

Is Virga Rain That Evaporates before Reaching the Ground?

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ABSTRACT

The visual phenomenon called virga, a sudden change in the brightness of a precipitation shaft below a cloud, is commonly attributed to evaporation of raindrops. It is said to be rain that does not reach the ground. The optical thickness of an evaporating rain shaft, however, decreases gradually from cloud base to ground. Thus, it is more likely that virga results from snowflakes melting in descent. Horizontal optical-thickness decreases of more than ten can occur in the short distance over which a snowflake is transformed into a raindrop. This decrease is caused by two factors: a smaller number density of hydrometeors because of the greater fall velocity of raindrops than of equivalent snowflakes, and a smaller scattering cross section: the first of these is dominant. An alternative explanation of virga is that it is precipitation that has not yet reached (rather than does not reach) the ground. This is a plausible explanation given the long time periods it may take hydrometeors, especially snowflakes, to descend from cloud to ground.

1. Introduction

A typical definition of virga is given by Petterssen (1958): "Once in a while, rain is seen to fall out of clouds and evaporate completely before it reaches the ground. Such streaks of precipitation are called *virga*." This definition or slight variations on it can be found in many meteorological textbooks in which virga is mentioned. When we turn to more scholarly monographs, however, we may encounter broader definitions. For example, Mason (1957) avers that "ice clouds . . . often appear as fibrous streaks . . . known as Fallstreifen or virga, which may be regarded as precipitation." The *Glossary of Meteorology* (Huschke 1959) repeats in its essentials Mason's definition of virga as a synonym for fall streaks, but adds that "virga . . . is discernible below the bases of high-level cumuliform clouds from which precipitation is falling into a dry subcloud layer."

By virga we do not mean fall streaks. A useful distinction can be and usually is made between fall streaks and virga. By virga we mean a sudden change in brightness of a precipitation shaft below a cloud. It seems that the widespread attribution of this brightness change to evaporation of raindrops has never been put to a test, either experimental, observational, or theoretical. The assertion of Kon et al. (1973), that "ice crystals began to melt and the virga became invisible at the 0°C layer, because of the decrease in size and

the increase in fall velocity when the ice crystals were melted to small rain drops besides the rapid vaporization," however, contains a hint of doubt about the explanation of virga as solely a consequence of rain suddenly evaporating before it reaches the ground.

In the following sections we analyze in detail this explanation and offer two alternative ones. Our analysis necessarily entails approximations and assumptions, some explicit, others implicit. In making them we try to give the evaporating-rain explanation of virga the benefit of the doubt. Some of the ideas explored in quantitative detail in this paper have been given in outline form in a recent popular article by Bohren (1991).

2. Optical thickness and rainfall rate at cloud base

All of our analysis is based on the Marshall–Palmer (1947) distribution for the number of raindrops $N(D)\Delta D$ per unit volume with diameters in the range between D and $D + \Delta D$:

$$N(D) = N_0 e^{-\Lambda D}. \quad (1)$$

Here $N_0 = 0.08 \text{ cm}^{-4}$ and $\Lambda = 4.1 R^{-0.21} \text{ mm}^{-1}$, where R is the rainfall rate (mm h^{-1}). The parameter Λ is the inverse of the mean diameter $\langle D \rangle$ (the first moment of the size distribution). We are aware of the limitations of this distribution, but use it because it is simple and familiar. Moreover, raindrop-size distributions at cloud base obtained by Blanchard (1953) have the same form as the Marshall–Palmer distribution: a monotonically decreasing number of drops with increasing diameter. One of the drawbacks of this distribution is that it applies to raindrop diameters at the

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ground whereas we are interested in their diameters at cloud base. Unfortunately, it is not possible to infer the size distribution there knowing that at the ground. This is because at cloud base the number of drops smaller than the largest drop that just makes it to the ground can have any value whatever and still give the observed distribution there. As a crude way of accounting for this indeterminacy, we used somewhat smaller mean diameters in the Marshall-Palmer distribution than might have been used otherwise. This was done on the assumption that evaporation of drops from cloud base to ground shifts the size distribution to larger values; hence, the mean diameter at the ground is larger than that at cloud base. In making this assumption, we err on the conservative side. Although our detailed analysis incorporates the Marshall-Palmer size distribution of raindrops at cloud base, our general conclusions do not require the strict applicability of this distribution to the problem considered.

