Monotone Flux Limitation in the Area-preserving Flux-form Advection Algorithm

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ABSTRACT

The area-preserving flux-form advection algorithm is extended to monotonicity. For this, the nonlinear positive-definite flux limitation of the original approach is replaced by new monotone flux limiters. The monotone fluxes are derived for one-dimensional constant transport velocities. The deformation occurring in divergent flow is accounted for by adding to the monotone advection fluxes a correction term, which has been derived from the deformation of the upstream method. The final algorithm is applicable to arbitrary multidimensional transport problems. However, due to the use of the time-splitting method, it is strictly monotone only in uniform flow fields.

Results of different one- and two-dimensional advection experiments are presented, demonstrating that the monotone flux limitation is an attractive alternative to the positive-definite algorithm. Amplitude and phase speed errors are somewhat larger in the monotone advection scheme. The computational effort of the new version is not much larger than that of the positive definite scheme. Thus, it is concluded that for many applications of atmospheric modeling, the monotone area-preserving flux-form advection algorithm is an accurate and numerically efficient method for the solution of the transport equation.

1. Introduction

In past decades, a large variety of different methods has been developed for the numerical solution of the transport equation in atmospheric models (e.g., Lax and Wendroff 1964; Crowley 1968; Boris and Book 1973; Smolarkiewicz 1983; Tremback et al. 1987; Williamson and Rasch 1989). However, many of these advection schemes show different disadvantages, which sometimes yield undesirably poor results. For instance, the upstream method is known to be very diffusive. The higher-order versions of the advection schemes of Tremback et al. (1987) are much less diffusive. Unfortunately, they lack positive definiteness, thus yielding in some cases negative values of positive-definite quantities, such as the concentrations of chemical species in the atmosphere. The advection scheme of Smolarkiewicz (1983) is positive definite, and at the same time it produces much less numerical diffusion than the upstream method. However, the scheme is not oscillation free. Therefore, in situations with strong spatial gradients of the transported quantity, the scheme produces over- and undershooting values, which may become a serious problem in particular applications—for example, the transport of highly water-soluble gas-phase chemical species in cloudy atmospheres. Based on the flux-corrected transport method of Boris and Book (1973) and Zalesak (1979), Smolarkiewicz and Grabowski (1990) extended the original positive-definite approach to monotonicity. In this treatment, the advective flux of the higher-order scheme is subdivided into the flux from a monotone lower-order scheme and a residual. The residual is then multiplied by a correction coefficient yielding monotonicity as well as low truncation errors of the higher-order scheme.

This paper presents the monotone version of the area-preserving flux-form advection algorithm of Bott (1989a,b). In contrast to the approach of Smolarkiewicz and Grabowski (1990), the monotone fluxes are not calculated as a function of the advective fluxes of a lower-order monotone scheme. Instead, they are directly obtained by replacing the positive-definite flux limiters of the original approach by new monotone flux limiters. The monotone fluxes are derived for constant one-dimensional transport velocities. Utilizing the deformation produced by the upstream method, in divergent flow fields correction terms are added to the monotone fluxes.

In multidimensional flows, the time-splitting method is applied. Due to this treatment, the algorithm is strictly monotone only in uniform flow fields. In applications with deformational but nondivergent velocity fields, the scheme still produces little over- and undershooting. These errors depend on the Courant number and on the strength of deformation. However, as will be shown in the numerical results of this paper, even in very strong deformational flow fields, the ripples are relatively small, so that under typical atmospheric conditions the results of the monotone advection scheme are still satisfactory.

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Recently, Bott et al. (1990) presented the one-dimensional semi-Lagrangian version of the positive-definite advection scheme by calculating the diffusional growth of aerosols and cloud droplets in a microphysical fog model for arbitrary Courant numbers. In contrast to the positive-definite algorithm, at present the monotone advection scheme is restricted by the Courant-Friedrich-Lewy criterion. The extension to arbitrary Courant numbers will be the subject of future investigations. Due to the time-splitting approach, this version will also be restricted to one-dimensional flows.

