A Two Time-Level, Three-Dimensional, Semi-Lagrangian, Semi-implicit, Limited-Area Gridpoint Model of the Primitive Equations. Part II: Extension to Hybrid Vertical Coordinates

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ABSTRACT

A two time-level, three-dimensional, semi-Lagrangian semi-implicit primitive equation gridpoint model that incorporates a sophisticated physics package and uses hybrid coordinates in the vertical is derived. A simple filter, which is needed to stabilize large time-step forecasts, is introduced. Using it, the model is shown to give accurate 24-h forecasts when integrated over a limited area using a $1.5^\circ \times 1.5^\circ$ Arakawa C grid in the horizontal and 16 levels in the vertical for time steps up to 2 h. Also, it is shown to give accurate forecasts on a $0.5^\circ \times 0.5^\circ$ horizontal grid, again using 16 vertical levels, for time steps up to 40 min, and to be as accurate as, and approximately twice as efficient as, a three time-level semi-Lagrangian scheme.

1. Introduction

In a previous paper (McDonald and Haugen 1992, hereafter called MH), we demonstrated an efficient two time-level semi-implicit and semi-Lagrangian gridpoint model of the primitive meteorological equations, which used the $\sigma$ coordinates of Phillips (1957) in the vertical. In this paper we wish to show how to generalize the semi-Lagrangian discretization to the more problematical system of equations that uses the hybrid vertical coordinate $\eta$ of Simmons and Burridge (1981). In section 2 it is shown how to overcome the difficulties caused by the $\eta$ coordinate by introducing a new vertical velocity $\dot{s}$ and a discretized semi-Lagrangian set of equations is derived. In section 3 an additional filter is introduced that is necessary for the successful implementation of the model. In section 4 the new system is used to integrate the baroclinic equations over a limited area and the forecasts are compared with those of the HIRLAM (High-Resolution Limited-Area Model) Eulerian model [see Kallberg (1990), henceforth called K90]. They are also compared with those of a three time-level $\eta$-coordinate semi-Lagrangian model and of the $\sigma$-coordinate semi-Lagrangian model of MH.

2. Semi-Lagrangian discretization using the $\eta$ coordinate

In this section a derivation is presented of a semi-Lagrangian discretization of the multilevel primitive equations using the $\eta$ coordinate in the vertical. Three guiding principles have been used: (a) that the linearized system is formally the same as that of the $\eta$-coordinate HIRLAM Eulerian model; (b) that in the $\sigma$ limit the set of equations should be as close as possible to those of MH; and (c) that the operator $\nabla \cdot \nabla \nabla$ should not be used explicitly even in the unlinearized terms.

Unless otherwise stated, the notation is conventional. See appendix A for definitions of the symbols.

As far as the vertical discretization is concerned, the wind $\mathbf{v}_k$, temperature $T_k$, humidity $q_k$, and linearized geopotential height $G_k$ are defined at the full levels ($k = 1, K$). The pressure $p_{k+1/2}$, geopotential height $\Phi_{k+1/2}$, and vertical velocities $\eta_{k+1/2}$ and $z_{k+1/2}$ are defined at the half-levels. For the horizontal discretization, the Arakawa C grid is used.

The hybrid coordinate is defined in terms of the pressure as follows:

$$p_{k+1/2} = A_{k+1/2}(\eta) + B_{k+1/2}(\eta) p_s(\lambda, \theta) \quad (1)$$

where the choice of $A$ and $B$ defines the closeness of the system to the $\sigma$ coordinates ($A = 0$) or $p$ coordinates ($B = 0$). See appendix B for the values used in tests in this paper.

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a. The continuity equation

The continuity equation in hybrid coordinates is

$$\left( \frac{d_t H}{dt} + D \right) \frac{\partial p}{\partial n} + \partial \left( \frac{\partial p}{\partial \eta} \right) = 0, \tag{2}$$

where $D$ is the divergence and

$$\frac{d_t H}{dt} = \frac{\partial}{\partial t} + \frac{u}{a \cos \theta} \frac{\partial}{\partial \lambda} + \frac{v}{a} \frac{\partial}{\partial \theta}. \tag{3}$$

When discretized in the vertical, Eq. (2) gives

$$\frac{d_t H \Delta p_k}{dt_k} = -D_k \Delta p_k - \left( \frac{\partial p}{\partial \eta} \right)_{k+1/2} + \left( \frac{\partial p}{\partial \eta} \right)_{k-1/2}. \tag{4}$$

Substituting $\Delta p_k = \Delta A_k + \Delta B_k p_s$, and dividing by $p_s$ yields

$$\Delta B_k \frac{d_t H \ln p_s}{dt_k} = -D_k \frac{\Delta p_k}{p_s} - (\hat{s}_{k+1/2} - \hat{s}_{k-1/2}) \tag{5}$$

where

$$\hat{s}_{k-1/2} = \frac{1}{p_s} \left( \frac{\partial p}{\partial \eta} \right)_{k-1/2}. \tag{6}$$

Separating into “linear” and nonlinear terms gives, for Eq. (5),

$$\left( p' \Delta B_k \right) \frac{d_t H \ln p_s}{dt_k} = (L_p + N_p)_k, \tag{7}$$

where $p'$ is a constant reference pressure and $\Delta p_k = \Delta A_k + \Delta B_k p'$,

$$(L_p)_k = -D_k - p' \left( \frac{\Delta s}{\Delta p'} \right)_k, \tag{8}$$

with

$$\left( \Delta s \right)_k = \hat{s}_{k+1/2} - \hat{s}_{k-1/2} \tag{9}$$

and

$$(N_p)_k = \left( 1 - \frac{\Delta p_k p'}{\Delta p'_k p_s} \right) D_k. \tag{10}$$

Although the second term in $L_p$ is not linear, this will not cause any problem when discretizing; see Eq. (13) below.

