Nonlinear Normal Mode Initialization of a Limited-Area Model: Inclusion of All Beta Terms in the Linearized Model Equations

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ABSTRACT

A nonlinear normal mode initialization method with all of the beta terms included in the linearized model equations is formulated for a limited-area model. It is the extension of an earlier method examining the sensitivity of nonlinear normal mode initialization to the inclusion of nonstationary Rossby modes. It is shown that the eigenmodes of the two methods for sufficiently large equivalent depths do coincide approximately. Results of the two methods are compared. They show the equivalence between the *f*-plane approach and the inclusion of all or some of the beta terms for midlatitude initialization.

1. Introduction

In large-scale atmospheric models (global or hemispheric) the distinction between gravity and Rossby modes is unambiguous. In limited-area models this is not so. The distinction depends on the linearized model equations used. The nonlinear normal mode initialization method of Machenhauer (1977) and Baer (1977), which requires the construction of the normal modes of the linearized model equations, was first applied to a limited-area model by Brière (1982). However, no beta terms were included in the linear part and the Coriolis parameter was given a constant value. Under these conditions the Rossby modes are stationary. Transformation of the method from normal mode space to physical space and changing to a variable Coriolis parameter afterwards gives the method of Bourke and McGregor (1983), closely related to the bounded derivative method applied by Browning et al. (1980). The relationship between these three methods is discussed in Bijlsma and Hafkenscheid (1986). The inclusion of the beta terms in the linear system might be important in determining the structure of the normal modes, especially for small equivalent depths in the tropics (Ballish 1979). Recently Temperton (1988) and Juvanon du Vachat (1988) generalized the method of Bourke and McGregor (1983) in physical space by including most of the beta terms. Their methods are very similar and can be considered as applications of the nonlinear normal mode method without knowing the normal modes. However these methods cannot be derived in the case of nonstationary Rossby modes.

In a previous paper Bijlsma (1989) examined the sensitivity of nonlinear normal mode initialization to the inclusion of nonstationary Rossby modes. This paper presents a nonlinear normal mode method with *all* beta terms included, generalizing the method of Brière (1982) in normal mode space. This is of course at the expense of the computational simplicity of the latter method. However, the computational effort may be well comparable with that of the implicit methods of Temperton and Juvanon du Vachat, if the normal modes are approximated in a natural way, which still reflects the influence of the beta terms (see section 3).

An outline of the method is given in section 2. Section 3 compares the method with the approximate method which is described in Bijlsma (1989). Conclusions are presented in section 4.

2. Modifications of the gravity and Rossby modes by the inclusion of all linear terms

In a limited-area model gravity and Rossby modes are defined in different ways depending on the linearization of the model equations. This paper considers the shallow-water equations in differentiated form on the sphere, with all linear terms included in the linear part. To be more specific, if λ and θ are longitude and latitude then the equations on a discrete grid $\lambda_m = \lambda_0$

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$+ m\Delta\lambda$, $\theta_n = \theta_0 + n\Delta\theta$ with $M \times N$ interior grid points are given by

$$\frac{\partial}{\partial t}\nabla_d^2\chi - 2\Omega\sin\theta_n\nabla_d^2\psi + \frac{2\Omega}{r^2}\left[\left(\frac{\partial}{\partial\lambda}\right)_d\chi - \cos\theta_n\left(\frac{\partial}{\partial\theta}\right)_d\psi\right] + \nabla_d^2\phi = Q_\chi, \quad (1)$$

$$\frac{\partial}{\partial t}\nabla_d^2\psi + 2\Omega\sin\theta_n\nabla_d^2\chi + \frac{2\Omega}{r^2}\left[\left(\frac{\partial}{\partial\lambda}\right)_d\psi + \cos\theta_n\left(\frac{\partial}{\partial\theta}\right)_d\chi\right] = Q_\psi, \quad (2)$$

$$\frac{\partial}{\partial t}\phi + d\nabla_d^2\chi = Q_\phi \quad (3)$$

where χ, ψ, ϕ, and d are the velocity potential, streamfunction, geopotential, and mean-free geopotential height, and r and Ω the radius and angular velocity of the earth. Further ∇_d^2 is the usual five-point discrete Laplacian operator in spherical coordinates, and $(\partial/\partial\lambda)_d$ and $(\partial/\partial\theta)_d$ are the centered difference operators in the λ and θ directions. Finally Q_χ, Q_ψ, and Q_ϕ are the nonlinear terms. Note that the eigenfunctions of the linear spatial operator on the left-hand side of Eqs. (1)–(3) are complex trigonometric functions in the east–west direction, if periodic boundary conditions are imposed. This property of separability is an essential condition for the application of normal mode initialization. Other boundary conditions such as homogeneous Dirichlet or Neumann boundary conditions are not compatible with the existence of eigenfunctions in the east–west (E–W) direction. Therefore the solution is written in the form

$$\chi = \chi_0 + \hat{\chi}, \quad \psi = \psi_0 + \hat{\psi}, \quad \phi = \phi_0 + \hat{\phi},$$

where the fields $\hat{\chi}$, $\hat{\psi}$, and $\hat{\phi}$ at the initial time, have periodic boundary conditions in the E–W directions and zero boundary conditions in the north–south (N–S) direction, and where the fields χ_0, ψ_0, and ϕ_0 satisfy the discrete Laplace equations

$$\nabla_d^2\chi_0 = 0, \quad \nabla_d^2\psi_0 = 0, \quad \nabla_d^2\phi_0 = 0$$

and are equal to the initial values of $\chi - \hat{\chi}$, $\psi - \hat{\psi}$, and