Our first task was to determine the horizontal optical thickness and rainfall rate at cloud base. Optical thickness is physical thickness measured in units of scattering mean free path, which is inversely proportional to the product of the number density of scatterers and their scattering cross section. Because raindrops are large compared with the wavelengths of visible light and are weakly absorbing, their scattering cross section is, to good approximation, twice their geometrical cross section. The total optical thickness τ per unit horizontal distance of a rain shaft is therefore proportional to the second moment of the size distribution

$$\tau = \frac{\pi}{2} \int_0^\infty N(D) D^2 dD. \quad (2)$$

The rainfall rate R depends on a moment of the size distribution higher than the third determined by how

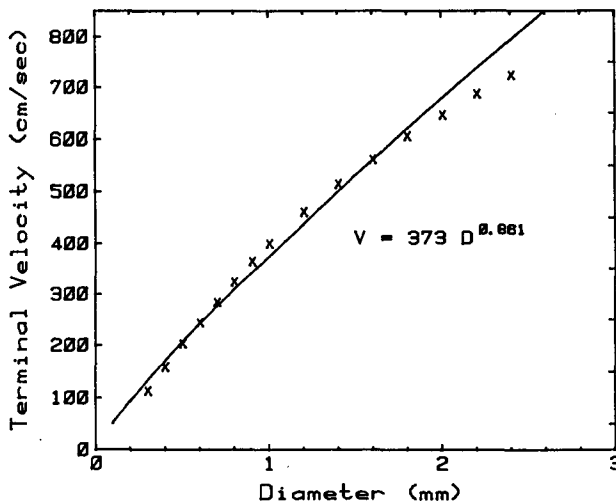


FIG. 1. Least-squares fit of measured raindrop terminal velocities (Gunn and Kinzer 1949) to a power law.

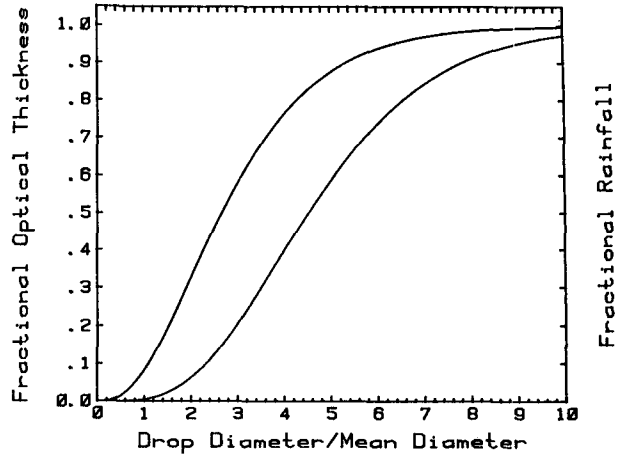


FIG. 2. The left curve shows the fraction of the total horizontal optical thickness of a rain shaft contributed by drops with diameter (relative to the mean) less than the abscissa. The right curve shows the fraction of the rainfall rate contributed by drops with diameter (relative to the mean) less than the abscissa. Both curves were obtained for the Marshall-Palmer (1947) distribution.

the terminal velocity $V(D)$ of a raindrop depends on its diameter:

$$R = \frac{\pi}{6} \int_0^\infty N(D) D^3 V(D) dD. \quad (3)$$

To obtain $V(D)$ we fit a power law to the terminal velocity measurements of Gunn and Kinzer (1949) shown in Fig. 1. The precise value of the exponent of D is of no great moment. Suffice it to say that it is somewhat less than one depending on the range of diameters over which the best fit is required. This means that the rainfall rate is given approximately by the fourth moment of the size distribution.

All drops do not contribute equally to the optical thickness and rainfall rate. Figure 2 shows the fraction of the total optical thickness and rainfall rate contributed by drops with diameters less than any given diameter relative to the mean.

The median diameter for optical thickness is about three times the mean diameter, whereas the median diameter for rainfall is slightly less than five times the mean diameter. Thus, without having done more than a few simple calculations, we already arrive at food for thought: a rain shaft can disappear optically before disappearing pluviometrically. That is, if all drops smaller than twice the mean diameter were to be removed from the distribution, the optical thickness would be reduced by 30%, whereas the rainfall rate would be reduced by only 5%. This result by itself sows seeds of doubt that virga is rain that literally does not make it to the ground.