A semi-Lagrangian discretization of Eq. (7) that uses the approximation called “centered Lagrangian explicit (2)” in MH for the nonlinear term is

$$\left( p' \Delta B_k \right) [(\ln p_s)^{n+1} - (\ln p_s)_s]_{s_{k+1/2}}$$

$$= \left( \frac{\Delta t}{2} \right) [(L_p)^{n+1} + (N_p)^{n+1/2}]_{s_{k+1/2}}$$

$$+ \left( \frac{\Delta t}{2} \right) [(L_p)^n + (N_p)^{n+1/2}]_{s_{k}}, \tag{11}$$

where the subscript $s_{k+1/2}$ (s_k) indicates a two- (three-) dimensional interpolation to evaluate the value of the field at the departure point. The uncentered scheme of Tanguay et al. (1992) has been included to damp gravity waves and $\Delta t_k = (1 + \epsilon_k) \Delta t$.

First summing over all levels and then summing partially yields the following two equations:

$$\left( \frac{\Delta t}{2} \right) [(\ln p_s)^{n+1} - (\ln p_s)_s]_{s_{k+1/2}}$$

$$= \sum_{j=1}^{K} \left[ A_p - \left( \frac{\Delta t}{2} \right) D^n_{j+1/2} \right] \frac{\Delta p_j}{p'} \tag{12}$$

and

$$\left( \frac{\Delta t}{2} \right) [(\ln p_s)^{n+1} - (\ln p_s)_s]_{s_{k+1/2}}$$

$$= \sum_{j=1}^{K} \left[ A_p - \left( \frac{\Delta t}{2} \right) D^n_{j+1/2} \right] \frac{\Delta p_j}{p'}, \tag{13}$$

where

$$(A_p)_k = \left( p' \Delta B_k \right) [(\ln p_s)_s]_{s_{k+1/2}} + \left( \frac{\Delta t}{2} \right)$$

$$\times \left[ -D^n - p' \left( \frac{\Delta s}{\Delta p'} \right)^n + (N_p)^{n+1/2} \right]_{s_{k}}, \tag{14}$$

with

$$(A_p)_k = \left( p' \Delta B_k \right) [(\ln p_s)_s]_{s_{k+1/2}} + \left( \frac{\Delta t}{2} \right) (N_p)^{n+1/2}. \tag{15}$$

Notice that the term $N_p$ of Eq. (10), which is being integrated explicitly in Eq. (11), could, for large $\Delta t$, be a possible source of instability. In fact, Simmons and Burridge (1981), when integrating an Eulerian $\eta-$coordinate model, found instabilities for some choices of $p'$ for time steps much smaller than are in principle allowed for a semi-Lagrangian scheme. Recall that in the $\sigma$ system no terms in the continuity equation need be integrated explicitly.

The field $\hat{s}$ is computed diagnostically using Eq. (7) at time step zero. At every subsequent time step, Eq. (13) is used to compute $\hat{s}^{n+1}$. The fields $\hat{s}^n$ and $\hat{s}^{n-1}$ are stored for use in the computation of $A_p$ [Eq. (14)], $A_T$ [Eq. (26)], and of the displacements [Eq. (45)].

b. The temperature equation

In the hybrid coordinate system the temperature equation can be written as

$$dT_k = \left[ \frac{\kappa T_v}{1 + (\delta - 1) q} \right] \frac{(\omega/p)_k}{T_v}, \tag{15}$$

where $T_v$ is as defined in MH,

$$\frac{d}{dt} = \frac{d_t H}{dt} + \frac{\partial}{\partial \theta}, \tag{16}$$

and

$$\frac{(\omega/p)_k}{T_v} = \left[ \frac{d_t H}{dt} + \frac{\partial}{\partial \theta} \right] \frac{1}{T_v}. \tag{17}$$
The latter must be estimated at the full levels \( k \). Using the following definitions yields a \( (\omega/p)_k \) identical to that given in the ECMWF documentation manual (Staff 1985, p. 2.18), and in the HIRLAM documentation manual [K90, Eq. (1.3.12)], provided \( \frac{\partial \rho}{\partial \eta} \) at the half-levels is obtained by partially summing Eq. (5) and \( (\Delta \ln p)_k \) is defined as \( (\Delta \ln p)_k = \ln(p_{k+1/2}/p_{k-1/2}) \),

\[
\frac{\omega}{p}_k = W_k(B) \left( \frac{dH \ln p_k}{dt} + W_k(\bar{s}) \right),
\]

where \( W_k \) is an operator that acts as follows,

\[
W_k(F) = \frac{p_s}{\Delta p_k} \left[ (\Delta \ln p)_k F_{k-1/2} + \alpha_k \Delta F_k \right],
\]

where \( \alpha_k = 1 - (\Delta \ln p/\Delta p)_k p_{k-1/2} \) when \( k = 2 \cdot \cdot \cdot K \), and \( \alpha_1 = \ln 2 \). Equation (15) can now be rewritten as

\[
\frac{dT_k}{dt} = \kappa T^0 \left[ W'_k(B) \left( \frac{dH \ln p_s}{dt} + W'_k(\bar{s}) \right) \right] = (N_T)_k
\]

where

\[
(N_T)_k = \left[ \kappa T_v \frac{\omega}{1 + (\delta - 1)q} \right] \left( \frac{\omega}{p} \right)_k - \kappa T^0 \left( \frac{\omega}{p} \right)_k
\]

and

\[
\left( \frac{\omega}{p} \right)_k = \left[ W'(B) \frac{dH \ln p_s}{dt} + W'(\bar{s}) \right]_k,
\]

with

\[
W'_k(F) = \frac{p'_s}{\Delta p'_k} \left[ (\Delta \ln p)'_k F_{k-1/2} + \alpha'_k \Delta F_k \right].
\]