In section 2, the monotone advection scheme will be derived from the positive-definite approach. Numerical results of different advection experiments are presented in section 3, demonstrating that the monotone scheme yields very good results for many different applications. Since the computational effort of the monotone version is relatively small as compared to other methods (e.g. Williamson and Rasch 1989), the method is very attractive for use in atmospheric models that are dealing with a large number of different transported quantities.

2. Theory

a. The positive-definite advection algorithm

Since the area-preserving flux-form algorithm (APF) has already been described in detail in the original references (Bott 1989a,b), here only the basic ideas of APF will be summarized for the special case of positive transport velocities. The monotone version of APF (hereafter abbreviated as MAPF) will be derived for one-dimensional flow fields with arbitrary values of $u$. In multidimensional flows, the time-splitting method will be applied. The grid mesh $\Delta x$ and the time increments $\Delta t$ are assumed to be equidistant, even though APF and MAPF may also be applied in systems with variable grid spacing (Bott 1989b).

Neglecting diffusion processes and sources or sinks of the transported quantity $\psi$, the one-dimensional continuity equation is simply given by

$$\frac{\partial \psi}{\partial t} = -\frac{\partial u \psi}{\partial x}. \quad (1)$$

At time step $n$, the discrete flux form of this equation is

$$\psi_{j+1}^{n+1} = \psi_j^n - \frac{\Delta t}{\Delta x} [F_{j+1/2} - F_{j-1/2}], \quad (2)$$

where the advective fluxes $F_{\pm 1/2}$ are usually given as functions of the Courant number

$$c_{j\pm 1/2}^n = \frac{\Delta t}{\Delta x} u_j^n \psi_j^n. \quad (3)$$

Utilizing the first-order forward upstream method for the solution of (2) yields (for $u \geq 0$)

$$F_{j+1/2} = \frac{A_{j+1/2}}{\Delta t}, \quad (4)$$

with

$$A_{j+1/2} = c_{j+1/2}^n \Delta x \psi_j^n. \quad (5)$$

The quantity $A_{j+1/2}$ corresponds to an area within grid box $j$ as indicated in Fig. 1. From this figure, it can be seen that in the upstream method the $\psi$ distribution of each grid box is given by the constant value $\psi_j^n$. This simple treatment causes the strong numerical diffusion of the upstream scheme.

To obtain a better representation of the $\psi$ field in each grid box, the method of polynomial fitting is used in APF. This concept is based on previous works by Crowley (1968) and Tremback et al. (1987). In grid box $j$, the area $A_{j+1/2}$ is now calculated as a function of polynomial $\psi_j^p(x')$ of order $l$ (for $u \geq 0$):

$$A_{j+1/2} = \Delta x \int_{1/2 - c_{j+1/2}^n}^{1/2} \psi_j^p(x') dx'$$

$$= \sum_{k=0}^{l} \frac{a_{j,k}}{(k+1)2^{k+1}} [1 - (1 - 2c_{j+1/2}^n)^{k+1}], \quad (6)$$

with

$$\psi_j^p(x') = \sum_{k=0}^{l} a_{j,k} x'^k$$

$$x' = \left( \frac{x - x_j}{\Delta x} \right), \quad -1/2 \leq x' \leq 1/2. \quad (7)$$

The constants $a_{j,k}$ are functions of the $\psi$ values at neighboring grid points $j \pm 1, j \pm 2, \ldots$. In APF they are determined by solving

$$\psi_j^n \Delta x = \int_{x_j-1/2}^{x_j+1/2} \sum_{k=0}^{l} a_{j,k} x'^k dx$$

$$i = j, j \pm 1, j \pm 2 \cdots \quad (8)$$

Table 1 of Bott (1989b) lists values of $a_{j,k}$ for $l = 2$ and $l = 4$.

Due to the local approach of the polynomial fitting, that is, the separate evaluation of the polynomials in each grid box $j$, in this form the advection algorithm produces already very low numerical diffusion. How-

Fig. 1. The $\psi$ distribution in grid box $j$ for the upstream method.
ever, in some situations the use of polynomials of order $l \geq 2$ may result in $\psi$ distributions with negative values. Depending on the Courant number $c_{j+1/2}^n$, in these cases the area $A_{j+1/2}$ becomes negative or exceeds the maximum value of $\Delta x \psi_j^n$, resulting in the undesired evolution of negative values of $\psi$.