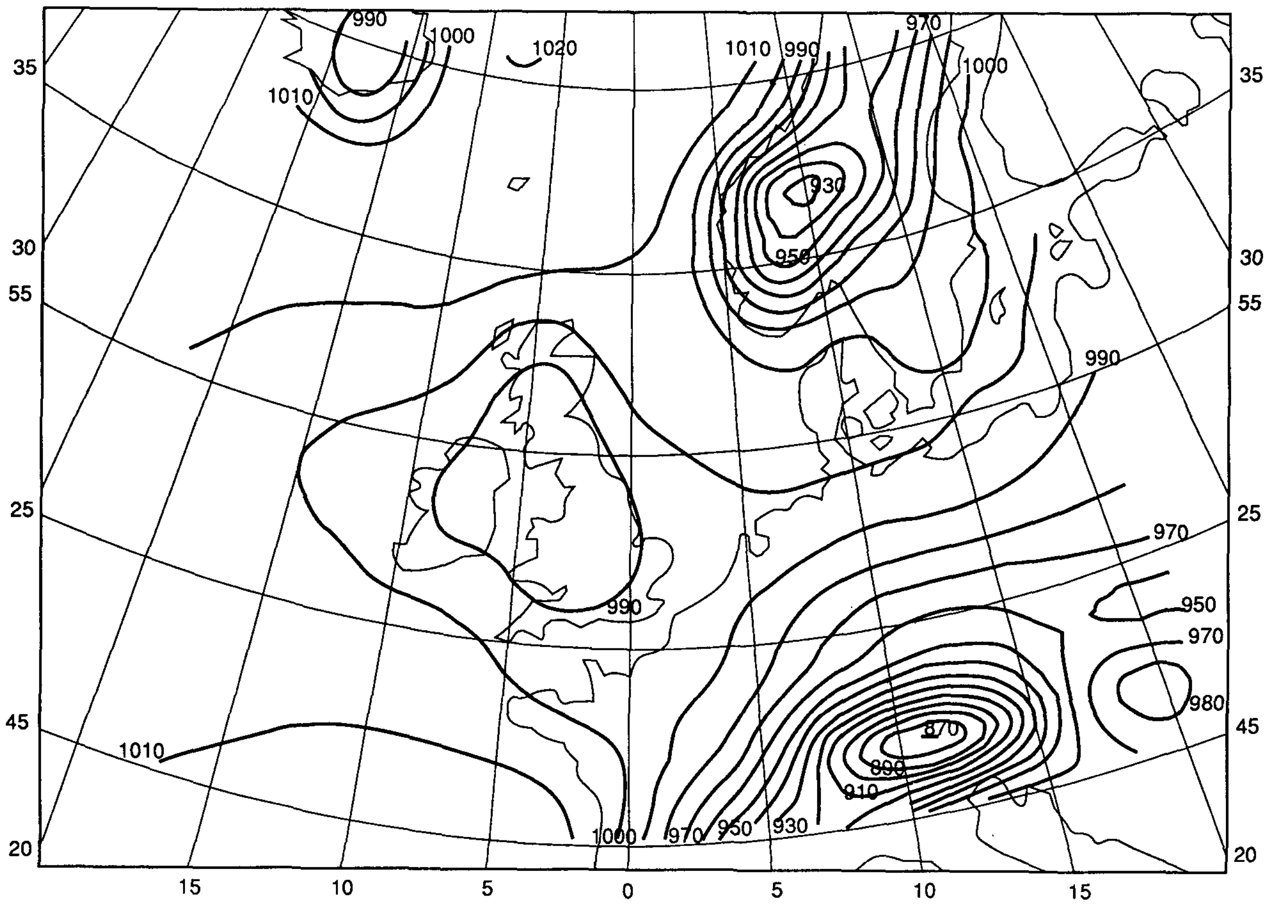

FIG. 1. Uninitialized surface pressure at 1200 UTC 5 January 1982. Units are millibars.

TABLE 1. Frequencies of the gravity waves of the method of section 2 and the approximate method for the external mode. For further explanation, see section 3.

k	σ_{kN+1} ($\times 10^{-3}$)	σ_{k2N+1} ($\times 10^{-3}$)	$\sigma_{k12}(\bar{f})$ ($\times 10^{-3}$)	$\sigma_{k13}(\bar{f})$ ($\times 10^{-3}$)	$\sigma_{k12}(\bar{f}_{kl})$ ($\times 10^{-3}$)	$\sigma_{k13}(\bar{f}_{kl})$ ($\times 10^{-3}$)
0	−0.42185	0.42185	−0.42233	0.42233	−0.42180	0.42180
1	−0.80627	0.80196	−0.80664	0.80233	−0.80625	0.80195
2	−1.3745	1.3717	−1.3750	1.3722	−1.3745	1.3717
3	−1.9191	1.9172	−1.9196	1.9176	−1.9191	1.9172
4	−2.4250	2.4235	−2.4254	2.4239	−2.4250	2.4235
5	−2.8838	2.8826	−2.8841	2.8830	−2.8838	2.8826
6	−3.2874	3.2866	−3.2878	3.2869	−3.2874	3.2866
7	−3.6286	3.6279	−3.6289	3.6283	−3.6286	3.6279
8	−3.9009	3.9004	−3.9012	3.9008	−3.9009	3.9004
9	−4.0992	4.0989	−4.0995	4.0992	−4.0992	4.0989
10	−4.2197	4.2195	−4.2200	4.2198	−4.2197	4.2195
11	−4.2600	4.2600	−4.2603	4.2603	−4.2600	4.2600

$\phi - \hat{\phi}$ on the boundary. In fact these periodic boundary values of $\hat{\chi}$, $\hat{\psi}$, and $\hat{\phi}$ are only known after completion of the initialization by performing a nonlinear iterative process (Bijlsma 1989; p. 2013). But at this moment it is sufficient to consider first approximations of χ_0, ψ_0, and ϕ_0 that satisfy the discrete Laplace equations and are equal to the initial values of χ, ψ, and ϕ on the boundary. This approach was first used by Brière (1982) for an f-plane approximation. In that case the fields $\hat{\chi}$, $\hat{\psi}$, and $\hat{\phi}$ are zero on the boundary, so that the fields χ_0, ψ_0, and ϕ_0 remain unchanged during the initialization process. The time tendencies of χ, ψ, and ϕ at the boundaries are set equal to zero.

Substitution in Eqs. (1)–(3) gives the set of equations

$$\frac{\partial}{\partial t}\nabla_d^2\hat{\chi} - 2\Omega\sin\theta_n\nabla_d^2\hat{\psi} + \frac{2\Omega}{r^2}\left[\left(\frac{\partial}{\partial\lambda}\right)_d\hat{\chi} - \cos\theta_n\left(\frac{\partial}{\partial\theta}\right)_d\hat{\psi}\right] + \nabla_d^2\hat{\phi} = Q_\chi - \frac{2\Omega}{r^2}\left[\left(\frac{\partial}{\partial\lambda}\right)_d\chi_0 - \cos\theta_n\left(\frac{\partial}{\partial\theta}\right)_d\psi_0\right], \quad (4)$$

$$\frac{\partial}{\partial t}\nabla_d^2\hat{\psi} + 2\Omega\sin\theta_n\nabla_d^2\hat{\chi} + \frac{2\Omega}{r^2}\left[\left(\frac{\partial}{\partial\lambda}\right)_d\hat{\psi} + \cos\theta_n\left(\frac{\partial}{\partial\theta}\right)_d\hat{\chi}\right] = Q_\psi - \frac{2\Omega}{r^2}\left[\left(\frac{\partial}{\partial\lambda}\right)_d\psi_0 + \cos\theta_n\left(\frac{\partial}{\partial\theta}\right)_d\chi_0\right], \quad (5)$$

$$\frac{\partial}{\partial t}\hat{\phi} + d\nabla_d^2\hat{\chi} = Q_\phi. \quad (6)$$

In order to find the eigenmodes of the Laplacian operator with periodic boundary conditions in the E–W and zero boundary conditions in the N–S direction, we substitute an expression of the form

$$S_{kl}(m, n) = f_{kl}(n)\exp[2\pi ikm/(M+1)],$$
$$k = 0, \cdots, M; \quad l = 1, \cdots, N$$

for each value of k leading to an eigenvalue problem for the real functions $f_{kl}(n)$, with $f_{kl}(0) = f_{kl}(N+1) = 0$ and eigenvalues $-\alpha_{kl}^2$, so that

$$\nabla_d^2 S_{kl}(m, n) = -\alpha_{kl}^2 S_{kl}(m, n).$$

TABLE 2. As in Table 1, but for the first internal mode.