The following semi-Lagrangian discretization has been chosen for Eq. (20),

\[
\left[ T^{n+1} - T^{n+1}_{n+1} \right] = \kappa T^0 W'_k(B)[\ln(p_s^{n+1} - (\ln p_s)'_{n+1/k}]
\]

\[
= \kappa T^0 \left( \frac{\Delta t}{2} \right) W'_k(\bar{s}^{n+1}) + \kappa T^0 \left( \frac{\Delta t}{2} \right) \left[ W'(\bar{s}) \right]_{n+1/k} + \left( \frac{\Delta t}{2} \right) N^{n+1/2}_T + \left( \frac{\Delta t}{2} \right) (N_T)^{n+1/2}_{n+1/k}.
\]

When evaluating \( W'(\bar{s}) \) at time level \( n+1 \), Eq. (13) is used; at other time levels, retained values of \( \bar{s} \) are used. Also, when evaluating \( dH \ln p_s/\Delta t \) in \( N_T \), substitution should be made from Eq. (5), using \( W'_k(B)/\Delta B_k = 0 \), when \( \Delta B_k = 0 \). In this way two desirable objectives are attained: (a) the operator \( (\nabla \cdot \nabla) \) is never used explicitly, and (b) the linearized set of equations is formally the same as the HIRLAM ECMWF Eulerian models [see Eq. (28) below]. The coefficient of \( \ln p_s \) on the left-hand side of Eq. (24) is zero when these substitutions are made and it becomes

\[
T^{n+1}_k + \kappa T^0 \left( \frac{\Delta t}{2} \right) \frac{\Delta \ln p}{\Delta p} \sum_{j,i=1}^{k-1} D^{\ast n+1}_j \Delta p_j + \alpha'_k D^{\ast n+1}_k = (A_T)_k
\]

\[
+ \kappa T^0 \left( \frac{\Delta \ln p}{\Delta p} \right) \sum_{j=1}^{k-1} (A_p)_{j} \Delta p_j + \alpha'_k (A_p)_k,
\]

where

\[
(A_T)_k = -\kappa T^0 W'_k(B)[\ln(p_s)'_{n+1/k} + \left\{ T^n + \left( \frac{\Delta t}{2} \right) \left[ T^n W'(\bar{s}) + (N_T)^{n+1/2}_k \right] \right\}^2_{n+1} + \left( \frac{\Delta t}{2} \right) (N_T)^{n+1/2}_k.
\]

If the variable \( G \) is defined by

\[
G_k = \Phi_s + R_d \sum_{j=1}^k (T \Delta \ln p_j)
\]

\[
+ \alpha' R_d T_k + R_d T^0 \ln p_s
\]

then Eqs. (25), (12), and (27) can be combined to get an equation for \( G^{n+1}_k \) and \( D^{n+1}_k \) in the standard form, written in matrix notation as follows:

\[
G^{n+1} + \left( \frac{\Delta t}{2} \right) (\gamma_T + R_d T^0 \nu \cdot) D^{n+1} = H,
\]

where

\[
H = \Phi_s + \gamma A_T + (\gamma_T + R_d T^0 \nu \cdot) A_p;
\]

\( \Phi, G, D, A_T, \) and \( A_p \) are \( K \)-dimensional vectors and \( \gamma, \tau, \) and \( \nu \) are matrix operators defined in appendix C.

c. The momentum equations

In the hybrid coordinate system, the momentum equations can be written as follows. Divergence damping (Sadourny 1975) has been included

\[
\frac{dv_k}{dt} = -f k \times \nu - \nabla (\Phi - cD) - R_d T_e \nabla \ln p_k,
\]

For consistency, \( \Phi \) and \( \nabla \ln p \) are defined at level \( k \) as

\[
\Phi_k = \Phi_s + R_d \sum_{j=1}^k (T \Delta \ln p_j) + R_d (\alpha T_e)_k
\]

and

\[
(\nabla \ln p)_k = \nabla \left( p_{k+1/2} \ln p_{k+1/2} - p_{k-1/2} \ln p_{k-1/2} \right).
\]
Substituting $G$ from Eq. (27) yields the following set of linearized equations:

$$\frac{du_k}{dt} = -\frac{\partial G_k}{a \cos \theta \partial \lambda} + fu_k + (N_u)_k$$

$$\frac{dv_k}{dt} = -\frac{\partial G_k}{a \partial \theta} - fu_k + (N_v)_k$$

where $G_k = G_k - c_k D_k$, and the nonlinear terms are defined as

$$(N_u)_k = -\frac{\partial(\Phi - G)}{a \cos \theta \partial \lambda}$$

$$- R_d(T_c)_k \frac{\partial \ln p}{a \partial \theta} + \left( \frac{uv}{a} \tan \frac{a}{k} \right),$$

$$(N_v)_k = -\frac{\partial(\Phi - G)}{a \partial \theta}$$

$$- R_d(T_c)_k \frac{\partial \ln p}{a \partial \theta} - \left( \frac{u^2}{a} \tan \frac{\theta}{a} \right).$$