Positive definiteness of the scheme is ensured (for $u \geq 0$), if

$$0 \leq A_{j+1/2} \leq \Delta x \psi_j^n. \quad (9)$$

To obtain the advection algorithm positive definite, therefore, the following nonlinear positive-definite flux limiters are introduced. In (4) the value of $A_{j+1/2}$ is set equal to zero whenever $A_{j+1/2} < 0$, and $A_{j+1/2}$ is held fixed at $\Delta x \psi_j^n$ whenever it exceeds this value. For $A_{j+1/2} < 0$ and $A_{j+1/2} > \Delta x \psi_j^n$, this procedure is graphically shown in Figs. 2a and 2b, respectively. For the mathematical description of the positive-definite flux limitation in arbitrary flow fields, the reader is referred to the original papers of Bott (1989a,b).

b. The monotone advection algorithm

Although for a large number of different numerical applications APF yields very good results, in situations with strong spatial gradients and nonzero background values of the transported quantity the algorithm may produce over- and undershooting values of $\psi$ that are not realistic and are, therefore, not desired. To eliminate this deficiency of the scheme, the positive-definite flux limiter of the previous section will now be replaced by the more restrictive monotone flux limitation.

1) Uniform flow

The analytical solution of (1) is deformation-free only if $u = \text{const}$. In divergent flow fields, a natural deformation of the initial $\psi$ distribution arises, eventually resulting in the evolution of over- and undershooting values of $\psi$. As a consequence of this, the monotone version of the advection scheme will first of all be derived for the special situation of constant $u$. The deformation of $\psi$ occurring in nonuniform flow fields will later be accounted for by adding a correction term to the monotone advective fluxes. Hence, in its final form, MAPF is applicable to the general case of arbitrary flow fields.

Obviously, positive definiteness of the advection scheme is ensured if $\psi_j^{n+1} \geq 0$. However, this requirement does not necessarily suppress over- and undershooting values of the transported quantity. From the analytical solution of the transport equation, it is known that for constant $u$ at time step $n + 1$, the $\psi$ values are constrained by

$$\min(\psi_j^{n-1}, \psi_j^n) \leq \psi_j^{n+1} \leq \max(\psi_j^{n-1}, \psi_j^n), \quad u \geq 0$$

$$\min(\psi_j^{n+1}, \psi_j^n) \leq \psi_j^{n+1} \leq \max(\psi_j^{n+1}, \psi_j^n), \quad u < 0.$$  \quad (10)

Due to these limitations, $\psi_j^{n+1}$ never exceeds the extreme values that are found at time step $n$ in grid box $j$ and in the neighboring grid box $j \pm 1$ by looking in an upstream direction. Thus, the evolution of ripples in the field of the transported quantity is completely suppressed if (10) is fulfilled.

The $\psi$ limits in (10) are now used to determine the monotone flux limiters. For convenience, the following quantities are defined:

$$F_{j+1/2}^+ = \alpha \int_{1/2-c_j^+}^{1/2} \psi_j^+(x')dx'$$

$$F_{j-1/2}^- = \alpha \int_{-1/2}^{-1/2+c_j^-} \psi_j^-(x')dx', \quad (11)$$

with $\alpha = \Delta x/\Delta t$ and $c_j^\pm = \pm(c_{j+1/2}^n \pm |c_{j+1/2}^n|)/2$. Now the advective flux at $j + 1/2$ is written as

$$F_{j+1/2} = F_{j+1/2}^+ - F_{j+1/2}^-.$$  \quad (12)

Since $F_{j+1/2}^+ = 0$ for $u \geq 0$ and $F_{j+1/2}^- = 0$ for $u \leq 0$, (2) may be reformulated as

$$\psi_j^{n+1} = \psi_j^n - \frac{1}{\alpha} [F_{j+1/2}^+ - F_{j-1/2}^-], \quad u \geq 0$$

$$\psi_j^{n+1} = \psi_j^n - \frac{1}{\alpha} [F_{j-1/2}^- - F_{j+1/2}^+], \quad u < 0.$$  \quad (13)

