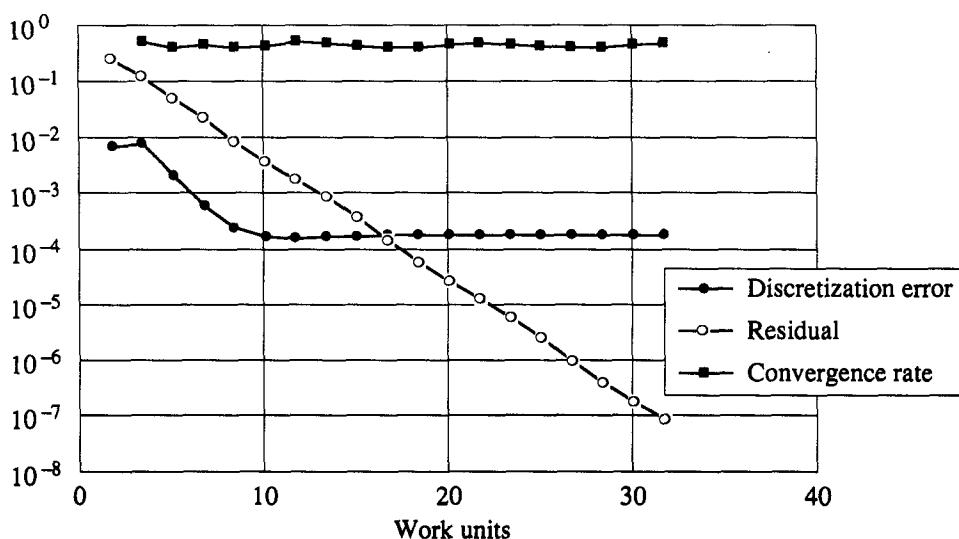


FIG. 3. Convergence of the multigrid iteration. Values are plotted at the end of each  $V$  cycle. The error is reduced to the discretization error within  $\sim 5$  cycles. The residual is reduced to  $10^{-7}$  within  $\sim 30$  work units, while the convergence rate is nearly constant at  $\sim 0.44$ .

The convergence rate is  $\sim 0.44$  per  $V$  cycle. Because of the remaining anisotropy of the grid, the convergence rate is worse than that achieved on a uniform, isotropic grid on a plane (0.25). A Fourier analysis of the region between  $60^\circ\text{S}$  and  $60^\circ\text{N}$ , where the grid is uniform, yields a smoothing factor of 0.64 (S. Barros, personal communication). With two relaxation sweeps per cycle, the predicted convergence rate is approximately  $(0.64)^2 = 0.41$ , indicating that the other components of the relaxation scheme are probably chosen correctly.

#### b. Discretization error and efficiency

A series of numerical experiments with varying grid resolution were carried out to determine the discretization error and efficiency of the multigrid algorithm. Results are shown in Fig. 4. Halving the gridpoint spacing reduces the rms error by  $\sim 0.28$ , rather than the ideal 0.25 expected from a second-order method. This is probably a result of the internal boundary treatment, since the local errors are somewhat larger near the internal boundaries than in the equatorial zone where the gridpoint spacing is uniform.

The increase in computation time resulting from a quadrupling of the number of grid points is 4.0, averaged over each of the quadruplings shown in Fig. 4. Computer times given are for calculations carried out in double precision on the Hewlett-Packard 835 computer in the Department of Atmospheric Sciences. There are slight fluctuations depending on variations in the convergence and imprecision in the timing routines. The work required to reduce the residuals to  $10^{-7}$

asymptotes to  $\sim 30$  relaxation sweeps (corresponding to  $< 20 V$  cycles), regardless of the grid resolution.

#### 4. Conclusions and recommendations

This paper has shown that the multigrid method can be successfully applied to Helmholtz-type problems on a semiregular grid in spherical coordinates by dividing the sphere into zonal subdomains with approximately uniform resolution in each subdomain. The cpu time per grid point required to reduce the residuals by a given amount is independent of grid resolution. Memory requirements of the method are  $O(N)$ , as is typical of multigrid methods.

If regular grids are suitable for the problem at hand, the methods developed by Barros (1990) and Adams (1989) are more efficient than the solver presented here, with convergence rates as low as 0.1 per  $V$  cycle. The MUDPACK routines are vectorized for use on the Cray and source code is available from NCAR. The scheme presented here for semiregular grids should be useful when regular grids cannot be used and should be adaptable to more general grids.

The development of the multigrid method makes finite-difference techniques quite competitive with or even superior to spectral methods in spherical coordinates (Barros et al. 1990). For problems where the forcing is local, and more easily represented on a grid than in terms of spectral basis functions, the multigrid method can have significant advantages in computational time and storage. The algorithms are not terribly complicated, involving only relaxation, averaging of

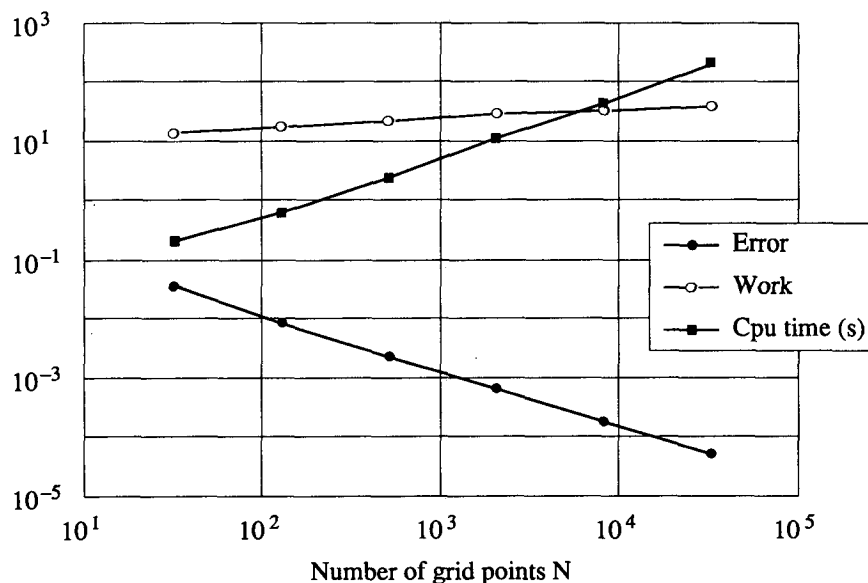


FIG. 4. The discretization error of the numerical solution, the cpu time in seconds, and the work units required to reduce the residual to  $10^{-7}$  as a function of the number of grid points  $N$ .

the residuals for transfer to the coarse grid, and interpolation of the correction terms back to the finer grid. With packages such as MUDPACK, these details are almost entirely hidden from the user, so efficient compact elliptic solvers can be developed quickly in spherical coordinates.

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