3. Optical thickness and rainfall as a function of altitude

The key to the visual phenomenon called virga is the optical thickness of a rain shaft as a function of

height from cloud base to ground. In turn, this requires knowing how a drop of diameter D_h at cloud height h is transformed by evaporation into a drop of diameter D_z at height z . We assume that only evaporation effects this transformation, that there are no other mechanisms shifting the size distribution.

With this assumption, the size distribution N_z at any height z is obtained from the distribution N_h at cloud base by a simple transformation of variables:

$$N_z(D_z) = N_h(D_h) \frac{dD_h}{dD_z}, \quad (4)$$

$$\int_{D_m}^{\infty} N_h(D_h) dD_h = \int_0^{\infty} N_z(D_z) dD_z, \quad (5)$$

where D_m is the diameter of the smallest drop at h that just makes it to z . The total number of drops at any height is not a conserved quantity because all drops with diameters less than some maximum value at cloud base disappear before reaching a given distance below cloud.

To determine the transformation of drops of diameter D_h to those of diameter D_z , we used Best's (1952) expression for the rate of change of drop diameter with height:

$$D \frac{dD}{dz} = C(1 - f)^{1.13} e^{-\alpha z} = \frac{g}{2}, \quad (6)$$

where f is the relative humidity. Best took the lapse rate to be $6.5^\circ\text{C km}^{-1}$ and considered two surface temperatures, 15° and 41°C . One temperature too low for our purposes, and the other too high, so we fit his values for the coefficients C and α to a linear function of surface temperature.

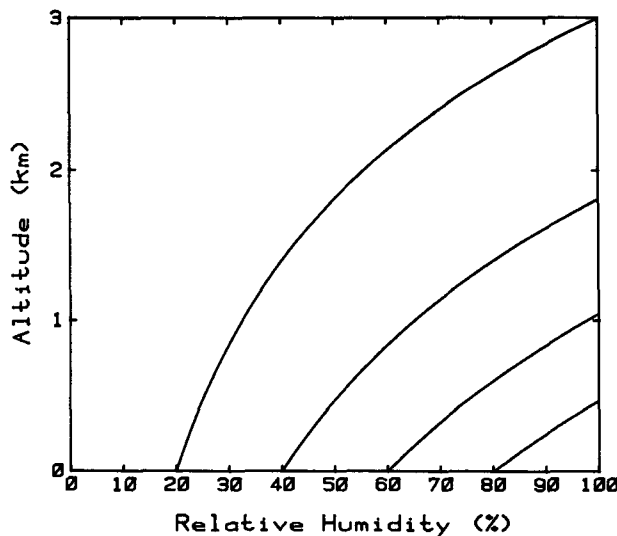


FIG. 3. Relative humidity profiles for constant mixing ratio from ground to cloud base. The surface temperature is 25°C .

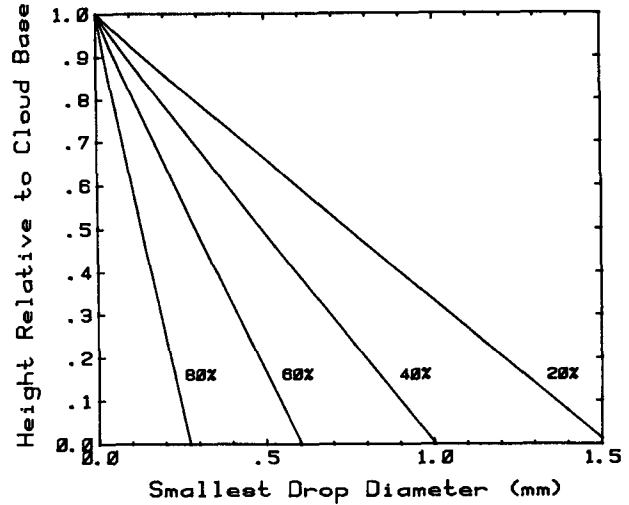


FIG. 4. Smallest raindrop that reaches a given height from cloud base for a surface temperature of 25°C and surface relative humidities between 20% and 80%.