Equations (33) and (34) are now discretized as in MH. Since the derivation is the same it will not be repeated here. The final discretized system is formally identical to that in MH:

$$\left[ u - \left( \frac{\Delta t}{2} \right)^2 \left( \frac{b \partial}{a \cos \theta \partial \lambda} + \frac{e \partial}{a \partial \theta} \right) (M, D) \right]^{n+1}_k = (Z_u)_k,$$

$$\left[ v - \left( \frac{\Delta t}{2} \right)^2 \left( \frac{b \partial}{a \partial \theta} - \frac{e \partial}{a \cos \partial \lambda} \right) (M, D) \right]^{n+1}_k = (Z_v)_k,$$

where

$$Z_u = b \left[ A_u - \left( \frac{\Delta t}{2} \right) \frac{\partial H}{a \cos \theta \partial \lambda} \right]$$

$$+ e \left[ A_v - \left( \frac{\Delta t}{2} \right) \frac{\partial H}{a \partial \theta} \right],$$

$$Z_v = b \left[ A_u - \left( \frac{\Delta t}{2} \right) \frac{\partial H}{a \partial \theta} \right] - e \left[ A_v - \left( \frac{\Delta t}{2} \right) \frac{\partial H}{a \cos \theta \partial \lambda} \right],$$

$$M = (\gamma \tau + R_d T^0 \nu) + 2c/\Delta t_+.$$

Equations (28), (37), and (38) are formally the same as Eqs. (49), (60), and (61) in MH and are solved in identical fashion. See section 3d of MH for a description. Last, the moisture equation is integrated as in MH.

d. Trajectory calculation

The model is encoded on an Arakawa C grid, meaning that $u$, $v$, and $T$ are staggered in the horizontal. The vertical velocity $\dot{s}$ is carried at the height points, that is, at the $T$ points as far as the horizontal discretization is concerned and at the half-levels in the vertical. The first step in doing the trajectory calculations is to interpolate all three velocity components to the $T$ points. Thus, $u$ and $v$ are interpolated in the horizontal using a cubic scheme and $\dot{s}$ is interpolated in the vertical to the "full" levels using a cubic scheme also.

All velocities having been defined at the same grid point, the procedure of Robert (1981) is used to compute the three-dimensional displacements by solving the following equation iteratively:

$$\hat{\alpha}^{(m+1)} = \alpha \left[ \lambda - \hat{\alpha}^{(m)}, \Theta - \hat{\beta}^{(m)}, \eta - \hat{\gamma}^{(m)}, t + \Delta t/2 \right],$$

where

$$\alpha = \left[ \frac{(\Delta t/2) \bar{u}}{a \cos \theta}, \frac{(\Delta t/2) \bar{v}}{a}, (\Delta t/2) \bar{p}, \left( \frac{\Delta \eta}{\Delta p} \right) \right],$$

and the bar indicates that the velocities are valid at the $T$ points. The first guess, $\hat{\alpha}^{(0)}$, is taken as $\alpha(\lambda, \Theta, \eta)$, $t + \Delta t/2$). The displacements are then in the appropriate form to compute the departure-point quantities in Eqs. (14), (26), and the humidity equation. For any field $\psi$ the departure point value is given by

$$\psi \frac{\hat{\gamma}^{(m+1)}}{\psi} = \psi \frac{\hat{\gamma}^{(m+1)}}{\psi} [\lambda - 2\hat{\alpha}^{(m+1)}, \Theta - 2\hat{\beta}^{(m+1)},$$

$$\eta - 2\hat{\gamma}^{(m+1)}].$$

To find the departure-point values of the quantities in Eqs. (42) and (43), the displacements are first interpolated in the horizontal to the $u$ and $v$ points, re-
respectively, using a cubic scheme. In the tests described in section 4, the fields at time level \((n + 1/2)\) are estimated using either a two-level or three-level extrapolation [see Eqs. (47) and (48)]. One iteration of a trilinear interpolation is used when solving Eq. (44).

3. Additional filter

When testing of the \(\eta\) model was begun, it was found to be unstable for large time steps. It was also found that this instability could not be controlled by increasing \(\varepsilon_f\), the uncentering coefficient, which had worked well for the \(\sigma\) model.

Since the quantities being evaluated at time level \((n + 1/2)\) are potential sources of instability, it was decided to filter them as follows. Any field, to be called \(\psi\), required at time level \((n + 1/2)\) is now computed as

\[
\psi(n + 1/2) = \frac{3\psi(n) - \psi(n - 1)}{2}
\]

or

\[
\psi(n + 1/2) = \frac{15\psi(n) - 10\psi(n - 1) + 3\psi(n - 2)}{8}
\]

where

\[
\psi_f(n) = \psi(n) + \varepsilon_N[\psi(n + 1) - 2\psi(n) + \psi(n - 1)].
\]

Only the nonlinear terms and centering to find the departure point are affected by this filter; the linear terms remain untouched. Thus, it should cause minimal decrease in accuracy.

Recall that the Robert (1966)–Asselin (1972) filter acts on the linear terms. Thus, although its filtering equation is formally the same as Eq. (49), it acts in a very different way.

Temperton and Staniforth (1987) demonstrated that using Eq. (48) (with \(\varepsilon_N = 0\)) to compute the velocities at time level \(n + 1/2\) yielded the most accurate trajectories. It does not necessarily provide the most accurate three time-level estimate of the nonlinear terms, however. Duran (1991) has examined this problem in the Eulerian context and has shown that a third-order Adams–Bashforth scheme is required. Thus, one expects only the estimate of the departure point to be improved by the use of Eq. (48). Since it does not degrade the accuracy of the nonlinear terms, however, Eq. (48) is used to compute both those and the trajectory velocities in tests described in section 4.