k	σ_{kN+1} ($\times 10^{-3}$)	σ_{k2N+1} ($\times 10^{-3}$)	$\sigma_{k12}(\bar{f})$ ($\times 10^{-3}$)	$\sigma_{k13}(\bar{f})$ ($\times 10^{-3}$)	$\sigma_{k12}(\bar{f}_{kl})$ ($\times 10^{-3}$)	$\sigma_{k13}(\bar{f}_{kl})$ ($\times 10^{-3}$)
0	−0.19038	0.19038	−0.19154	0.19154	−0.19036	0.19036
1	−0.31819	0.31339	−0.31916	0.31434	−0.31818	0.31337
2	−0.51843	0.51556	−0.51968	0.51680	−0.51842	0.51555
3	−0.71541	0.71341	−0.71663	0.71463	−0.71540	0.71340
4	−0.89987	0.89838	−0.90100	0.89951	−0.89987	0.89838
5	−1.6078	1.0666	−1.0688	1.0677	−1.0678	1.0666
6	−1.2157	1.2149	−1.2167	1.2158	−1.2157	1.2149
7	−1.3409	1.3403	−1.3418	1.3412	−1.3409	1.3403
8	−1.4409	1.4404	−1.4418	1.4413	−1.4409	1.4404
9	−1.5137	1.5134	−1.5146	1.5143	−1.5137	1.5134
10	−1.5580	1.5578	−1.5588	1.5586	−1.5580	1.5578
11	−1.5728	1.5728	−1.5736	1.5736	−1.5728	1.5728

TABLE 3. Projections of the diagonal part of the eigenmodes of (8), (9), and (10) on the eigenmodes of the approximate method (16) for $l = 1$ and $k = 1$ and $k = (M - 1)/2$, respectively, for the external mode. For further explanation, see section 3.

	$\mathbf{P}_{kl1}(\bar{f})$	$\mathbf{P}_{kl2}(\bar{f})$	$\mathbf{P}_{kl3}(\bar{f})$	$\mathbf{P}_{kl1}(\bar{f}_{kl})$	$\mathbf{P}_{kl2}(\bar{f}_{kl})$	$\mathbf{P}_{kl3}(\bar{f}_{kl})$
$\mathbf{P}_{kll}$	9.9998×10^{-1}	2.2730×10^{-3}	2.2565×10^{-3}	9.9998×10^{-1}	1.7077×10^{-5}	1.6898×10^{-5}
	1.0000	1.8748×10^{-3}	1.8747×10^{-3}	1.0000	9.3832×10^{-8}	9.5118×10^{-8}
$\mathbf{P}_{klN+l}$	-2.2679×10^{-3}	9.9998×10^{-1}	-1.4192×10^{-5}	-1.2012×10^{-5}	9.9998×10^{-1}	1.2248×10^{-5}
	-1.8748×10^{-3}	1.0000	-1.8439×10^{-6}	-7.9289×10^{-8}	1.0000	-8.7273×10^{-8}
$\mathbf{P}_{kl2N+l}$	-2.2513×10^{-3}	-1.5410×10^{-5}	9.9998×10^{-1}	-1.1707×10^{-5}	-1.2248×10^{-5}	9.9998×10^{-1}
	-1.8747×10^{-3}	-1.8470×10^{-6}	1.0000	-7.8848×10^{-8}	-8.9102×10^{-8}	1.0000

In addition it is assumed that the functions $f_{kl}(n)$ are normalized, i.e.,

$$\frac{1}{M+1}\sum_{m=0}^{M}\sum_{n=1}^{N} S_{kl}(m,n)S^{*}_{k'l'}(m,n)\cos\theta_n = \delta_{kk'}\delta_{ll'} \tag{7}$$

where $\delta_{kk'}$ is the Kronecker delta and the asterisk denotes complex conjugation.

We introduce the column vector $\hat{\eta} = (\hat{\chi}, \hat{\psi}, \hat{\phi})^T$ where T denotes the transpose of a vector, so that the following relation can be written as:

$$\hat{\eta} = \sum_{k=0}^{M}\sum_{l=1}^{N} \hat{\eta}_{kl} f_{kl}(n)\exp[2\pi ikm/(M+1)],$$

where $\hat{\eta}_{kl} = (i\hat{\chi}_{kl}, \hat{\psi}_{kl}, \hat{\phi}_{kl})^T$. The complex coefficients $i\hat{\chi}_{kl}$, $\hat{\psi}_{kl}$, and $\hat{\phi}_{kl}$ are the expansion coefficients in the orthonormal set $S_{kl}(m, n)$. It appears to be convenient to write $i\hat{\chi}_{kl}$ instead of $\hat{\chi}_{kl}$, when we expand $\hat{\eta}$ in the normal modes. In order to find the normal modes we substitute, for a particular value of k, an expression of the form

$$\sum_{l=1}^{N} \mathbf{A}_{kl} f_{kl}(n)\exp[2\pi ikm/(M+1)],$$

where $\mathbf{A}_{kl} = (iA_{kl}, B_{kl}, C_{kl})^T$ in Eqs. (4)–(6), with the right-hand sides set to zero, multiply these equations by $f_{kl}(n)\exp[-2\pi ikm/(M+1)]\cos\theta_n$ and sum over m and n. If we assume that the modes behave like $\exp(-i\sigma t)$ with respect to the time t, then using the orthonormality relation (7) leads to the following $3N \times 3N$ eigenvalue problem:

$$-\epsilon_{kl}A_{kl} + \frac{2\Omega}{\alpha_{kl}^2}\sum_{l'=1}^{N}\alpha_{kl'}^2 a_{kll'}B_{kl'} - \frac{2\Omega}{r^2\alpha_{kl}^2}\sum_{l'=1}^{N} b_{kll'}B_{kl'} - C_{kl} = \sigma A_{kl}, \tag{8}$$

$$\frac{2\Omega}{\alpha_{kl}^2}\sum_{l'=1}^{N}\alpha_{kl'}^2 a_{kll'}A_{kl'} - \frac{2\Omega}{r^2\alpha_{kl}^2}\sum_{l'=1}^{N} b_{kll'}A_{kl'} - \epsilon_{kl}B_{kl} = \sigma B_{kl}, \tag{9}$$

$$-\alpha_{kl}^2 dA_{kl} = \sigma C_{kl}, \tag{10}$$

where

$$\epsilon_{kl} = \frac{2\Omega}{r^2\Delta\lambda\alpha_{kl}^2}\sin\frac{2k\pi}{M+1},$$

$$a_{kll'} = \sum_{n=1}^{N}\sin\theta_n f_{kl'}(n) f_{kl}(n)\cos\theta_n,$$

$$b_{kll'} = \sum_{n=1}^{N}\cos\theta_n\left(\frac{\partial}{\partial\theta}\right)_d f_{kl'}(n) f_{kl}(n)\cos\theta_n.$$

The coefficients $a_{kll'}$, which are symmetric in l and l', and $b_{kll'}$ are the expansion coefficients of $\sin\theta_n f_{kl'}(n)$ and $\cos\theta_n(\partial/\partial\theta)_d f_{kl'}(n)$ in the orthonormal set $f_{kl}(n)$ [cf., the behavior of spherical harmonics in the global case, Longuet-Higgins 1968, Eq. (3.17)]. In order to establish orthogonality relations between the normal modes (and in order to show that the corresponding eigenvalues are real), this study basically follows Kasahara (1976, pp. 674–675) but interprets his results (4.8) and (4.9) in terms of streamfunction and velocity potential. This leads to the scalar product (20), defined

TABLE 4. As in Table 3, but for the first internal mode.

