Validity of the Tangent Linear Approximation in a Moist Convective Cloud Model

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ABSTRACT

The validity of the moist tangent linear model (TLM) derived from a time-dependent 1D Eulerian cloud model is investigated by comparing TLM solutions to differences between results from a nonlinear model identically perturbed. The TLM solutions are found to be highly sensitive to the amplitude of the applied perturbation, and thus the linear approximation is valid only for a specific range of perturbations. The TLM fails to describe the evolution of perturbations when the initial variation is given on a parameter in the vicinity of a nonlinear regime change, a result that has important implications for many cloud-scale processes. Uncertainty imposed on certain aspects of microphysical processes can have a significant influence on the behavior of the TLM solutions, and in some cases this behavior can be explained by the particular discretizations used to solve the equations. The frequency at which the nonlinear basic state is updated in the TLM can also have a profound effect on the TLM validity, though this sensitivity is in some cases modulated by the numerical scheme and model configuration used.

1. Introduction

The space–time evolution of errors or uncertainties in a numerical model (e.g., from observations, physical parameterizations, boundary conditions) is an important factor in determining its predictive capability. In numerical weather prediction (NWP), nonlinear error growth leads to an inherent predictability limit for a given atmospheric phenomenon, and this limit may be quantified in several ways.

For example, one can examine the dynamics of error evolution by running the nonlinear model (NLM) repeatedly for various initial perturbations on any input parameter (e.g., Smagorinsky et al. 1970). While this approach yields the true nonlinear evolution of the perturbations, it is limited in practice to a relatively small subset of perturbations due to the expense of running the NLM.

Conversely, one can evaluate model solution sensitivity in an approximate yet much more efficient manner through the use of a linearized version of the NLM, known as the tangent linear model (TLM), and its associated adjoint (e.g., Lorenz 1965; Vukicevic 1991; Ehrendorfer and Errico 1995; see discussion below). With this strategy, one obtains complete information on the evolution of all possible perturbations, though only within the context of linear dynamics. That is, the results are valid provided that the perturbations remain “reasonably small.” Establishing this validity and its generality is thus a vitally important preface in the application of this approach.

The TLM is based on the first-order derivative term of the Taylor expansion in the vicinity of NLM solutions. It describes the linear evolution of perturbations along the solution trajectories of the NLM. The TLM is useful not only for analyzing forward sensitivity (e.g., Park et al. 1994; Wang et al. 1995), but also in formulating the adjoint model (ADJM), which provides gradient information for both adjoint sensitivity analysis (e.g., Errico and Vukicevic 1992; Rabier et al. 1992; Langland et al. 1995) and variational data assimilation (e.g., Lewis and Derber 1985; Le Dimet and Talagrand 1986; Navon et al. 1992; Zou et al. 1993; Li and Droegemeier 1993; Li et al. 1994). The combined TLM/ADJM system is also used for investigating singular vectors (or optimal modes) to assess atmospheric predictability (e.g., Buizza et al. 1993; Palmer et al. 1994; Ehrendorfer and Errico 1995). However, one must note that the ADJM is an exact transpose of the TLM, and hence those applications that depend upon the ADJM are affected by the extent to which the TLM accurately characterizes the true nonlinear evolution of perturbations.

Most TLMs have been developed for a dry or “no physics” version of the corresponding NLM due to the complexity of treating diabatic physical processes (Vukicevic 1991; Navon et al. 1992; Errico and Vukicevic 1992; Errico et al. 1993). Although dry TLMs have been

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used successfully to describe nonlinear error behavior in a mesoscale model (Vukicevic 1991; Errico et al. 1993; Langland et al. 1995), it is desirable to implement all physical processes in the TLM in order to extract the greatest available information, especially for convection, where diabatic processes are of first-order importance. Toth and Kalnay (1993) recognized that neglecting physical processes could lead to important deficiencies in the TLM/ADJM system.