Duran (1991) also showed that a two-time-level Eulerian semi-implicit scheme using a third-order Adams–Bashforth extrapolation for the nonlinear terms yields a scheme whose physical mode is always unsteady. Gravel et al. (1992) have shown in a more general context that both Eqs. (47) and (48) with \(\varepsilon_N = 0\) produce unstable models if they are used to discretize the nonlinear terms. It is an open question whether there exists an \(O(\Delta t^2)\) three-time-level extrapolation for the nonlinear terms that yields an unconditionally stable scheme. Such a scheme might eliminate the need for filtering.

4. Numerical integrations

a. Testing on a \(1.5^\circ \times 1.5^\circ\) grid

The area, horizontal grid, and the treatment of the boundaries are all as described in section 4a of MH. In the vertical, the atmosphere has again been divided into 16 layers. The values of \(A\) and \(B\) that define the \(\eta\) levels are given in appendix B; \(T = 300\text{ K}\) and \(p = 800\text{ hPa}\). The same dataset, starting from 1200 UTC 29 December 1988, has been interpolated from the \(\sigma\) levels to these \(\eta\) levels, as have the required boundary datasets.

A broad description of the physical parameterization schemes in the model can be found in Gustafsson (1991) and all the fine details are in Kallberg (1990). Briefly, they work as follows.

The vertical diffusion scheme affects the horizontal velocity components, the dry static energy, and the specific humidity. Surface fluxes are determined by means of a drag coefficient formulation, using Monin–Obukhov similarity theory for the atmospheric surface layer. The calculation of fluxes above the lowest model level is based on mixing length formulation using exchange coefficients that depend on the static stability, which is described by means of a Richardson number. The analytic formulas describing the drag and exchange coefficients are those of Louis et al. (1981). The effect of shallow convection is included by defining a modified Richardson number.

The deep convection scheme is of the Kuo type. Evaporation of precipitation below convective layers can occur, but freezing or melting of precipitation is neglected. Stratiform condensation occurs when a grid-box specific humidity exceeds a saturation value.

In the radiation scheme the heating and cooling rates of model layers are expressed as a sum of a “cloudy” contribution and a “clear air” contribution. This applies to both short- and longwave radiation. Any model layer may contain clouds, which are diagnosed from relative humidity.

Surface processes are included by simple prognostic equations describing the evolution of the surface temperature, soil water content, and snow depth. Soil water and temperature are described in three layers, with climatological values assigned to the deepest layer.

As currently implemented, these processes are updated at the arrival points in a split-explicit—or, where appropriate, in a split-implicit—fashion after the semi-Lagrangian and semi-implicit stepping has been completed.
The new filter described by Eq. (47) in section 3 was tested by running a series of 24-h forecasts and comparing them with the verifying analysis. The root-mean-square (rms) height errors were used as a measure of the accuracy of these forecasts. The model was found to be unstable even for modestly large time steps when run with \( \epsilon_N = 0.0 \) and \( \epsilon_F = 0.0 \). With \( \epsilon_N = 0.0 \) and \( \epsilon_F = 0.05 \) stability is established for all time steps up to 3 h, although the rms height errors are larger than those from the forecasts using \( \sigma \) coordinates with these values. See Figs. 1a–d, where they are plotted as diamonds connected by full lines. To get a feeling for the impact of the new filter, these experiments were repeated with \( \epsilon_N = 0.25 \), \( \epsilon_F = 0.05 \). Again the rms height errors are plotted in Figs. 1a–d as squares connected by a dashed line. The improvement is mainly at 100 hPa for the \( \Delta t = 3 \) h and \( \Delta t = 2 \) h forecasts. There are also smaller improvements at the other levels for \( \Delta t = 3 \) h. It should be pointed out that for \( \Delta t = 3 \) h, a similar decrease in noise could have been attained by using \( \epsilon_F = 0.3 \) and \( \epsilon_N = 0.0 \), so that, strictly speaking, the new filter is not needed in this case.

Consider next a case where nonzero \( \epsilon_N \) is essential. In section 4b of MH it was pointed out that it was not possible to obtain a stable forecast using the three-time-level extrapolation

\[
\psi(n + \frac{1}{2}) = \frac{15\psi(n) - 10\psi(n - 1) + 3\psi(n - 2)}{8}
\]

(50)

to evaluate the fields at time level \((n + \frac{1}{2})\). (Even for \( \epsilon_F = 0.4 \) the forecasts were developing instabilities.) Consider instead the filtered extrapolation given by Eq. (48). This enables one to get stable forecasts. For ex-

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**Fig. 1.** (a)–(d) Coarse mesh with physics. Graphs of root-mean-square height differences (m) between the analysis of 1200 UTC 30 December 1988 and the 24-h forecasts verifying at the same time. On the y axis is plotted the rms error (m). On the x axis is plotted \( \Delta t \), the time step. (a) The rms errors at 100 hPa; (b) at 300 hPa; (c) at 500 hPa; and (d) at 1000 hPa. The rms height errors of the HIRLAM Eulerian model are represented by the diamond enclosed in a circle. For the two-time-level semi-Lagrangian model using Eq. (47) and \( \epsilon_F = 0.05 \), and \( \epsilon_N = 0.0 \), see the diamonds; with \( \epsilon_F = 0.05 \), and \( \epsilon_N = 0.25 \), see the squares. For the two-time-level semi-Lagrangian model using Eq. (48) and \( \epsilon_F = 0.1 \), and \( \epsilon_N = 0.2 \), see the pluses; with \( \epsilon_F = 0.1 \), and \( \epsilon_N = 0.3 \), see the X's.
ample, using $\epsilon_\eta = 0.1$ and $\epsilon_\eta = 0.2$ yields the rms height errors displayed in Figs. 1a–d by the pluses joined by dots. Not only are the forecasts stable, but they show a reduction in errors at 300 hPa for time steps larger than or equal to 1.5 h.