Substituting these expressions into (10) yields

$$\alpha[\psi_j^n - \max(\psi_j^{n-1}, \psi_j^n)] + F_{j+1/2}^+ \leq F_{j+1/2}^+ \leq \alpha[\psi_j^n - \min(\psi_j^{n-1}, \psi_j^n)] + F_{j-1/2}^- \quad u \geq 0$$

$$\alpha[\psi_j^n - \max(\psi_j^{n+1}, \psi_j^n)] + F_{j+1/2}^+ \leq F_{j-1/2}^- \leq \alpha[\psi_j^n - \min(\psi_j^{n+1}, \psi_j^n)] + F_{j+1/2}^-, \quad u < 0.$$  \quad (14)
Finally, the fluxes are to be limited by
\begin{align}
0 & \leq F_{j+1/2}^+ \leq \alpha \psi_j^+ \\
0 & \leq F_{j-1/2}^- \leq \alpha \psi_j^-.
\end{align} 
This requirement inhibits advective fluxes in the opposite flow direction (if $F_{j+1/2}^- < 0$) and the outflux of more than the total $\psi$ content of grid box $j$ (if $F_{j+1/2}^+ > \alpha \psi_j^+$). The expressions (14), together with (15), are the complete monotone flux limiters.

From (14) it follows that in each grid box the outgoing monotone fluxes depend on the incoming fluxes of the neighboring grid box in an upstream direction. Thus, for $u > 0$, $F_{j+1/2}^+$ is calculated by starting in grid box $j = 2$ and increasing $j$ up to the right boundary $j = m$ of the numerical grid mesh. The outgoing flux $F_{j+1/2}^+$ of grid box $j = 1$ is determined from the original positive-definite approach of the advection scheme. For negative velocities, the procedure is reversed, that is, the calculation of $F_{j-1/2}^-$ starts at $j = m - 1$ and $j$ is decreased to $j = 1$. The outgoing flux $F_{m-1/2}^-$ of grid box $j = m$ is again taken from APF. Due to this treatment, the monotonicity of the advection scheme is restricted to the inner $\psi$ field of the model domain. However, by choosing for the transport calculations appropriate boundary conditions in the first and in the last grid box (for example, constant values of $\psi_1$ and $\psi_m$), monotonicity of the scheme is achieved in the whole model domain.

2) DIVERGENT FLOW

To apply the monotone flux limitation to arbitrary flow fields, a correction term, accounting for the deformation of the $\psi$ field, is added to the monotone fluxes. This term will now be derived from the deformational part of the upstream advective fluxes:
\begin{align}
F_{j+1/2}^+ & = \alpha c_j^+ \psi_j^+ \\
& = \alpha (c_j^- \psi_j^- + D_{j+1/2}^-) \\
F_{j-1/2}^- & = \alpha c_j^- \psi_j^- \\
& = \alpha (c_j^+ \psi_j^+ + D_{j-1/2}^+). 
\end{align}
Here, the following deformation terms have been introduced:

\[ D_{j+1/2}^+ = \psi_j^+ (c_j^+ - c_{j-1}^+) \]
\[ D_{j-1/2}^- = \psi_j^- (c_{j-1}^- - c_j^-). \]  \hspace{1cm} (17)

From (16) and (17), it is seen that the upstream solution is deformation-free if at the same time \( D_{j+1/2}^+ = 0 \) and \( D_{j-1/2}^- = 0 \). This is only the case for constant values of the transport velocity.

For the monotone version of APF, the procedure is as follows: (i) The upstream fluxes \( \alpha c_{j-1}^+ \psi_j^+ \) and \( \alpha c_j^- \psi_j^- \) of (16) are replaced by the corresponding monotone fluxes of the present approach. (ii) To these fluxes the deformation terms of (17) are added. (iii) The resulting fluxes are restricted to the limits of (15), yielding the final advective fluxes of MAPF for arbitrary flow fields.