Until recently, only a few studies have investigated the effects of moist processes in the TLM/ADJM system (Douady and Talagrand 1990; Bao and Warner 1993; Verlind and Cotton 1993; Vukicevic and Errico 1993; Zou et al. 1993; Zupanski 1993; Zupanski and Messinger 1995). Such processes typically involve numerical representations that contain discontinuities, that is, on–off switches associated with “IF tests.” While there exist no strict limitations in performing data assimilation with discontinuous (in time and space) physical processes (e.g., Zou et al. 1993; Zupanski 1993; Verlind and Cotton 1993; Bao and Warner 1993), some processes in practice, especially those involving convective adjustment, can lead to a detrimental impact if the model includes on–off switches (Douady and Talagrand 1990; Zupanski 1993). Further, it has been shown that the moist TLM of a mesoscale model does not accurately describe nonlinear perturbations in regions of frequent discontinuous transitions associated with moist diabatic processes (Vukicevic and Errico 1993; Bischof and Pusch 1994). Xu (1996a,b) discusses various impacts of the on–off switching process on the evaluation of derivative information and suggests that simple strategies for dealing with numerically discontinuous processes may be inadequate.

A major concern in NWP is an accurate determination of the “true” initial state of the atmosphere and the depiction of this state by the model. In the case of storm-scale NWP, the challenges are particularly acute (Droegemeier 1990; Lilly 1990). Since the details of convective initiation are poorly understood, the uncertainty in portraying the initial state is the major factor in storm-scale predictability (McPherson and Droegemeier 1991; Brooks 1992; Brooks et al. 1992). Due to the lack of high-resolution observational data, most storm simulations have begun with highly idealized and unbalanced perturbations in the mass or wind fields. However, recent real-time storm-scale predictions (e.g., Droegemeier et al. 1996a,b; Xue et al. 1996a,b) suggest that individual convective elements can be predicted in realistic settings without resorting to ad hoc initialization procedures.

In an effort to understand the dynamical evolution of model and observational errors on the storm scale, and to establish a basis for using the TLM/ADJM as a tool for related sensitivity analysis, we investigate the validity of the TLM derived from a 1D moist Eulerian cloud model. The most important goal of our study is to determine whether the linear perturbation fields from the TLM reasonably describe the true nonlinear evolution of perturbations as determined by comparative runs with the NLM. This assessment should precede any application of the TLM/ADJM system to sensitivity analysis or variational data assimilation. If the TLM is not valid for reasonably small perturbations, the sensitivity results based upon the TLM/ADJM approach will be meaningless.

It is important to recognize that the TLM results obtained here are valid only for a very simple 1D cloud model that lacks the physical realism needed to simulate certain types of deep convective storms that are inherently three-dimensional. Nevertheless, we felt it appropriate to start with a 1D model because it embodies the basic physics processes of more sophisticated models and represents a more tractable problem for initial study. Results gleaned from the present investigation have been used in a TLM sensitivity analysis of a fully 3D cloud model (Park 1996) and those findings will be reported in a separate paper.

Section 2 describes the numerical model and methodology of the study, while section 3 defines and presents results from the control experiment. Various sources of uncertainty in the TLM are discussed in section 4, and in sections 5–9, we investigate TLM validity for several types of errors in the storm and its environment, changes to the updating period of the TLM basic state, and specific aspects of the model formulation. Summary and discussion are provided in section 10.

2. Numerical model and methodology

a. Formulation of the nonlinear model

Our 1D warm-rain Eulerian cloud model is based on that of Ogura and Takahashi (1971, hereafter OT71), in which the cloud is modeled as a cylindrical air column with a time-independent radius $R$. The downward motion in the environment, which compensates the updraft inside the cloud, is not considered. That is, all environmental variables are assumed to be stationary and functions of height only. We assume that the pressure adjusts instantaneously at any level to that of the environment, which is in hydrostatic equilibrium. All cloud variables are expressed in terms of the mean values averaged over the horizontal cross section of the cloud (Asai and Kasahara 1967). Further details of our model, including the equations and some features that differ from OT71, are provided in the appendix.

It is noteworthy that entrainment in this model is composed of two parts; lateral mixing (turbulent exchange), which depends on gradients at the boundary of the rising cloud, and dynamical entrainment (Houghton and Cramer 1951), which is associated with systematic flow into or out of the cloud through the side wall, according to mass continuity. In this respect, the present model may be regarded as “one and a half” dimensional. In other 1D cloud models, the dynamic inter-
action between the cloud and its environment is represented in terms of entrainment that is inversely proportional to the cloud radius (e.g., Weinstein 1970; Wisner et al. 1972).

b. Computational methodology

As will be shown in section 5b, the numerical scheme used in the model can have a significant impact on the validity of the TLM. For vertical advection, we use the first-order upstream-in-space and forward-in-time scheme. In contrast, OT71 used an upstream-averaged advection velocity in order to produce convective motion over the 2-km layer when vertical velocity $w$ was set to zero in their initial wind profile. Wisner et al. (1972) pointed out that the manner in which the advection velocity is defined can be critical to the cloud model solutions. To address this concern, we test three advection velocities, shown by an overbar, that depend upon the initial $w$ field:

- no averaging (case A1);
  \[ \overline{w}_k = w_k, \]  

- upstream averaging (case A2):
  \[ \overline{w}_k = \frac{1}{2}(w_k + w_{k+1}) \quad \text{if } w_k \geq 0, \]  
  \[ \overline{w}_k = \frac{1}{2}(w_k + w_{k-1}) \quad \text{if } w_k < 0, \]  

- 1-2-1 averaging (case A3):
  \[ \overline{w}_k = \frac{1}{4}(w_{k-1} + 2w_k + w_{k+1}), \]  

where $w_k$ is the vertical advection velocity at vertical level $k$. The control case (section 4) is the same as case A1 except for the initial sounding, which is described in the next subsection. For computation of the rainwater flux (see appendix), we use a centered-in-space scheme.

c. Initial and boundary conditions

The initial and boundary conditions are identical in all experiments except for the initial sounding of the control case, in which the environmental temperature decreases dry adiabatically from a surface value of $28^\circ$C up to 1 km, and thereafter at the moist-adiabatic rate ($6.3^\circ$C km$^{-1}$) to the top of the model (10 km). The initial sounding for cases A1 through A3 is modified so that the lower layer ($\leq 1$ km) is more stable (lapse rate of $8.6^\circ$C km$^{-1}$) than the control case (dry adiabatic). In all experiments, the surface relative humidity is 94.5% and increases linearly to 99.5% at 1 km, and then decreases at a rate of 5% km$^{-1}$ thereafter. Mixing ratios of cloud water ($Q_c$) and rainwater ($Q_r$) are assumed to be zero initially.

Convective motion is initiated by introducing a small updraft of parabolic shape from the ground to 1.75 km, with a maximum value of $w = 1$ m s$^{-1}$ at $z = 1$ km where $w$ is the vertical velocity and $z$ is the height. The profile is given by

\[ w(t_0) = w_0 \left( \frac{z}{z_0} \right)^2 \quad \text{for } z \leq 1.75 \text{ km}, \]  
\[ w(t_0) = w_1 (\frac{z_1}{z})^2 \quad \text{for } z > 1.75 \text{ km}. \]  

Here, $w_0 = 1$ m s$^{-1}$, $w_1 = w(t_0)_{\text{1.75 km}}$, $z_0 = 1$ km, and $z_1 = 1.75$ km. At both the upper and lower boundaries, $w$ and $Q$ are assumed to be zero, while the temperature $T$ and water vapor mixing ratio $Q$ are fixed at their environmental values. Here, $Q$ at the upper boundary is set to zero and, at the bottom, is obtained via linear extrapolation from the point above. The domain is divided into 41 levels with a grid interval of 250 m. The time step is 1 s for the control simulation, which lasts for 100 min.

d. The tangent linear model

The model equations (see appendix) can be represented as an integration of a dynamical system expressed symbolically as

\[ \frac{d}{dt} \delta \mathbf{X}(t) = \mathbf{F}(\mathbf{X}, \alpha, t), \]  

where $\mathbf{X}$ is an $m$-dimensional vector of model states, $\alpha$ is an $l$-vector of input parameters including model initial conditions and physical/computational parameters, and $\mathbf{F}$ is generally a nonlinear function of $\mathbf{X}$, $\alpha$ and time $t$. In our case, $\mathbf{X} = (w, T, Q_c, Q_r, Q)^T$. Taking the first-order (linear) term of the Taylor expansion of (5) for small perturbations $\delta \mathbf{X}$ and $\delta \alpha$, respectively, yields the TLM

\[ \frac{d}{dt} \delta \mathbf{X}(t) = \mathbf{A}(t) \delta \mathbf{X}(t) + \mathbf{F}_n(t) \delta \alpha(t), \]  

where the elements of the $m \times m$ Jacobian $\mathbf{A}$ are given by $a_{ii} = \partial F/\partial X_i$, and $\mathbf{F}_n(t) = \partial \mathbf{F}(t)/\partial \alpha$. Equation (6) describes a TLM constructed in a generalized manner; that is, in our model, the linearization is performed with respect to both model parameters and states. To simplify the discussion, we assume that $\alpha$ is a component of the initial conditions of $\mathbf{X}$. Then, in discrete form, (5) can be expressed recursively without loss of generality as