From the expression g , which reflects the change of temperature and humidity with altitude, we can find by integration the transformation of a drop of diameter D_h at cloud base to a drop of diameter D_z at height z . This transformation is determined solely by the diameter D_m of the smallest drop at cloud base that just makes it—with zero diameter—to height z :

$$D_h^2 = D_z^2 + D_m^2, \quad (7)$$

$$D_m^2 = \int_z^h g(z) dz. \quad (8)$$

Best assumed a constant relative humidity from cloud base to ground. This was both unnecessary and a bit unrealistic, so in integrating the function g , we included a humidity profile.

The humidity profile is that appropriate to a constant mixing ratio from ground to cloud base. This is a reasonable assumption for the convective clouds of interest here. With the assumption of a well-mixed subcloud layer, the humidity profile depends on only the surface temperature and relative humidity (Fig. 3). We took the lapse rate to be $9.8^\circ\text{C km}^{-1}$, which is greater than the lapse rate underlying (6). By combining Best's expression for drop evaporation based on a lapse rate of $6.5^\circ\text{C km}^{-1}$ with a humidity profile corresponding to a lapse rate of $9.8^\circ\text{C km}^{-1}$, we overestimate the rate of drop evaporation.

Not surprisingly, the minimum drop diameter varies more over the range of surface humidities than over the range of surface temperatures (Figs. 4 and 5). For a surface relative humidity of 0%, no drop can make it from cloud to ground because the cloud base is at infinity. For a surface relative humidity of 100% all drops can make it to ground. Because the dependence of minimum drop diameter on surface temperature is

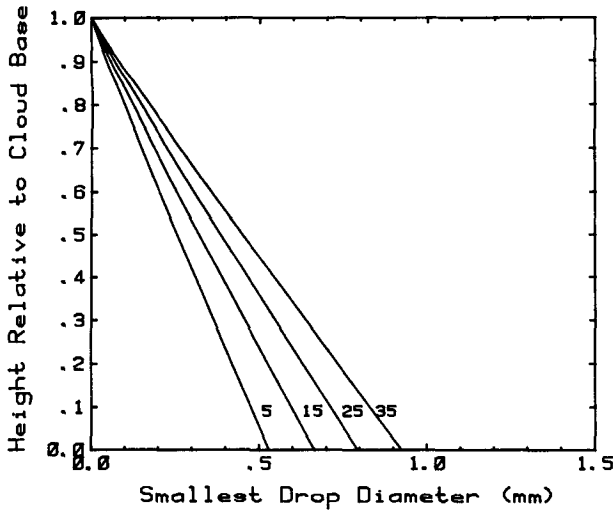


FIG. 5. Smallest raindrop that reaches a given height from cloud base for a surface relative humidity of 50% and surface temperatures between 5° and 35°C.

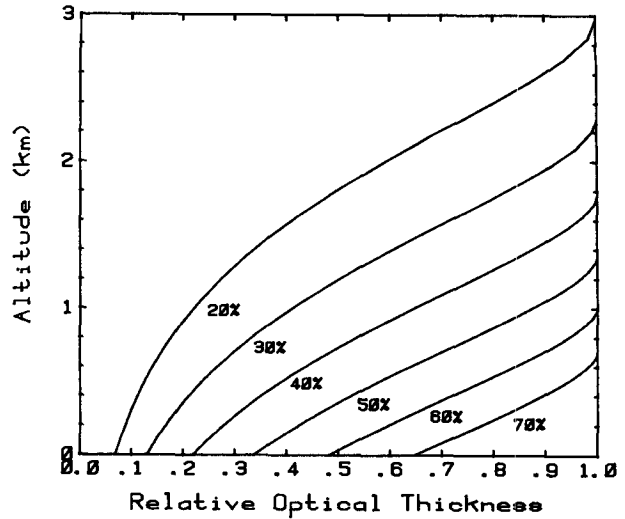


FIG. 6. Horizontal optical thickness relative to the value at cloud base for surface relative humidities between 20% and 70%. The mean drop size is 0.35 mm, and the surface temperature is 25°C.

not so great, we took the surface temperature to be 25°C for all the results presented.