	$\mathbf{P}_{kl1}(\bar{f})$	$\mathbf{P}_{kl2}(\bar{f})$	$\mathbf{P}_{kl3}(\bar{f})$	$\mathbf{P}_{kl1}(\bar{f}_{kl})$	$\mathbf{P}_{kl2}(\bar{f}_{kl})$	$\mathbf{P}_{kl3}(\bar{f}_{kl})$
$\mathbf{P}_{kll}$	9.9967×10^{-1}	5.6800×10^{-3}	5.6239×10^{-3}	9.9988×10^{-1}	2.9988×10^{-4}	2.9534×10^{-4}
	9.9997×10^{-1}	5.0665×10^{-3}	5.0648×10^{-3}	1.0000	1.7039×10^{-6}	1.7039×10^{-6}
$\mathbf{P}_{klN+l}$	-5.5940×10^{-3}	9.9974×10^{-1}	-8.5724×10^{-5}	-2.1393×10^{-4}	9.9975×10^{-1}	-7.9037×10^{-5}
	-5.0646×10^{-3}	9.9999×10^{-1}	-1.3468×10^{-5}	-1.5713×10^{-6}	1.0000	-6.4786×10^{-7}
$\mathbf{P}_{kl2N+l}$	-5.5300×10^{-3}	-1.0366×10^{-4}	9.9975×10^{-1}	-2.0163×10^{-4}	-7.9442×10^{-5}	9.9977×10^{-1}
	-5.0657×10^{-3}	-1.3493×10^{-5}	9.9999×10^{-1}	-1.5722×10^{-6}	-6.4748×10^{-7}	1.0000

in Juvanon du Vachat (1988, p. 2018), which is valid for both global and limited-area models with appropriate boundary conditions (such as periodic boundary conditions in the E–W and zero boundary conditions in the N–S direction). Application of Green's theorem to this latter expression gives the scalar product used by Daley (1978, p. 204), which reads in terms of the discrete functions $\hat{\chi}$, $\hat{\psi}$, and $\hat{\phi}$

$$\langle \hat{\eta}_1, \hat{\eta}_2 \rangle = \frac{1}{M+1} \sum_{m=0}^{M} \sum_{n=1}^{N} [\hat{\phi}_1 \hat{\phi}_2^* - d(\hat{\chi}_1 \nabla_d^2 \hat{\chi}_2^* + \hat{\psi}_1 \nabla_d^2 \hat{\psi}_2^*)] \cos\theta_n. \quad (11)$$

The horizontal structure of a particular normal mode is then given by

$$\mathbf{P}_{kj}(m, n) = \sum_{l=1}^{N} \mathbf{A}_{kl}(j) f_{kl}(n) \times \exp[2\pi i k m/(M+1)], \quad j = 1, \cdots, 3N$$

where j stands for one of the $3N$ normalized eigenvectors $(\mathbf{A}_{k1}, \ldots, \mathbf{A}_{kN})^T$ of system (8) to (10).

In addition, the quantity $\langle \hat{\eta}, \hat{\eta} \rangle$ is related to the total energy of the linearized equations. To distinguish between the N low-frequency westward Rossby modes, and the N high-frequency eastward and N high-frequency westward gravity modes the eigenvalues (frequencies) must be considered. For large values of the equivalent depth there is a large separation between the frequencies of the Rossby waves and the westward gravity waves. We will denote the Rossby modes, and westward and eastward gravity modes, ordered according to increasing gravity mode frequencies [for $k = 0$ and $k = (M+1)/2$, if M is odd, the Rossby modes have zero frequencies and are therefore not unique because the diagonal elements ϵ_{kl} in (9) vanish] by

$$\mathbf{P}_{kl'}, \mathbf{P}_{kN+l'}, \mathbf{P}_{k2N+l'}, \quad l' = 1, \cdots, N. \quad (12)$$

Expanding the field $\hat{\eta}$, at the initial time in the normal modes, initialization can be performed following the method exposed in Bijlsma (1989, p. 2013). The expansion coefficient $\hat{\gamma}_{kj}$ of $\hat{\eta}$ in the mode $\mathbf{P}_{kj} = (P_{kj1}, P_{kj2}, P_{kj3})^T$ which is defined as