The 300-hPa errors at $\Delta t = 3$ h continue to disappoint, however. In MH we speculated that using three time levels to extrapolate to level $(n + \frac{1}{2})$ would reduce the error to about 14 m. A possible explanation is that these dampers, although controlling the gravity waves, are also affecting the meteorological waves adversely for very large time steps.

The values of $\epsilon_\eta$ and $\epsilon_\eta$ can be tuned to produce the best results for a given time step. Larger values are needed to maintain stability at larger time steps. For instance, increasing $\epsilon_\eta$ to 0.3 improves the rms height errors at 500 and 1000 hPa for $\Delta t = 3$ h; however, it disimproves the forecast at 300 hPa for $\Delta t = 2$ h and $\Delta t = 1.5$ h. See the “×” joined by dot-dashes in Figs. 1a–d.

For comparison, the rms height errors for the Eulerian model using a time step of 15 min are displayed as diamonds surrounded by circles in Figs. 1a–d.

The filter can, of course, be applied to the model using sigma coordinates in the vertical. Comparing the two, the main point to emerge is that the forecasts are more unstable when the $\eta$ coordinates are used. This may not be surprising if one considers the term $N_p$ defined in Eq. (10), which is integrated explicitly; see Eq. (11). At the first level in the model, $[1 + \Delta p_0/(\Delta p_0 p)]$ is equal to 0.2 when $p_0 = 1000$ hPa and is equal to -0.333 when $p_0 = 600$ hPa. Recall that in $\sigma$ coordinates this term is equal to zero.

The values of $\epsilon_\eta$ and $\epsilon_\eta$ can also be tuned for the model using sigma coordinates. The outcome is that the level of accuracy attained is the same with either vertical coordinate even though larger values of $\epsilon_\eta$ and $\epsilon_\eta$ may be required when using $\eta$ coordinates. The rms height errors of forecasts using $\sigma$ and $\eta$ coordinates are compared in Fig. 2. (The values of $A$ and $B$ for the $\sigma$ coordinates are...
coordinates are given in appendix B.) For all of these forecasts, $\epsilon_\sigma = 0.1$ and $\epsilon_N = 0.2$. The forecasts using the $\sigma$ coordinates (the "t" joined by dots) are very similar to those using $\eta$ coordinates (the diamonds joined by solid lines). Minor differences appear for $\Delta t = 3 \text{ h}$, but no obvious systematic effect has been found.

For comparison, the rms height errors for the Eulerian model using a time step of 15 min are displayed as diamonds surrounded by circles ($\sigma$) and pluses surrounded by circles ($\eta$) in Figs. 2a–d.

b. Testing on a $0.5^\circ \times 0.5^\circ$ grid

The analysis fields of 1200 UTC 29 December 1988 and the appropriate boundary datasets used in MH were interpolated from the 16 $\sigma$ levels to the 16 $\eta$ levels given in appendix B. Thus, the area and horizontal grid are the same as in MH, where the 500-hPa and sea level analysis charts are displayed. Thus, the forecasts should reach the same level of accuracy as those in MH.

Even more so than with the coarse mesh, the new filter is needed. For example, with $\Delta t = 30 \text{ min}$, stable forecasts were produced using $\sigma$ coordinates with $\epsilon_\sigma = 0.05$ and $\epsilon_N = 0.0$. For the $\eta$ coordinates these parameters must be increased to $\epsilon_\sigma = 0.20$ and $\epsilon_N = 0.30$ to maintain stability. Figure 3 displays the 500-hPa and sea level pressure charts using these parameters. Compare them with Fig. 7 in MH, which displays the same forecasts using the same time step, but using $\sigma$ coordinates and $\epsilon_\sigma = 0.05$, and $\epsilon_N = 0.0$.

For smaller time steps, less damping is required. For example, with $\Delta t = 20 \text{ min}$, $\epsilon_\sigma = 0.1$, $\epsilon_N = 0.1$ gives stable, accurate forecasts. In Table 1 the rms height errors are presented for the various time steps. For $\Delta t$ less than or equal to 20 min, $\epsilon_\sigma = 0.1$ and $\epsilon_N = 0.1$. For $\Delta t$ greater than 20 min $\epsilon_\sigma = 0.2$ and $\epsilon_N = 0.3$. No combination of $\epsilon_\sigma$ and $\epsilon_N$ could be found that stabilized the $\Delta t = 1 \text{ h}$ forecast.

When the two time-level model described in section 2 was being encoded, a three time-level semi-Lagrangian model was also encoded. In section 2, if one replaces $\Delta t/2$ with $\Delta t$ and time level $n$ with $n - 1$, and time
level \( n + \frac{1}{2} \) with \( n \), then the equations describe the three time-level model, except for the Coriolis terms, which are now integrated explicitly. Also, a Robert-Asselin filter was added for the three time-level model. The rms height errors produced by this three time-level model using \( \eta \) coordinates are presented in Table 2. These results indicate that the two time-level model is as accurate as the three time-level model for time steps up to 40 min. Since the computational cost per time step is the same, the two-level model has a big advantage over the three time-level model.