The optical thickness τ_z as a function of altitude is determined from (2) with N replaced by N_z :

$$\tau_z = \frac{\pi}{2} \int_0^\infty N_z(D_z) D_z^2 dD_z. \quad (9)$$

At cloud base the size distribution is taken to be the Marshall–Palmer distribution. Because of unequal rates of evaporation of falling raindrops, this distribution is not maintained. The size distribution N_z at z is obtained from the Marshall–Palmer distribution N_h at h by (4), where the relation between D_h and D_z is given by (7) and (8). The resulting optical thickness as a function of altitude is given by

$$\tau_z = \frac{\pi}{2} N_0 \int_0^\infty \frac{D_z^3}{(D_z^2 + D_m^2)^{1/2}} \times \exp[-\Lambda(D_z^2 + D_m^2)^{1/2}] dD_z. \quad (10)$$

We do not account for the fact that the number density of drops increases because their velocity decreases as they evaporate. This errs on the side of underestimating the optical thickness. The rainfall rate is determined from (3) in the same way that (10) was obtained from (1), (2), (4), (7), and (8).

Figure 6 shows rain-shaft optical thickness relative to the cloud-base value as a function of altitude for various surface relative humidities. The surface temperature is 25°C, and the mean drop diameter is 0.35 mm, which in the Marshall–Palmer scheme corresponds to a rainfall rate of 5 mm h⁻¹. The optical thickness decreases from cloud base downward, and this decrease depends on the surface humidity. But the change in optical thickness is smooth; it experiences no sharp changes. Note that the slope of these curves

is about the same for all humidities over much of the distance from cloud to ground.

As expected from the simple results in the second section of this paper, the rainfall rate does not decrease as quickly as the optical thickness (Fig. 7). As a rule of thumb, if the optical thickness at the ground is some fraction of its value at cloud base, the rainfall rate at the ground relative to that at cloud base is 50% greater than this fraction.

How does the changing *relative* optical thickness with altitude translate into what is observed? This depends very much on the *absolute* optical thickness. If the optical thickness at cloud base is 1000, a decrease of a factor of 10 from cloud base to ground will have

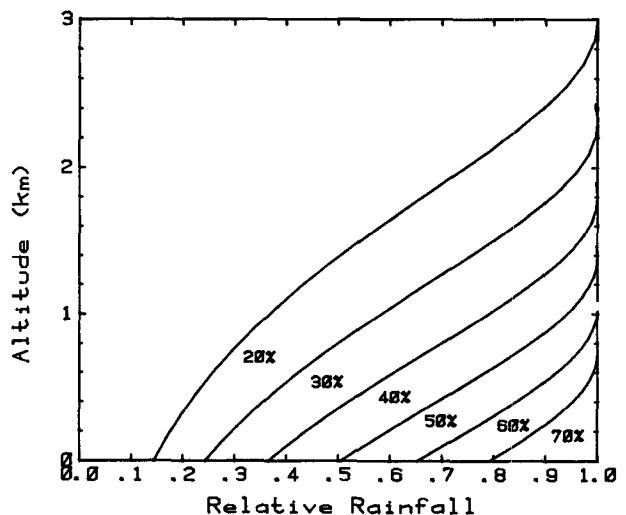


FIG. 7. Rainfall rate relative to the value at cloud base for surface relative humidities between 20% and 70%. The mean drop size is 0.35 mm, and the surface temperature is 25°C.

little observable effect. If the optical thickness at cloud base is 0.1, a decrease of a factor of 10 again corresponds to little change in the observable characteristics of the rainshaft.

According to the Marshall–Palmer size distribution, the optical thickness of rain with mean drop diameter 0.35 mm is about 1 km^{-1} . So we took the horizontal thickness of the rain shaft at cloud base to be 1 km—which is a reasonable value whereas 10 m and 100 km are not—and calculated the rain-shaft transmittance as a function of altitude (Fig. 8). The transmittance from cloud base to ground varies by at most a factor of 3, but this change occurs over several kilometers rather than abruptly. We also did calculations similar to those shown in Fig. 8 for rain-shaft optical thicknesses at cloud base of 2, 3, and 5. The maximum gradient of transmission versus altitude increased by no more than about 35%.

These calculations are based on exponential attenuation, which occurs only if multiple scattering is negligible. This is a good assumption for small optical thicknesses. If multiple scattering is accounted for, attenuation does not change as rapidly with optical thickness. Assuming exponential attenuation therefore overestimates the slope of the transmission curve. Consider reflection by a rain shaft. If it is everywhere optically thick, its reflectance does not change much along its length: this is another example where large optical thicknesses smooth out gradients.