$$\hat{\gamma}_{kj} = \langle \hat{\eta}, \mathbf{P}_{kj} \rangle$$

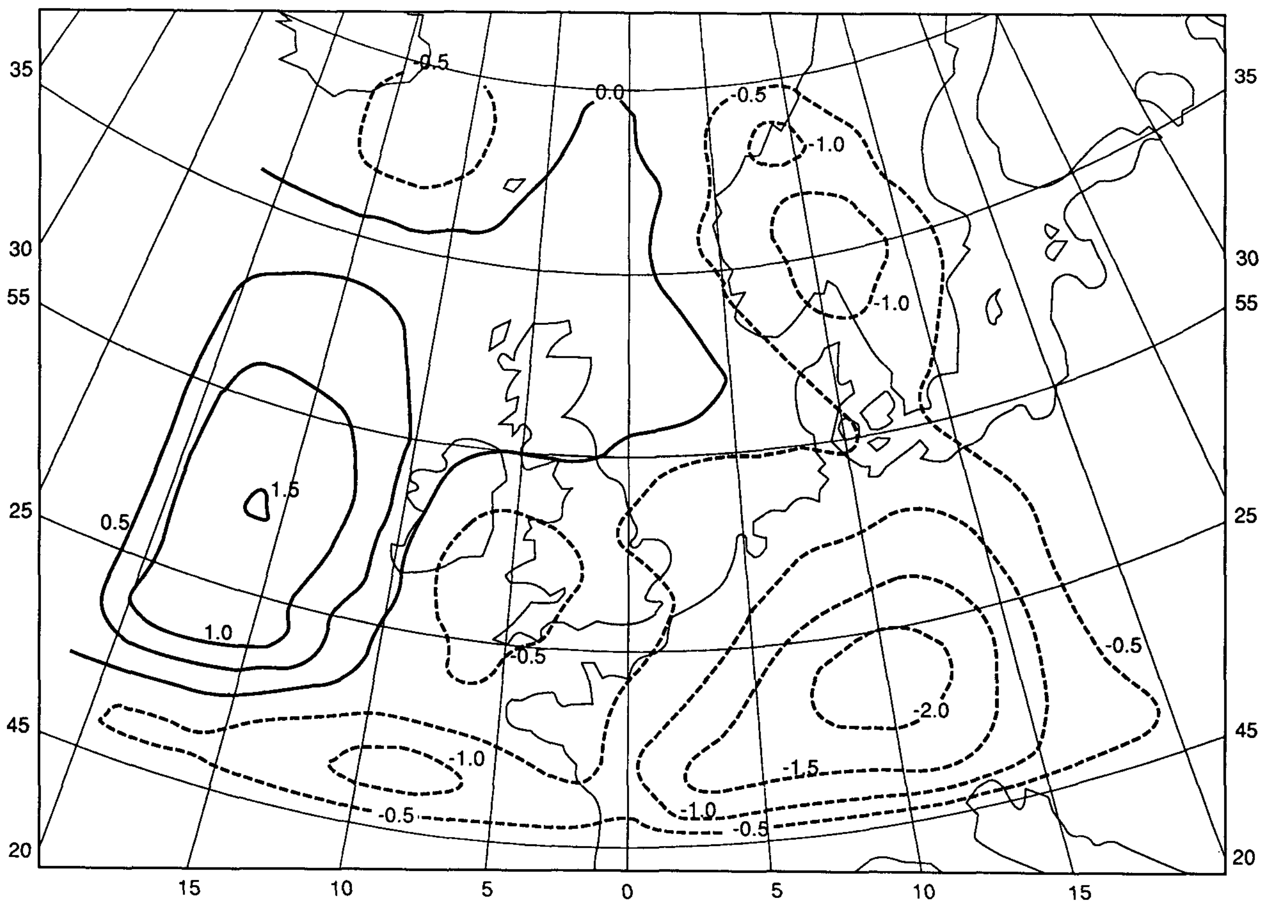

FIG. 2. Surface pressure changes at 1200 UTC 5 January 1982, after initialization of the external mode with two iterations with all beta terms taken into account. Units are millibars.

can be written as

$$\hat{\gamma}_{kj} = \sum_{l=1}^{N} \{ \hat{\phi}_{kl}C_{kl}(j) + \alpha_{kl}^2 d[\hat{\chi}_{kl}A_{kl}(j) + \hat{\psi}_{kl}B_{kl}(j)] \}$$

and the coefficient F_{kj} containing the nonlinear terms is given by

$$F_{kj} = \frac{1}{M+1} \sum_{m=0}^{M} \sum_{n=1}^{N} \Bigg(Q_\phi P_{kj3}^* - d \Bigg\langle \Bigg\{ Q_\psi - \frac{2\Omega}{r^2} \times \left[\left(\frac{\partial}{\partial \lambda} \right)_d \psi_0 + \cos\theta_n \left(\frac{\partial}{\partial \theta} \right)_d \chi_0 \right] \Bigg\} P_{kj2}^* + \Bigg\{ Q_\chi - \frac{2\Omega}{r^2} \left[\left(\frac{\partial}{\partial \lambda} \right)_d \chi_0 - \cos\theta_n \left(\frac{\partial}{\partial \theta} \right)_d \psi_0 \right] \Bigg\} P_{kj1}^* \Bigg\rangle \Bigg) \cos\theta_n .$$

3. Effect of the beta terms

In order to investigate the effect of the beta terms on the initialization a set of experiments was performed with the five-level limited-area version of the ECMWF gridpoint model employed by Temperton and Williamson (1981) on a grid having a mesh spacing $(\Delta\lambda, \Delta\theta) = (2°, 1°)$ and covering approximately the area between 45° and 65°N and 20°W and 20°E, and with initial data 1200 UTC 5 January 1982. The surface pressure is given in Fig. 1, showing a cyclone over the British Isles. For further details the reader is referred to Bijlsma and Hafkenscheid (1986).

First compare the eigenmodes of foregoing method with the modes of the approximate method (Bijlsma 1989), which are determined by the linear part of the model equations (1), (2), and (3), without the terms $(-2\Omega/r^2)\cos\theta_n(\partial/\partial\theta)_d\psi$ in (1) and $(2\Omega/r^2)\cos\theta_n(\partial/\partial\theta)_d\chi$ in (2) and with $\sin\theta_n$ in (1) and (2) replaced by the constant value $\sin\bar{\theta}$. Using the notation of the foregoing section, the 3×3 eigenvalue problem in this case can be written as