It is interesting to note that the three time-level model is not immune to stability problems. This may not be unexpected because the nonlinear terms are also integrated explicitly for this scheme. For example, with \( \Delta t = 20 \) and 30 min, it was found necessary to use \( \epsilon_g \)

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**Table 1.** Fine mesh with physics. Root-mean-square height differences (m) between the analysis of 1200 UTC 30 December 1988 and the 24-h forecasts verifying at the same time produced by the different models. E03 refers to the HIRLAM Eulerian forecast with \( \Delta t = 300 \) s. The forecasts produced by the semi-Lagrangian model with \( \Delta t = 600 \) s, 1200 s, \( \cdots \) are referred to as L06, L12, \( \cdots \). The latter is two time level, with \( \epsilon_g = 0.1 \) and \( \epsilon_h = 0.1 \) for L06, L12, and \( \epsilon_g = 0.2, \epsilon_h = 0.3 \) for L18 and L24.

<table>
<thead>
<tr>
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<th>L06</th>
<th>L12</th>
<th>L18</th>
<th>L24</th>
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<tr>
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<td>16.1</td>
<td>16.2</td>
<td>15.9</td>
<td>15.7</td>
</tr>
<tr>
<td>300 hPa</td>
<td>17.5</td>
<td>16.9</td>
<td>17.1</td>
<td>17.4</td>
<td>17.7</td>
</tr>
<tr>
<td>500 hPa</td>
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<td>12.0</td>
<td>12.3</td>
<td>12.4</td>
<td>12.6</td>
</tr>
<tr>
<td>1000 hPa</td>
<td>17.0</td>
<td>16.4</td>
<td>17.2</td>
<td>17.7</td>
<td></td>
</tr>
</tbody>
</table>

**Table 2.** Fine mesh with physics. Root-mean-square height differences (m) between the analysis of 1200 UTC 30 December 1988 and the 24-h forecasts verifying at the same time produced by the different models. E03 refers to the HIRLAM Eulerian forecast with \( \Delta t = 300 \) s. The forecasts produced by the semi-Lagrangian model with \( \Delta t = 300 \) s, 600 s, \( \cdots \) are referred to as L03, L06, \( \cdots \). The latter is three time level, with \( \epsilon_g = 0.0 \) and \( \epsilon_h = 0.05 \) for L03, L06, L09, and \( \epsilon_g = 0.1 \) and \( \epsilon_h = 0.1 \) for L12 and L18.

<table>
<thead>
<tr>
<th></th>
<th>E03</th>
<th>L03</th>
<th>L06</th>
<th>L09</th>
<th>L12</th>
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<tr>
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<td>12.7</td>
<td>12.9</td>
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<tr>
<td>1000 hPa</td>
<td>17.0</td>
<td>16.5</td>
<td>17.4</td>
<td>18.1</td>
<td>18.6</td>
<td>19.3</td>
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</tbody>
</table>
Table 3. Same as Table 1 but using the two-dimensional instead of the three-dimensional interpolation in Eq. (11).

<table>
<thead>
<tr>
<th></th>
<th>E03</th>
<th>L06</th>
<th>L12</th>
<th>L18</th>
<th>L24</th>
</tr>
</thead>
<tbody>
<tr>
<td>100 hPa</td>
<td>17.1</td>
<td>16.1</td>
<td>16.1</td>
<td>16.0</td>
<td>15.9</td>
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<td>300 hPa</td>
<td>17.5</td>
<td>17.0</td>
<td>17.5</td>
<td>17.9</td>
<td>18.7</td>
</tr>
<tr>
<td>500 hPa</td>
<td>13.2</td>
<td>12.2</td>
<td>12.6</td>
<td>12.9</td>
<td>13.4</td>
</tr>
<tr>
<td>1000 hPa</td>
<td>17.0</td>
<td>16.5</td>
<td>17.4</td>
<td>18.1</td>
<td>18.3</td>
</tr>
</tbody>
</table>

= 0.1 and the coefficient of the Robert–Asselin filter ($\varepsilon_R$) had to be increased to 0.1; the latter was equal to 0.05 for the smaller time steps, and $\varepsilon_R = 0.0$. Also worthy of note is the fact that a time step of 30 min was feasible for the three-level scheme, whereas $\Delta t = 1$ h was not for the two time-level scheme.

It would seem more natural to use a two-dimensional interpolation in Eq. (11). This option was tested and was found to be as accurate as the original in tests using the coarse-mesh dataset. When tested with the fine-mesh dataset, however, it was found to give slightly larger errors for large time steps. See Table 3, where the results are listed for the two time-level model. The same pattern of errors was found for the three time-level model.

5. Conclusions

The two time-level, gridpoint, three-dimensional semi-Lagrangian and semi-implicit primitive equation model that incorporates a sophisticated physics package presented in MH has been extended to incorporate the hybrid coordinate $\eta$ in the vertical.

The model was found to be more prone to instability if the $\eta$ coordinate rather than the $\sigma$ coordinate was used. (See appendix B for the values of each.) A new filter was introduced to control this instability and it was shown to work satisfactorily on the $1.5^\circ \times 1.5^\circ$ grid, bringing the level of noise down to that of the forecasts with the $\sigma$ coordinate. Also, it enabled us to use the more accurate approximation of the $(\eta + 1/2)$-level fields used to estimate the departure-point position, thus bringing about better forecasts for the larger time steps, although, for $\Delta t = 3$ h, not as accurate as one would anticipate at 300 hPa, in particular.