First, the outgoing monotone fluxes \( \tilde{F}_{j+1/2}^+ \) and \( \tilde{F}_{j-1/2}^- \) are calculated for constant values of \( u \) at the cell walls of grid box \( j \). This is done by defining

\[ \tilde{F}_{j+1/2}^+ = \begin{cases} \alpha \int_{c_j^-}^{1/2} \psi_j^+(x') dx', & u_{j+1/2}^+ > 0 \\ 0, & u_{j+1/2}^+ \leq 0 \end{cases} \]
\[ \tilde{F}_{j-1/2}^- = \begin{cases} \alpha \int_{-1/2}^{-1/2+c_j^-} \psi_j^- (x') dx', & u_{j-1/2}^- < 0 \\ 0, & u_{j-1/2}^- \geq 0. \end{cases} \]  \hspace{1cm} (18)

Fig. 5. (a) Numerical solution of the rotating cone test with APF4 after 628 iterations. (b) Same as (a) but after 3768 iterations. (c) Numerical solution of the rotating cone test with MAPF4 after 628 iterations. (d) Same as (c) but after 3768 iterations.
From this expression it follows that

\[ \tilde{F}_{j+1/2} = 0, \quad \text{if} \quad u^+_{j+1/2} \leq 0 \quad \text{or} \quad u^-_{j+1/2} = 0 \]

\[ \tilde{F}_{j-1/2} = 0, \quad \text{if} \quad u^-_{j-1/2} \geq 0 \quad \text{or} \quad u^+_{j+1/2} \geq 0. \quad (19) \]

Furthermore, it is seen that, according to (16), \( \tilde{F}_{j+1/2} \) is calculated as function of \( u^-_{j+1/2} \) (instead of \( u^+_{j+1/2} \)) and \( \tilde{F}_{j-1/2} \) is calculated as function of \( u^+_{j+1/2} \) (instead of \( u^-_{j+1/2} \)). The fluxes in (18) are now restricted to the monotone limiters:

\[ \alpha[\psi^-_{j} - \max(\psi^+_{j-1}, \psi^+_{j})] + F^+_{j+1/2} \leq \tilde{F}^+_{j+1/2} \leq \alpha[\psi^-_{j} - \min(\psi^-_{j-1}, \psi^-_{j})] + F^-_{j-1/2}, \]

if \( u^+_{j+1/2} < 0 \quad \text{and} \quad u^-_{j-1/2} > 0 \),

\[ \alpha[\psi^-_{j} - \max(\psi^+_{j+1}, \psi^+_{j})] + F^+_{j+1/2} < \tilde{F}^-_{j-1/2} \leq \alpha[\psi^-_{j} - \min(\psi^-_{j+1}, \psi^-_{j})] + F^-_{j-1/2}. \quad (20) \]

The deformation terms of (17) are added, yielding

\[ F^+_{j+1/2} = \tilde{F}^+_{j+1/2} + D^+_{j+1/2} \]

\[ F^-_{j-1/2} = \tilde{F}^-_{j-1/2} + D^-_{j-1/2}, \quad (21) \]

and the resulting fluxes are constrained to (15). In the special case of \( u^-_{j-1/2} < 0 \) and \( u^+_{j+1/2} > 0 \), (15) is replaced by

\[ F^-_{j-1/2} + F^+_{j+1/2} = D^-_{j-1/2} + D^+_{j+1/2} \leq \alpha \psi^+_{j}. \quad (22) \]

The flux limitations (15) or (22) are performed in the same way as already described in Bott (1989a). The final advective flux at \( j + \frac{1}{2} \) is again given by (12).

Note that due to the restriction (15), \( F^+_{j+1/2} \) is always equal to zero if \( u^+_{j+1/2} \leq 0 \), even though in this case \( D^+_{j+1/2} \leq 0 \). Correspondingly, \( F^-_{j+1/2} = 0 \) if \( u^-_{j+1/2} \geq 0 \).