On the basis of our analysis, and subject to the various caveats that underlie it, we conclude that a sudden change in the brightness of a rain shaft below cloud base cannot be attributed to the complete evaporation of raindrops. By a sudden change we mean one that occurs over a distance small compared with that from cloud to ground.

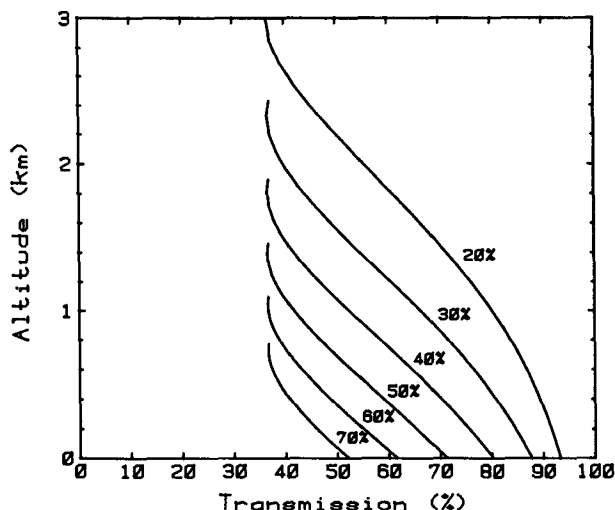


FIG. 8. Horizontal transmission by a rain shaft, assuming an optical thickness at cloud base of 1, for surface relative humidities between 20% and 70%. Surface temperature is 25°C.

4. Melting snowflakes: An alternative mechanism

We must turn elsewhere for an explanation of sudden brightness changes of what appears to be an evaporating rain shaft. In particular, it could just as well be a snow shaft. Therefore, we must ask ourselves how the optical thickness of a snow shaft changes as it descends below the 0°C isotherm.

There are two components to this change: the change in the scattering cross section of a single snowflake relative to that of the drop obtained by melting it and the change in the number density of scatterers because of a change in terminal velocity. We consider each of these in turn and combine them.

The scattering cross section of a weakly absorbing particle, large compared with the wavelength of the light illuminating it, is twice its geometrical cross section; the average projected area of a convex particle is one-fourth its surface area (van de Hulst 1957). We therefore took the average scattering cross section of the snowflakes considered to be one-half their surface area. The optical thickness per unit physical pathlength is the product of this cross section and the number density of snowflakes.

We considered three types of idealized snowflakes: hexagonal columns, plates, and stars. The stars were chosen to crudely represent a dendritic snowflake. Figure 9 shows the scattering cross section of a snowflake relative to that of the drop obtained by melting it as a function of aspect ratio, which is approximately the ratio of maximum to minimum linear dimension. Although the scattering cross section of a hexagonal column is almost equal to that of an equivolume sphere, the scattering cross sections of hexagonal plates and stars can be two or more times greater. Thus, the optical thickness of a snow shaft can decrease upon melting by about a factor of 2 because of a change in scattering cross section. The change in the optical thickness because of decreasing number density is generally larger.

When a snowflake melts, its aerodynamic drag decreases markedly; hence, its terminal velocity increases. One of the consequences of this is the *bright band* of radar meteorology (Battán 1973). Snow melts when it falls below the 0°C isotherm. At radar frequencies, the difference between the dielectric functions of ice and liquid water is sufficiently large that the backscattering cross section of raindrops is markedly greater than that of their parent snowflakes. But the greater fall velocity causes the number density of hydrometeors to decrease. The result is an abrupt increase in radar reflectivity because of the phase change followed by a decrease because of the reduced number density. At optical frequencies, in contrast, the snow-shaft optical thickness only decreases below the 0°C isotherm because of a lower scattering cross section and number density of hydrometeors.