$$-\epsilon_{kl}A_{kl} + \bar{f}B_{kl} - C_{kl} = \sigma A_{kl}, \quad (13)$$

$$\bar{f}A_{kl} - \epsilon_{kl}B_{kl} = \sigma B_{kl}, \quad (14)$$

$$-\alpha_{kl}^2 dA_{kl} = \sigma C_{kl}, \quad (15)$$

where $\bar{f} = 2\Omega\sin\bar{\theta}$. The values of σ corresponding to

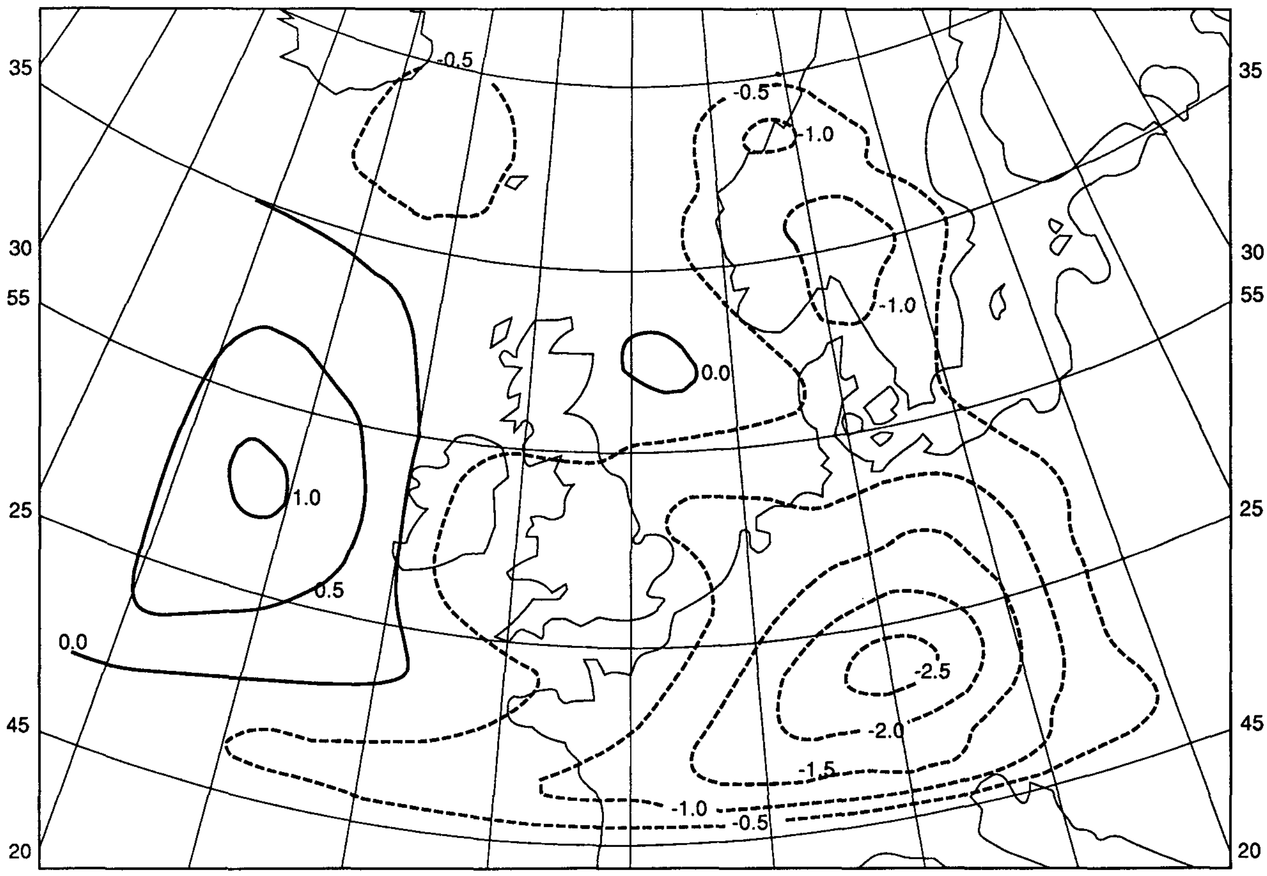

FIG. 3. As in Fig. 2, but after initialization of the external and first internal mode with two iterations.

Rossby waves and westward and eastward gravity waves are for $\epsilon_{kl} \geqslant 0$ (we consider geopotential heights, so that $3\alpha_{kl}^2 d - 6\bar{f}^2 + \frac{2}{3}\epsilon_{kl}^2 > 0$)

$$\sigma_{kl1} = -\frac{2}{3}\epsilon_{kl} - \frac{2}{3}(3p)^{1/2}\cos(\pi + \mu)/3,$$

$$\sigma_{kl2} = -\frac{2}{3}\epsilon_{kl} - \frac{2}{3}(3p)^{1/2}\cos(\pi - \mu)/3,$$

$$\sigma_{kl3} = -\frac{2}{3}\epsilon_{kl} + \frac{2}{3}(3p)^{1/2}\cos\mu/3,$$

$k = 0, \cdots, M/2$ (M even) or

$k = 0, \cdots, (M+1)/2$ (M odd);

$l = 1, \cdots, N$

where

$$\mu = \arctan[(12p^3 - q^2)^{1/2}/q], \quad 0 < \mu \leqslant \pi/2,$$

$$p = \bar{f}^2 + \alpha_{kl}^2 d + \frac{1}{3}\epsilon_{kl}^2,$$

$$q = 3\epsilon_{kl}\alpha_{kl}^2 d - 6\epsilon_{kl}\bar{f}^2 + \frac{2}{3}\epsilon_{kl}^3$$

and for $\epsilon_{kl} < 0$

$$\sigma_{M+1-kl1} = -\sigma_{kl1},\ \sigma_{M+1-kl3} = -\sigma_{kl2},\ \sigma_{M+1-kl2} = -\sigma_{kl3},$$

$k = 1, \cdots, M/2$ (M even) or

$k = 1, \cdots, (M-1)/2$ (M odd);

$l = 1, \cdots, N$.

The eigenmodes normalized with the scalar product (11) can be written as

$$\mathbf{P}_{klr} = N_{klr}^{-1/2}\mathbf{A}_{klr}S_{kl}(m, n) \qquad (16)$$

with

$$\mathbf{A}_{klr} = [i(\sigma_{klr} + \epsilon_{kl}), \bar{f}, \bar{f}^2 - (\sigma_{klr} + \epsilon_{kl})^2]^T,$$

$$N_{klr} = [\bar{f}^2 - (\sigma_{klr} + \epsilon_{kl})^2]^2 + \alpha_{kl}^2 d[(\sigma_{klr} + \epsilon_{kl})^2 + \bar{f}^2], \quad k = 0, \cdots, M; \quad l = 1, \cdots, N; \quad r = 1, 2, 3.$$