3. Numerical results

To examine the performance of the monotone area-preserving flux-form algorithm, a large number of different numerical advection experiments has been conducted. In this section, a small selection of these model runs will be presented. For brevity, the discussion is restricted to the fourth-order versions of the advection schemes (henceforth abbreviated APF4 and MAPF4). However, the conclusions from the numerical results of these two algorithms also apply to model versions of APF and MAPF with lower-order polynomials. To clarify the effect of the monotone flux limiters, the results obtained with MAPF4 will be compared to the corresponding results of APF4.

a. Uniform flow fields

Since the mathematical structure of the advection schemes is one-dimensional, it is appropriate to first present the results of some one-dimensional model runs. Figures 3a and 3b depict the analytical solution as well as the numerical results of the advection of a triangle and a square wave, respectively, which are obtained after 50 iterations with constant Courant number \( c = 0.4 \). These figures show the main characteristics of MAPF as compared to APF. (i) Due to the more restrictive monotone flux limiters, MAPF produces somewhat larger phase and amplitude errors than APF. (ii) If the spatial gradients of the transported quantity are not too large, APF produces very little over- and undershooting, so that for certain applications APF may be preferred to MAPF (see Fig. 3a). (iii) In situations with strong spatial gradients of the transported quantity, the over- and undershooting of the positive-definite approach may become so large that MAPF yields much better results than APF (see Fig. 3b).

The findings of the one-dimensional advection tests are confirmed by the model runs in two-dimensional flow fields. Figure 4 depicts the initial \( z \) distribution of the well-known rotating-cone test, which is also the analytical solution of the problem. In contrast to previous model runs, now the cone is superimposed on a constant background value of \( z = 1 \). The transport velocities \( u \) and \( v \) in \( x \) and \( y \) directions are the same.

![Fig. 6. Initial distribution and analytical solution of the rotating-cylinder test. Background value of z is 10.](image-url)
as in Bott (1989a). Results are shown in Fig. 5 after one and after six revolutions of the cone, corresponding to 628 and 3768 iterations, respectively.

Figures 5a,b, depicting the distributions of the positive-definite approach, show that in this particular situation very little over- and undershooting occurs. Due to the low phase-speed errors of APF4, the distributions are very symmetrical. The corresponding results of MAPF4 are shown in Figs. 5c,d. In contrast to the positive definite approach, in MAPF4 no over- and undershooting of $z$ is observed during the whole simulation. (Note that in APF4 at some time steps $\max [z_{n,j}^p] < \max [z_{n,j}^{p+1}]$.) However, now the maxima of the curves are somewhat lower and the distributions are not as symmetrical as in APF4. Nevertheless, the amplitude and phase-speed errors of MAPF4 are still so small that even in situations with moderate spatial gradients of the transported quantities, this model version yields very good results.

By comparing Figs. 5a–d with the analytical solution of Fig. 4, it can be seen that in both advection schemes the largest difference between the numerical and the analytical solution evolves during the first rotation of the cone. The additional numerical diffusion produced after the first rotation is very small, resulting in very similar distributions after 628 and 3768 iterations. This is, for instance, reflected by the maxima of $z$, which for APF4 (MAPF4) are 92% (86%) after 628 iterations and 90.3% (79.2%) after 3140 additional iterations. The observed phenomenon is typical, especially for all versions of APF and MAPF with higher-order polynomials. It is explained by the fact that after a few

Fig. 7. (a) Numerical solution of the rotating-cylinder test with APF4 after 628 iterations. (b) Same as (a) but after 3768 iterations. (c) Numerical solution of the rotating-cylinder test with MAPF4 after 628 iterations. (d) Same as (c) but after 3768 iterations.
advection steps, the sharp contours of the initial distributions are smoothed by the polynomial approach of the advection schemes. In the remaining advection steps, the smoothed distributions are very well reproduced by the polynomials so that the numerical diffusion drastically ceases. These findings are in close agreement with the results of the one-dimensional advection tests with single Fourier modes of different wavelength, as presented in Bott (1989a). Here it has been shown that phase and amplitude errors are largest for the short wavelengths.