\[ \mathbf{X}_n = \mathbf{F}(\mathbf{X}_{n-1}), \]  

where the subscript $n$ denotes a time index with $t_n = t_0 + n \Delta t$, $t_0$ the initial time, and $\Delta t$ the time step. The TLM of (7) can be expressed as

\[ \delta \mathbf{X}_n = \mathbf{M}_{n-1} \delta \mathbf{X}_{n-1}, \]  

where $\mathbf{M}_{n-1}$ is a Jacobian matrix whose components are
m,(t_{n-1}) = \partial F(t_{n-1})/\partial X(t_{n-1}). Since this is a recursive system, (8) can be expressed in terms of initial perturbations (e.g., Errico and Vukicevic 1992) as

\[ \delta X_i = M_{i-1}M_{i-2}\cdots M_0\delta X_0 = Q_i\delta X_0, \]  

(9)

where the elements of \(Q_i\) are expressed as \(q_{ij}(t_n) = \partial X(t_n)/\partial X(t_0)\).

Because the solutions of the TLM [Eq. (9)] are linear along the trajectories of the corresponding nonlinear state variables (\(X_i\)), they indeed describe the tangent linear evolution of initial perturbations (\(\delta X_i\)).

e. Measuring TLM validity

Because the TLM is the linear first-order term of the Taylor expansion, the nonlinear higher-order terms (HOT) must remain “small” in order for the TLM to be valid. Therefore, the behavior of the HOT is extremely important in assessing TLM validity. One measure of nonlinearity using the HOT is the “distance” between the TLM solution and the difference between an NLM control and NLM perturbation solution. If the latter difference is denoted NLP (i.e., nonlinear perturbation or perturbation solution minus control solution), then norm \([\text{NLP} - \text{TLM}]\) would be an appropriate measure of this “distance.”

However, when on-off switching processes (e.g., condensation, evaporation, cloud-to-rain conversion) are involved, one must consider the associated nonlinear effects as well. The switching times may be different for different perturbation sizes, and this can result in different evolutions for the simulated convective storms. With a very simple mathematical model, Xu (1996a) showed that when the variation of the switching time is considered the associated nonlinearity is not only local but also a half-order lower than the quadratic nonlinearity associated with advection processes. For a complex model, however, it is difficult to discern the nonlinearity due to variations in the switching time from those associated with the HOTs. One possibility is to force the switching times of a perturbation run to be identical to those of the control run. This would allow for a comparison of two perturbation runs: one with “free” switching times and the other with fixed switching times (Xu 1996, personal communication). Unfortunately, this would require checking the switching times of all physical processes for all grid points in the control run, which is an almost impossible task even for a simple 1D model.

As one method for establishing the validity of the TLM, we define the nonlinearity coefficient (NLC), which is the ratio between the \(L_1\) norm (least deviation) of the nonlinear HOTs and the linear first-order term (TLM solutions) of the Taylor expansion. That is,

\[ \text{NLC} = \frac{\|\text{NLP} - \text{TLM}\|}{\|\text{TLM}\|}, \]  

(10)

where \(\|\|\) indicates the Euclidian \(L_1\) norm. Thus, the NLC measures the relative importance of the first and higher-order terms in the Taylor series. When a switching process is involved, Eq. (10) includes the nonlinearity arising from variations in the switching times because they are included in the NLP fields.

When the NLC is much greater than unity, nonlinear effects dominate in the perturbation solution and the TLM is “not valid.” Similarly, when the NLC is much less than unity, the perturbation is principally under the control of linear dynamics and the TLM is “valid.” In practice, the dividing line between these two extremes is difficult to establish. For example, when the NLC stays near unity (i.e., slightly larger or less than unity) for a fairly long time, one might consider the TLM to be “marginally valid.”