If $\bar{f}$ is replaced by

$$\bar{f}_{kl} = 2\Omega(a_{kll} - b_{kll}/r^2\alpha_{kl}^2),$$

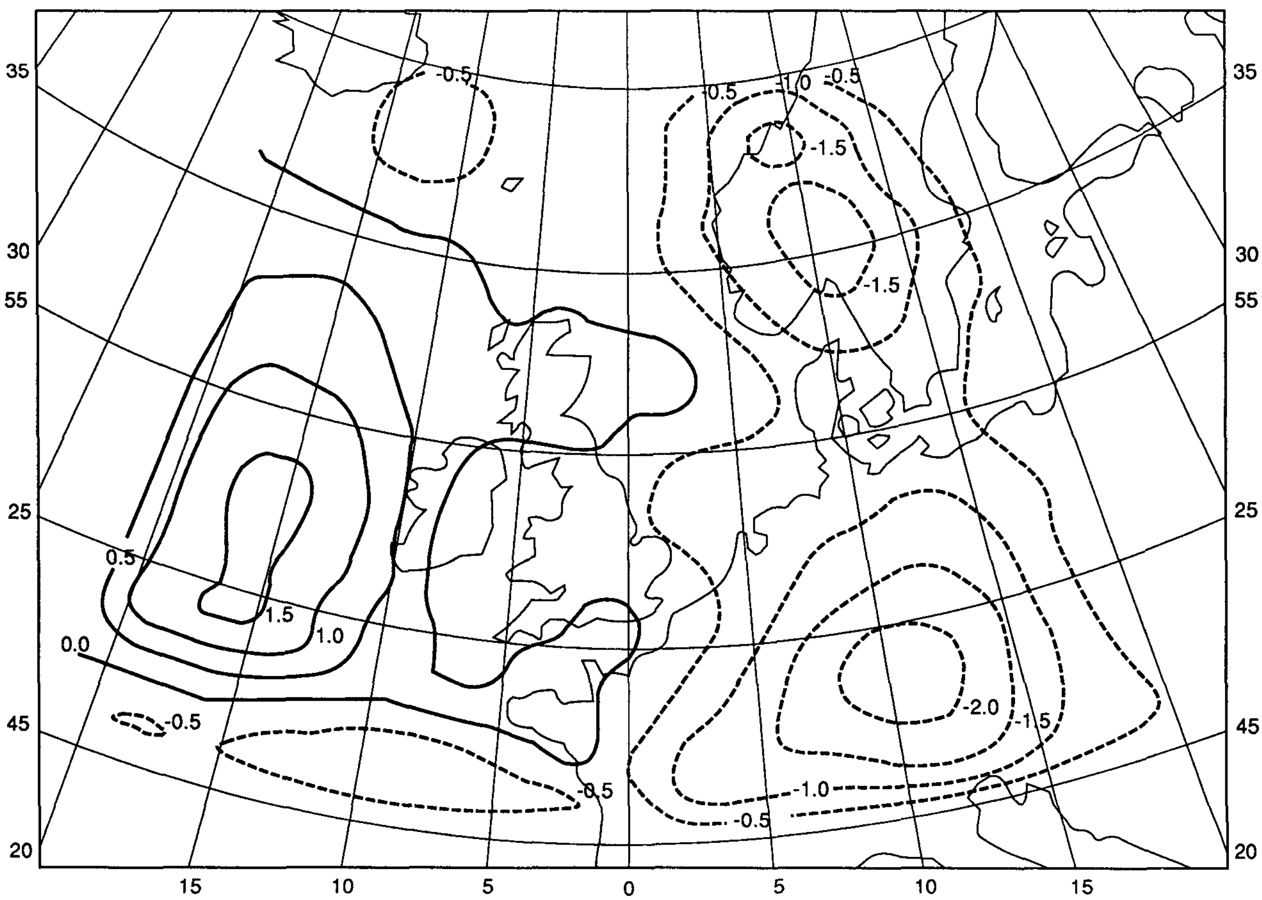

FIG. 4. As in Fig. 2, but after initialization of the external mode with two iterations, with some beta terms taken into account.

Eqs. (13), (14), and (15) determine the eigenvalues and eigenvectors of the diagonal submatrices of (8), (9), and (10). In order to compare the modes (12) and (16) the following expressions must be considered:

$$\mathbf{A}_{kl}(l')S_{kl}(m, n), \mathbf{A}_{kl}(N + l')S_{kl}(m, n),$$
$$\mathbf{A}_{kl}(2N + l')S_{kl}(m, n), \quad l' = 1, \cdots, N \quad (17)$$

for the same k and l as in (16). The parameters used have the following values: the geopotential heights corresponding to the external and first internal vertical modes are $d = 91\,932.53$, $12\,478.39\ \mathrm{m^2\ s^{-2}}$, respectively, the angular velocity $\Omega = 7.29 \times 10^{-5}\ \mathrm{s^{-1}}$, the radius of the earth $r = 6367 \times 10^3$ m, $M = 21$, $N = 20$, and $\bar{\theta} = 55.5°$N.

Since $S_{M+1-kl} = S^*_{kl}$ for $k = 1, \cdots, (M - 1)/2$, it is sufficient to consider the wavenumbers $k = 0, \cdots, (M + 1)/2$.

For the first two vertical modes the eigenvalues of Eqs. (8), (9), (10), and (13), (14), (15) are very similar. In Tables 1 and 2, the eigenvalues are given for the gravity modes denoted by σ_{kN+l}, σ_{k2N+l} and $\sigma_{kl2}(\bar{f})$, $\sigma_{kl3}(\bar{f})$, respectively, for $k = 0, \cdots, (M + 1)/2$ and $l = 1$ (lowest frequency for each value of k).

If the constant Coriolis parameter $\bar{f}$ in (13) and (14) is replaced by $\bar{f}_{kl}$, denoting the gravity wave frequencies by $\sigma_{kl2}(\bar{f}_{kl})$ and $\sigma_{kl3}(\bar{f}_{kl})$, the resemblance is closer.

Also, approximate orthogonality relations between systems (16) and (17) exist. For instance, if we set $l' = l$ in (17) (the vector components are relatively small for $l' \neq l$), the projections of $\mathbf{P}_{klr}(\bar{f})$, $r = 1, 2, 3$ on (17), denoted by $\mathbf{P}_{kll}$, $\mathbf{P}_{klN+l}$, $\mathbf{P}_{kl2N+l}$ are given in Tables 3 and 4 for the external and first internal vertical modes, and $k = 1, (M - 1)/2$ and $l = 1$. Also, values of these projections are given if $\bar{f}$ in (13) and (14) is replaced by $\bar{f}_{kl}$. Apparently the three-component vectors in (17) are, for $l' = l$, the approximate eigenvectors of the diagonal submatrices of (8), (9), and (10). These results suggest that the initialized fields obtained with the eigenmodes (16) are similar to those where (12) are used as normal modes. The likeness between the two methods can be made even closer if the constant Coriolis parameter $\bar{f}$ in the former method is replaced by $\bar{f}_{kl}$. The effect of the beta terms is investigated by initializing with the modes (16), computed from Eqs. (13), (14), and (15) with constant Coriolis parameter $\bar{f}$ and variable Coriolis parameter $\bar{f}_{kl}$, respectively.