To clarify the advantages of MAPF4 as compared to APF4, the cone of Fig. 4 has been replaced by a cylinder with the same base radius and a height of \( z = 110 \). The background value is now \( z = 10 \) (see Fig. 6). The numerical solutions after one and after six revolutions of the cylinder are shown in Fig. 7. In contrast to the rotating-cone test, now APF4 yields distinctly higher over- and undershooting values of \( z \). In the first advective steps, the minimum in the \( z \) field decreases to zero. After the first revolution, \( \min(z_{ij}^{28}) = 2.46 \) and is slightly increasing to 2.91 at the end of the simulations. In the beginning, the maximum overshooting is more than 10% of the initial maximum, decreasing to approximately 9% after one and after six rotations.

Obviously, in the test with the rotating cylinder, the results obtained with MAPF4 are much better than those of APF4, because no ripples are observed in the \( z \) distributions, (see Figs. 7c,d). However, as in the previous model runs, the numerical diffusion is somewhat larger, yielding at the boundaries of the cylinder not-so-steep gradients as in the positive-definite approach. Similar to the rotating-cone test, during the first advective steps in both model versions, a typical \( z \) distribution evolves, which remains relatively constant in the long-term run.

b. Arbitrary flow fields

In uniform flow fields, the quality of the model results can easily be examined by comparing them to the analytical solutions. In deformational flow fields, this is not necessarily the case, because very often the analytical solution of the problem is not known. However, for the deformational flow field test of Smolarkiewicz (1982), Staniforth et al. (1987) provide the analytical solution so that this test is a very good tool to investigate the performance of MAPF4 in arbitrary flow fields.

In Bott (1989a), the deformational flow field test has already been carried out for the positive definite version of the integrated flux form of Tremback et al. (1987). By comparing at different time steps the numerical results with the corresponding solutions of Staniforth et al., it has been shown that in the short-term solution the positive-definite approach reproduces the analytical solution very well. In the long-term run— that is, when the analytical solution is no more resolvable by the numerical grid mesh—very weak numerical instabilities evolved in the model simulations.

The deformational flow field test has been repeated with APF4 and MAPF4. Since the numerical results are very similar to the results obtained in Bott (1989a), their presentation will be omitted here. Instead, the test has been performed after superimposing the cone on a constant background value of \( z = 1 \). Thus, the initial \( z \) distribution is similar to Fig. 4, with the only difference that now the cone is located in the center of the model domain. The numerical time step is \( \Delta t = 0.7 \).

From Fig. 8, it is seen that the field of the streamfunction is given by six central vortices, three vortices at the left and right boundary and eight half-vortices at the upper and lower boundary of the model domain. In the case of cyclic boundary conditions, the eight half-vortices are linked, yielding four vortices. Model results will be presented for MAPF4 with cyclic boundary conditions.

Since at the boundaries of the vortices no outflow occurs, in each vortex the \( z \) distribution does not depend on the values in the neighboring vortices. As a consequence, in the central vortices it is expected that the solution behaves analogously to the situation with zero background value of \( z \). In the remaining model domain, the analytical solution is simply \( z(t) = z(t_0) = 1 \) because the deformational flow field is nondivergent, that is, \( \nabla \cdot \mathbf{v} = 0 \).

Figure 9 shows the resulting \( z \) distributions at different timesteps of the simulations. As can be clearly seen from this figure, the short-term solutions of the six central vortices behave very similarly to the situation

![Fig. 8. Streamlines of the deformational flow field test.](image-url)
with zero background value of $z$ [compare Figs. 9a–d with the corresponding Fig. 3 of Staniforth et al. (1987)]. In each vortex, the $z$ content is rotating along the streamlines, clockwise in $H$ vortices and counterclockwise in $L$ vortices. However, in the outer vortices of the model domain, the initially constant $z$ distributions start to evolve into the typical egg-cup structure. This is explained by the fact that the monotone flux limitation has been derived for one-dimensional flow fields and that in multidimensional flows the time-splitting method is applied. Due to this treatment, the scheme is strictly monotone only if in each direction the transport velocities are constant, as in the uniform flow field experiments presented above. In contrast to this, in the deformational flow field test, $\partial u/\partial x = -\partial v/\partial y \neq 0$. Hence, in the first step of the time-splitting approach, the constant $z$ distribution is deformed by the divergent flow in $x$ (or $y$) direction. If in the second step the $z$ field is moved in the $y$ (or $x$) direction, the deformation of the first step is not canceled because the modified $z$ values yield in each grid box different deformation terms as compared to the first step.