Figure 10 shows the terminal velocity of snowflakes relative to that of equivolume spheres obtained from Langleben's (1954) measurements. Here diameter re-

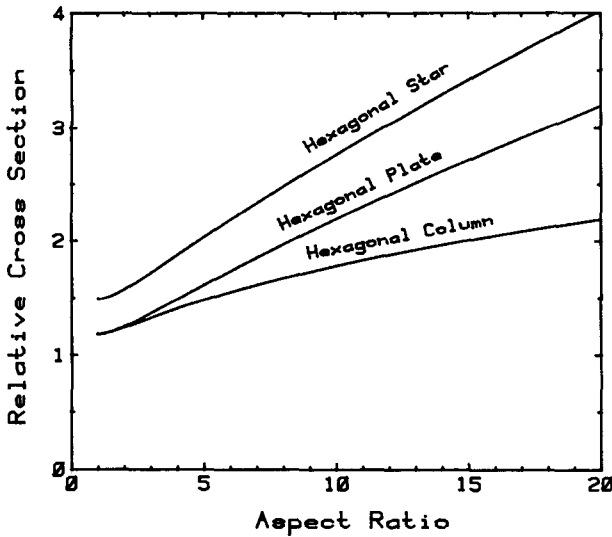


FIG. 9. Scattering cross sections of hexagonal stars, plates, and columns relative to that of equivolume spheres. Aspect ratio is approximately the ratio of maximum to minimum linear dimension.

fers to that of the drop obtained by melting the snowflake. For dendrites especially, the change in terminal velocity can be quite large, almost a factor of 10 depending on snowflake volume.

If snowflakes fall one by one separated by a fixed distance, the separation between the drops that results from melting increases in the ratio of the terminal velocity of the drop to that of its parent snowflake. Thus, the number density of drops decreases by this ratio. Optical thickness is compounded of both cross section and number density. If melting decreases the scattering cross section by a factor of 2 and decreases the number density of scatters by 5, the decrease in optical thickness is 10.

Figure 11 shows the total optical thickness change on melting of snowflakes resulting from both a decrease in scattering cross section and in number density. To obtain these curves we combined Langleben's terminal-velocity data for dendrites with our calculations of scattering cross sections for hexagonal stars. For an equivalent drop diameter of 1 mm, the optical thickness of a monodisperse collection of dendrites is more than ten times that of the drops obtained by melting them. Although this kind of optical thickness can be obtained by evaporation of raindrops, it requires a fall from cloud base to ground, whereas the change in optical thickness upon melting occurs over a small fraction of this distance, the region over which snowflakes melt. Thus, we conclude that abrupt changes in the optical thickness of what appears to be a rain shaft below a cloud is more likely attributable to melting of snowflakes than to evaporation of raindrops.

Our calculations were done for ice crystals with simple shapes. If the crystals are more complex, the change in terminal velocity and scattering cross section on melting are even greater. The measurements of Jayaweera (1972) and of Kajikawa (1972) show that stellar and dendritic crystals have smaller terminal velocities than hexagonal plates with the same mass. Moreover, stellar and dendritic crystals will have larger surface to volume ratios and, hence, larger scattering cross sections per unit volume than hexagonal plates. When we turn to crystal aggregates, the changes upon melting can be even greater. Pruppacher and Klett (1980) assert that early measurements of terminal velocities of spatial ice crystals and aggregates "show inconsistencies and cannot be considered reliable. . . . However, these early data did show that most crystal aggregates fall with speeds of 1.0–1.5 cm s⁻¹, which increases very little with increasing size." (The units given here should have been meters per second.) Aggregates that melt to

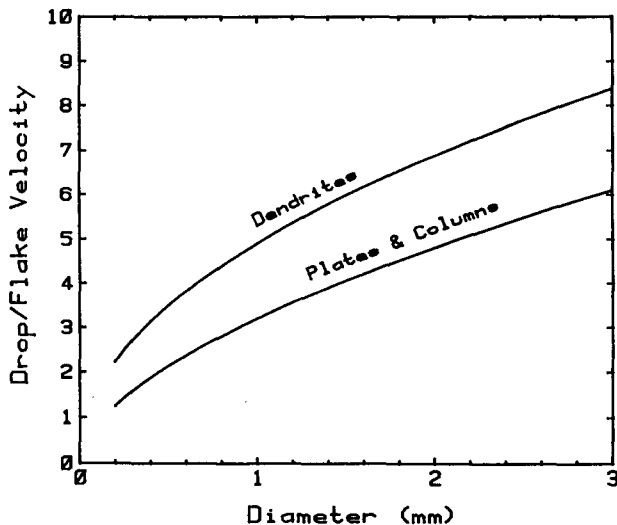


FIG. 10. Terminal velocity of raindrop relative to that of parent snowflake. Data are from Langleben (1954).