Correlation coefficients between the TLM solutions and the NLP fields are also used to quantify the validity of the TLM, along with dissipation and dispersion errors (Takacs 1985), the latter as a way of depicting the effective amplitude and phase errors between the TLM and NLP fields. Following Takacs (1985), we define the total error (ETOT) to be the mean square error between any two fields

\[ \text{ETOT} = \frac{1}{M} \sum_i (P_N - P_T)^2, \]  

(11)

where, in our case, \(P_N\) is the NLP field, \(P_T\) is the TLM field, and \(M\) is the total number of grid points (index \(k\)). Equation (11) can be expressed as the variance of \((P_N - P_T)\) plus the squared difference of the means,

\[ \frac{1}{M} \sum_i (P_N - P_T)^2 = \sigma^2(P_N - P_T) + (\bar{P}_N - \bar{P}_T)^2, \]  

(12)

where \(\sigma^2\) is the variance and overbarred quantities are means. After some simplification (Takacs 1985), Eq. (11) can be expressed as

\[ \text{ETOT} = [\sigma(P_N) - \sigma(P_T)]^2 + (\bar{P}_N - \bar{P}_T)^2 \]  

\[ + 2(1 - \gamma)\sigma(P_N)\sigma(P_T), \]  

(13)

where \(\gamma\) is the correlation coefficient between \(P_N\) and \(P_T\). When \(P_T\) and \(P_N\) are exactly correlated (i.e., \(\gamma = 1\)), only dissipation error is present. Thus, the dissipation error (EDISS) is defined as the sum of the first two terms on the right-hand side (rhs) of Eq. (13). For \(\gamma \neq 1\), an additional error is introduced due to dispersion (EDISP), defined as the third term on the rhs of Eq. (13).

We have verified the correctness of our TLM code by starting with a zero initial perturbation to ensure that it remained zero for all time. Additionally, we compared results from our hand-coded TLM with those produced by an automatically generated TLM code using ADIFOR (Automatic Differentiation of FORTRAN; Bischof et al. 1992). They were identical down to machine precision.

To validate the correctness of our 1D NLM, we attempted to reproduce the control case in OT71.
the same temperature and moisture sounding, our model produced very similar though not identical results.

3. Sources of error in the TLM

The first step in sensitivity analysis or variational data assimilation with the TLM/ADJM system is to investigate the ability of the TLM to describe the true nonlinear evolution of perturbations, and to specify the range of this validity in terms of the integration time, magnitude of the perturbations, model parameters and states, and characteristics of the flow being simulated. Because the TLM is constructed on the basis of linear perturbation theory, the fundamental requirement is that the magnitude of a perturbation be small compared to the basic state. But how small is small? If the TLM is valid only for perturbations below the accuracies of current observing systems, then sensitivity and data assimilation results based on the TLM/ADJM system would be of little practical use.

In addition to the linear approximation, several other differences between the TLM and NLP might lead to failure of the former in describing the latter (Errico et al. 1993; Vukicevic and Errico 1993), including: 1) The possibility of nondifferentiable, discontinuous solutions, especially in physical processes, which cannot be implemented in the TLM in a global sense; 2) the variation in switching time of discontinuous processes; 3) The highly nonlinear property of solutions, which may invalidate the linear approximation and induce regime changes. Many nonlinear chaotic systems involve bifurcation points, where solutions started from slightly different conditions take completely different trajectories (Thompson and Stewart 1986). Since the TLM is linearized along a particular nonlinear solution trajectory, its solutions near the boundary of a regime change or bifurcation point may fail to describe the true nonlinear evolution even with small perturbations; 4) the presence of terms like $Q^2$, where $Q$ is a water variable and $\gamma$ a constant, for which the TLM may give unreasonably large derivatives when $0 < \gamma < 1$ (see section 6c) and the magnitude of $Q$ is exceptionally small. In this situation, the linear approximation may be violated (e.g., Vukicevic and Errico 1993); 5) the infrequent update of the nonlinear basic fields in the TLM may be an important source of error (Errico et al. 1993), especially in convective storms where solution behavior may change significantly in a short time.

Additionally, the TLM results can be affected by specific formulations of parameterizations in both microphysical and dynamical processes (Park and Droegemeier 1995). As shown in section 9, perturbations computed from the TLM can differ substantially from the corresponding NLP solution, even for a small change in a coefficient in the parameterization.

In constructing our TLM, we deal with discontinuous processes by taking a local directional derivative in the same vein as the on–off switch method suggested by Bao and Warner (1993) and Zou et al. (1993). To eliminate the uncertainty associated with the update of the basic fields (item 5 above), we implement the TLM line by line in the original NLP so that both can be run simultaneously. In this manner, the basic state of the TLM is updated every time step.