Tests on a $0.5^\circ \times 0.5^\circ$ horizontal grid, again using 16 levels in the vertical, showed that stable, accurate forecasts are attainable with a 30-min time step, giving a large saving in CPU. The values of the off-centering and filter coefficients needed for stability were $\varepsilon_R = 0.2$ and $\varepsilon_N = 0.3$, in contrast with $\varepsilon_R = 0.05$ and $\varepsilon_N = 0.0$ when $\sigma$ coordinates were used. Nevertheless, the rms errors remain satisfactory. The two time-level model was shown to be as accurate, and almost twice as efficient, as a three time-level model for effective time steps up to 40 min, thus giving the former a big CPU advantage over the latter. These tests were performed on a single dataset. Assuming that the accuracy obtained is representative, the model offers the prospect of a large saving of computer time.

No other damping was used aside from the nonzero $\varepsilon_\eta$ and $\varepsilon_N$. Divergence damping was tested but proved much less satisfactory. See Gravel et al. (1992) for an analysis of the relative merits of uncentering versus divergence damping.

It is interesting to ask what makes the two time-level model become increasingly noisy for large time steps. It is possible that this is caused by the explicit integration of the nonlinear terms (see Gravel et al. 1992), although the three time-level integrations displayed similar tendencies. The unphysical nature of the upper boundary condition, ($\eta = 0$), and the lack of balance between the boundary fields and the model fields in the boundary zone are other possible sources of noise. Another possibility is the following. The three-dimensional discretization of the temperature and continuity equations is problematical in $\eta$ coordinates. It is possible that our choice of Eqs. (11) and (24) is not optimal, thus leading to some imbalance when large time steps are used. For an alternative discretization see Temperton (1992).

Acknowledgments. Thanks to Jim Hamilton for assistance with the graphics.

APPENDIX A

List of Variables and Constants

- $a$: radius of the earth ($6.371 \times 10^6$ m)
- $c$: divergence damping coefficient
- $c_{pd}$: specific heat of dry air (1004.64 J kg$^{-1}$ K$^{-1}$)
- $c_{pw}$: specific heat of moist air (1869.46 J kg$^{-1}$ K$^{-1}$)
- $D$: divergence (s$^{-1}$)
- $\delta$: $c_{pw}/c_{pd}$
- $\varepsilon_R$: uncentered damping coefficient
- $\varepsilon_N$: filter coefficient
- $\varepsilon_R$: Robert–Asselin filter coefficient
- $\eta$: vertical coordinate
- $\eta$: $\eta$ vertical velocity
- $f = 2\Omega \sin \theta$: Coriolis parameter (s$^{-1}$)
- $g$: acceleration due to gravity (9.80665 m s$^{-2}$)
- $\kappa$: $R_d/c_{pd}$
- $\lambda$: longitude
- $\Omega$: angular speed of the earth ($7.292 \times 10^{-5}$ s$^{-1}$)
- $p$: pressure (Pa)
- $p'$: reference pressure for semi-implicit scheme (Pa)
- $p_s$: surface pressure (Pa)
- $\Phi$: geopotential ($g \times$ height) (m$^2$ s$^{-2}$)
- $\Phi_s$: surface geopotential (m$^2$ s$^{-2}$)
APPENDIX B

List of the Values of $A$ and $B$ That Define the $\eta$ and $\sigma$ Coordinates Used in Testing the Model

For the $\eta$ system, the values of $A_{k+1/2} (k = 0, 16)$ are as follows.

0.00, 4999.90, 9890.62, 14 166.18, 17 346.22, 19 120.98, 19 371.42, 18 164.38, 15 742.22, 12 487.98, 8 881.82, 5 437.54, 2 626.26, 783.30, 0.00, 0.00, 0.00.

For the $\eta$ system, the values of $B_{k+1/2} (k = 0, 16)$ are as follows.

0.000000, 0.0000002, 0.0017206, 0.0131978, 0.0422170, 0.0937618, 0.1695702, 0.2680178, 0.3842722, 0.5108339, 0.6382661, 0.7563879, 0.8556100, 0.9287480, 0.9729840, 0.9922820, 1.0000000.

For the $\sigma$ system, the values of $A_{k+1/2} (k = 0, 16)$ are all zero.

For the $\sigma$ system, the values of $B_{k+1/2} (k = 0, 16)$ are as follows.

0.000000, 0.0472790, 0.0968020, 0.1503064, 0.2089848, 0.2734752, 0.3438725, 0.4197155, 0.5000004, 0.5831677, 0.6671145, 0.7491835, 0.8261721, 0.8943251, 0.9493408, 0.9863662, 1.0000000.

APPENDIX C

Definition of the Matrix Operators

$$(\gamma T)_k = (\alpha' R_d T)_k + R_d \sum_{j=k}^{K} (T \Delta \ln p_j')$$

$$(\tau D)_k = \kappa T^0 \left[ \left( \frac{\Delta \ln p}{\Delta p} \right)_k \sum_{j=1}^{K} D_j \Delta p_j' + \alpha_x D_k \right]$$

$$v \cdot D = \sum_{j=1}^{K} D_j \frac{\Delta p_j'}{p_j' \Delta p}$$

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Staff of the ECMWF research department, 1985: Research manual 2, ECMWF forecast model adiabatic part. 2.1–2.28. [Available from ECMWF, Shinfield Park, Reading, Berkshire RG2 9AX, England.]