Since in the deformational flow field test over- and undershooting $z$ values result from the use of the time-splitting approach, it is expected that the ripples can be reduced by choosing smaller Courant numbers in the advection experiment. To examine the performance of MAPF4 in this case, the same test has been repeated with the exception that now the time step is $\Delta t = 0.35$. Results are shown in Fig. 10 at time steps corresponding to those of Fig. 9. Obviously, now the numerical instabilities are distinctly lower than in the

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**Fig. 9.** (a) Numerical results of the deformational flow field test with MAPF4 and $\Delta t = 0.7$ after 19 iterations. (b) Same as (a) but after 38 iterations. (c) Same as (a) but after 57 iterations. (d) Same as (a) but after 75 iterations.
previous model run. In the outer vortices, the $z$ distributions are much smoother with values closer to unity. Similar to Fig. 9, the largest deviations from the analytical solution are observed at the boundaries between the vortices. The distributions in the central vortices are again very satisfactory.

As already mentioned by Smolarkiewicz (1982), in the present test the deformation of the flow field is extremely strong. Since under typical atmospheric conditions the deformations are several orders of magnitude smaller, it is concluded that in these situations the performance of MAPF4 is much better than in the advection test presented above. Hence, the monotone advection scheme may be used in arbitrary atmospheric flow fields yielding satisfactory results.

4. Conclusions

On the basis of the area-preserving flux-form advection algorithm of Bott (1989a,b), the monotone version of the scheme has been developed. For this, the non-linear positive-definite flux limiters of the original approach have been replaced by the more restrictive monotone flux limitation. The monotone flux limiters have been determined for the special situation of constant transport velocities in one dimension. To apply the monotone advection scheme in arbitrary flow fields, a correction term has been added. This term has been deduced from the deformation that is produced in the upstream method. In multidimensional flow fields, the time-splitting method is applied. A comprehensive se-

Fig. 10. (a) Numerical results of the deformational flow field test with MAPF4 and $\Delta t = 0.35$ after 38 iterations. (b) Same as (a) but after 76 iterations. (c) Same as (a), but after 114 iterations. (d) Same as (a) but after 150 iterations.
ries of numerical advection experiments has been carried out. From the numerical results of these tests, the following conclusions are drawn.

1) MAPF produces somewhat larger phase and amplitude errors than the positive-definite approach because the monotone flux limitation is more rigorous than the positive-definite flux limiters.

2) In the case of moderate spatial gradients of the transported quantity, APF and MAPF yield very good results because the ripples produced by APF are relatively small and the numerical diffusion of MAPF is not too large.

3) In uniform flow fields with strong spatial gradients, MAPF is clearly superior to APF because the over- and undershooting of APF is undesirably high.

4) In both model versions, the largest numerical diffusion is produced during the first advection steps when the discontinuities of the initial distributions are smoothed by the polynomial approach. In the long-term run, the algorithms are distinctly less diffusive.

5) Due to the time-splitting approach, in multidimensional applications, MAPF is strictly monotone only if the flow fields are uniform. In strong deformational flows, little over- and undershooting occurs. These ripples become smaller for decreasing values of the Courant number. However, under typical atmospheric conditions, the numerical instabilities of the scheme are still so small that MAPF may also be applied in these situations.

Since the monotone advection scheme has certain advantages and disadvantages as compared to the positive-definite approach, it cannot generally be concluded which version is superior to the other. This decision depends on the particular transport problem being involved. However, the mathematical structure of both advection schemes is very similar so that it causes no problems to switch from one model version to the other. Thus, in particular atmospheric applications, it is very easy to find out if APF or MAPF yields better results.

The computational effort of MAPF4 is approximately 10% higher than that of APF4. Since APF4 is only about four times slower than the usual upstream scheme, all versions of APF and MAPF are very attractive for a large variety of applications in atmospheric modeling.

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