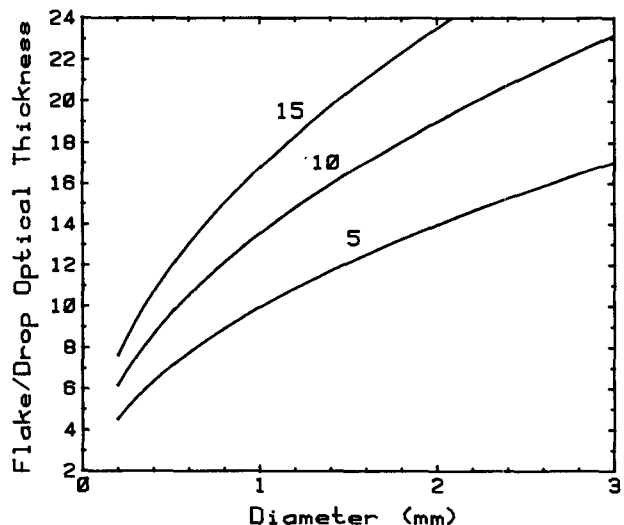


FIG. 11. Optical thickness of snowflakes (hexagonal stars) with aspect ratios of 5, 10, and 15 relative to that of equivolume spheres.

form drops with diameters in the range 1–2 mm undergo an increase in terminal velocity of factors between about 3 and 7 (see Fig. 1). When the concomitant decrease of number density is coupled with a decrease in scattering cross section of more than a factor of 3, the result is an optical thickness decrease upon melting of factors between about 10 and 20. The other extreme is graupel. In contrast with large crystal aggregates, graupel is not expected to change markedly in optical thickness when it melts because neither its shape nor its terminal velocity change appreciably.

There is another possible explanation of what is observed: virga is rain that has not yet reached the ground. Everything takes time, including the descent of raindrops and snowflakes. We took 0.35 mm as the mean diameter for the Marshall–Palmer distribution. As far as optical thickness is concerned, the median diameter is about three times the mean, about 1 mm. According to the Gunn and Kinzer data, a raindrop with a diameter of 1 mm has a terminal velocity of about 400 cm s⁻¹ (see Fig. 1). If cloud base is at 3000 m, it takes about 750 s for such a drop to reach ground, assuming that its terminal velocity does not change (which, of course, it does: it decreases because of evaporation). Thus, at least 15 min are required for a 1 mm drop to fall 3000 m, and more than an hour is required for a dendritic snowflake of the same volume to fall the same distance. Even 15 min is long compared with the time that anyone but a dedicated and patient observer would devote to staring at rain or snow falling from a cloud. To the casual observer, a precipitation shaft that appears to disappear before reaching the ground might instead be one that has not yet reached it.

The mere fact that it takes time for a raindrop (or snowflake) to reach the ground would not, in itself, account for an abrupt change in optical thickness. The range of fall velocities would, in the absence of any other influences, result in a smoothly varying optical thickness with height contributed to by a few big drops at the bottom of the shaft and by more and a broader distribution of drops upward from it. Yet, as Scorer (1958) points out, there is a simple mechanism that sharpens the transition across the base of a shaft. Falling rain drags air down with it so that within a shaft the terminal velocity of each drop is the sum of its terminal velocity in still air and the downdraft velocity. Having a downward velocity greater than the descending air, raindrops constantly fall out of the leading edge of the shaft and, hence, decelerate in the still air below. This results in a convergence of rain at the leading edge of the shaft. Although there would be a smooth transition in the absence of a downdraft, the accumulation of rain at the base of a downdraft may produce the abrupt visual transition that characterizes virga resulting from rain that has yet to reach the ground. The convergence of raindrops because of a velocity decrease across the leading edge of a descending shaft is the reverse of the

divergence of snowflakes because of a velocity increase upon melting.

5. Conclusions

Evaporating rain shafts do change in optical thickness with distance below cloud, which can result in observable consequences. But this change is gradual rather than abrupt. Our analysis suggests that evaporation of hydrometeors is not responsible for a sudden change in brightness of a precipitation shaft below a cloud. A more plausible explanation is that this change marks the melting line. The optical thickness and, hence, the amount of transmitted background light of a snow shaft can easily change by a factor of 10 or more over the short distance required for snowflakes to melt, whereas the same change resulting from evaporation of raindrops occurs only over the entire distance from cloud base to ground.

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