For each experiment, we prescribe an error or perturbation in the NLM initiating updraft. We then compute the difference between the solution so obtained and the NLM control solution (i.e., perturbation run minus control run) at each time and grid point to obtain the nonlinear perturbation (NLP) fields for comparison against the TLM fields. In the following sections, we discuss the effects of these perturbations on the accuracy of TLM solutions.

4. The control experiment

The environmental sounding for our experiments allows for the development of deep convection (see section 2c). The A1 advection velocity [Eq. (1)] is used for the control experiment. All simulated fields depict the typical three-stage life cycle of a single cell storm: development, maturity, and dissipation. To illustrate, we discuss the $w$ and $Q_r$ fields, shown in Fig. 1, as they will be the main focus of the subsequent analysis.

Starting from the initial updraft, a region of maximum updraft grows vertically, reaching a maximum of 14.6 m s$^{-1}$ at 34 min near $z = 3.5$ km (Fig. 1a). The downdraft develops first in the lower part of the cloud at approximately 26 min and spreads to higher altitudes, finally occupying a major portion of the cloud after about 78 min with a maximum intensity of $-5.9$ m s$^{-1}$ near the ground.

In the developing stage (0–16 min), updraft is present throughout the cloud and grows upward at an appreciable rate. Although no precipitation has yet reached the ground, hydrometeors are present inside of the cloud. Our results show that the rainwater steadily increases with time, with local maxima developing first at low levels (Fig. 1b) because the updraft is so weak that it cannot overcome the terminal velocity of the rain. Thus, rain reaches the ground as early as about 16 min, even before appreciable downdraft is observed in the region above.

The mature stage (16–78 min), where regions of updraft and downdraft coexist, begins when rain first reaches the ground ($\sim 16$ min). As the downdraft spreads and occupies a major portion of the cloud, the dissipating stage sets in ($\sim 78$ min). At this time, there is no appreciable source of water vapor to maintain condensation and a sharp tilt of the axis of the maximum $Q_r$ (AMQ) occurs. Here, $Q_r$ reaches a local maximum of 5.2 g kg$^{-1}$ around 76 min at $z = 5$ km and decreases very rapidly thereafter.

The stair-step pattern near cloud top in Fig. 1 is observed in all fields and is due to the particular finite-difference scheme applied here. (For example, it dis-
appeared completely in case A3.) The pattern is also related to gravitational oscillations generated in the unsaturated stable layer above cloud top (Ogura and Takahashi 1973). These oscillations are initiated by adiabatic cooling above the rising cloud (Soong and Ogura 1973), and are not observed in the cloud water and rainwater fields. Also evident are unrealistically strong gradients at cloud top due to well-known deficiencies in this type of model (e.g., OT71; Wisner et al. 1972; Ogura and Takahashi 1973; Lopez 1973; Silverman and Glass 1973). In particular, the present model does not include perturbation pressure or vertical mixing, both of which tend to reduce the magnitude of all gradients at cloud top. By including the former in a 1D model, Holton (1973) demonstrated that strong gradients in $w$ can be reduced.

5. Impact of perturbation amplitude and model environment

a. Perturbation amplitude

To test the effect of the amplitude of perturbations on the validity of the TLM, we perform 12 experiments by adding positive bias-type perturbations of 5%–60% amplitude uniformly to the initial updraft profile in the control run. Theoretically, the TLM solutions should converge to the NLP fields as the perturbation size approaches zero. These experiments are meant to test such asymptotic behavior for this particular type of perturbation. For much smaller perturbations of 0.1% and 1%, the TLM solutions matched the NLP fields very well. We show in Fig. 2 the TLM and NLP $Q_r$ fields for 10% and 20% perturbations. The TLM results for a 10% perturbation have exactly the same pattern as for 20% perturbation, though the magnitudes are halved, as they should be, since the results are linear. Comparing Fig. 2a and Fig. 2b, the TLM solution for a 10% error describes the nonlinear perturbation very well, both in amplitude and phase, up to approximately 50 min, though it fails to predict the amplitude thereafter. Nonetheless, the locations of the local extrema match very well throughout the entire simulation. In addition, the zero contour of the TLM solution during the decaying stage ($\approx 78$ min; Fig. 2a) exactly matches the AMQr in Fig. 1b, which means that the TLM gives correct derivative information regarding that feature.