Figure 2 gives the surface pressure changes after ini-

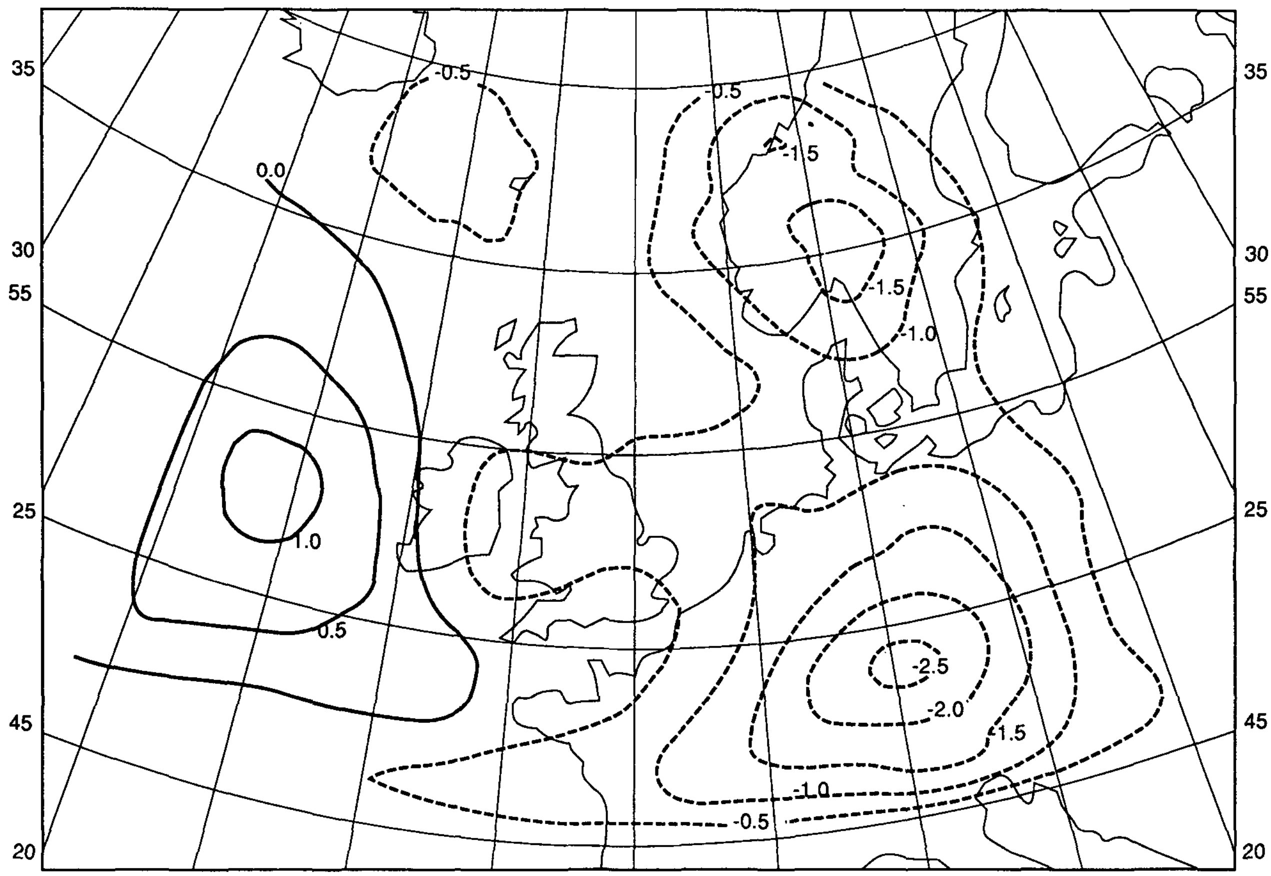

FIG. 5. As in Fig. 4, but after initialization of the first two vertical modes with two iterations.

tialization of the first vertical mode with two iterations using the eigenmodes $\mathbf{P}_{klr}(\bar{f}_{kl})$, $r = 2, 3$. Figure 3 shows these changes after initialization of the first two vertical modes with two iterations. Corresponding results after initialization with the $\mathbf{P}_{klr}(\bar{f})$, $r = 2, 3$ modes are shown in Figs. 4 and 5. A similar result is obtained for the f-plane approach, using modes (16) with ϵ_{kl} set to zero, as shown in Fig. 6. From these and previous experiments (cf., Bijlsma 1989) it may be concluded that the effect of the beta terms on a midlatitude application of the nonlinear normal mode initialization is negligible.

4. Conclusions

A nonlinear normal mode method with all beta terms included in the linearized model equations is formulated for a limited-area model. It is shown that the normal modes of this method for sufficiently large equivalent depths do coincide approximately with the modes of a previous method (Bijlsma 1989), in which certain approximations were made with respect to the Coriolis parameter.

The likeness between the two methods can be made even closer if the constant Coriolis parameter in the latter method is replaced by a parameter that depends on the zonal and meridional wavenumbers. Experiments show that the initialized fields obtained with the two methods are similar, demonstrating the equivalence between the f-plane approach and the inclusion of some or all beta terms in the linearized model equations. It is emphasized that these conclusions concern midlatitude application of nonlinear normal mode initialization. For applications to equatorial regions (Tribbia 1979; DeMaria and Schubert 1984) things will be different, due to the completely different parameter values.

In a sense the method is more general than the implicit normal or nonnormal mode methods of Temperton (1988) and Juvanon du Vachat (1988), but also requires more computational work, if it is not allowed to make suitable approximations (possibly in the case of small equivalent depths in the tropics). On the other hand, these latter methods can be applied if the linearized model equations are nonseparable, so that the normal modes cannot be found easily.

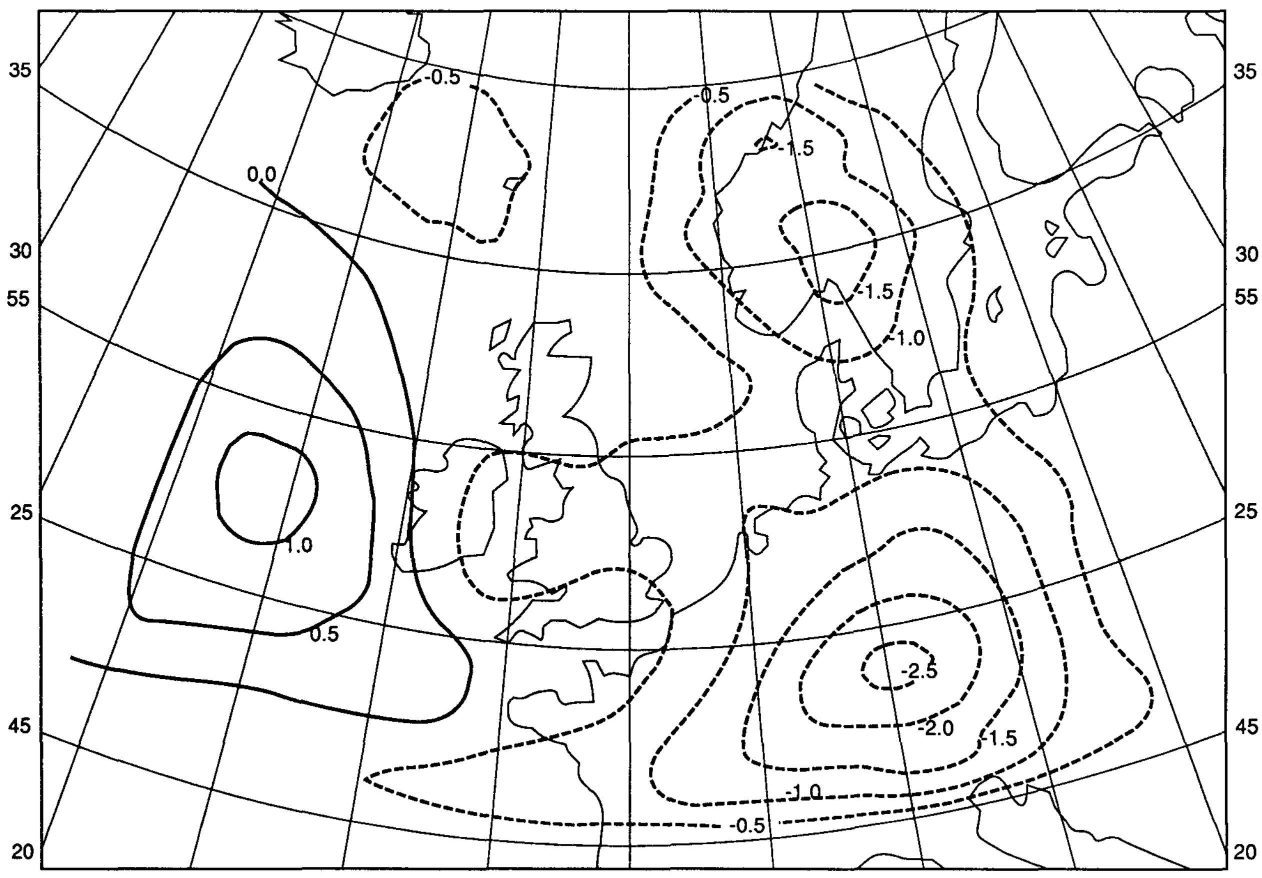

FIG. 6. As in Fig. 2, but after initialization of the first two vertical modes with two iterations with a constant Coriolis parameter.

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