In Fig. 2a, the TLM $Q_r$ solution predicts that an increase in the initial updraft leads to an increase in $Q_r$ near cloud top, where the jump occurs in the stepwise pattern (except for 30–38 min, where negative values appear). This is consistent with the nonlinear differences in Fig. 2b. Also note the abrupt change in sign across the zero contour (i.e., the AMQr), where the axis of positive maximum precedes and that of the negative maximum succeeds the zero contour by 2 min. This implies that an increase in the initial updraft will result in an earlier occurrence of the AMQr, and hence an earlier decay in the cloud system (by about 2 min) relative to the control run.

For a 20% initial updraft perturbation (Figs. 2c and 2d), a greater difference occurs near cloud top, especially during the mature stage. (In the control run, the strongest gradient near cloud top occurs during the mature stage for all variables.) Most terms in the model equations showed strong gradients near cloud top throughout the mature stage, except for rain evaporation, which had its strongest gradients in the AMQr and near the ground, especially during the decaying stage.

b. Environmental sounding and advection velocity

We quantify the nonlinearity of the $w$ fields in Fig. 3 for various magnitudes (10%, 20%, 30%, and 40%)
of initial updraft perturbation in the control, A1, A2, and A3 experiments. In the control case, the NLCs remain less than unity before approximately 23 min for all magnitudes of perturbations (Fig. 3a). Thereafter, for perturbations of 20%–40%, the NLCs show sharp increases followed by oscillations, with several peaks during the mature stage of cloud development. The latter time matches very well the time when the large disagreement between the TLM and NLP first appears near cloud top (cf. Figs. 2c and 2d). For a perturbation of 10%, the NLC fluctuates around unity up to about 55 min by which time the TLM is thought to describe the nonlinear perturbations fairly accurately. If we neglect the cloud-top region, the NLC is much smaller than unity everywhere else in the domain.

With a more stable sounding at lower levels (lapse rate of 8.6°C km⁻¹ below 1 km—case A1; Fig. 3b), the NLCs remain well below unity throughout the simulation for all perturbations. In this case, the cloud does not develop as vigorously and the cloud top does not rise during the mature stage for the given perturbations and advection velocity formulation (no averaging). The abrupt increase in the NLC near 38 min, though below unity, is accompanied by sudden increases in both the dissipation and dispersion errors between the TLM and NLP fields (not shown). The oscillation and peaks are not related to the stair-step pattern, which implies that the behavior of the high-order error terms is not affected by cloud-top lifting as in the control case.

For the same sounding as case A1, but with an upstream-averaged advection velocity (case A2; Fig. 3c), the NLCs exceed unity as early as 4 min for perturbations of 20%–40%. One disadvantage of using this advection velocity formulation is that the advection is
highly sensitive to the sign and magnitude of the velocity at the upstream point. This may cause a decoupling of the advection for upstream and downstream grid points (Wisner et al. 1972), which might lead to diverging solutions. Therefore, even with a more stable lower layer, the cloud top may lift and the NLCs may increase above unity. The oscillations are strongly related to the stair-step pattern near cloud top during the developing and early mature stage. Even with a 10% perturbation, the TLM is valid only for the first 13 min.

For case A3 (Fig. 3d), in which the advection velocities are 1-2-1 averaged while the sounding is the same as case A1, the NLCs for 10%-40% errors remain below unity throughout the simulation and show a consistent pattern of behavior.

It is interesting to note that the magnitude of the initial perturbation is not necessarily proportional to the NLC in some cases. For example, in the control and A2 experiments (Figs. 3a and 3c), a smaller perturbation leads to larger NLCs, especially after the NLC abruptly exceeds unity. In experiments A1 and A3 (Figs. 3b and 3d), the NLC is smaller for smaller initial perturbations.

This implies that the NLM solutions show a highly nonlinear response to perturbation size in the control and A2 cases, while they exhibit a more linear response in cases A1 and A3. It is apparent that changes in the NLM solutions due to different advection velocity formulations lead to different behavior of the higher-order error terms, and thus affect the validity of the TLM.

c. Relation among analysis parameters

We now investigate relations among the NLC, the correlation coefficient (COR) between the TLM and NLP fields, and the dissipation (EDISS) and dispersion (EDISP) errors. They are depicted in Fig. 4 for $Q_r$ from case A1 with a 30% perturbation in the initial $w$ field.

Here, EDISS and EDisp are normalized by their maximum values (the actual magnitude of the EDISS is about three orders of magnitude larger than that of EDisp). With a correct TLM, we expect the NLC to be much less than unity with the COR remaining near unity for all time.

Although the NLC remains below unity throughout...
the simulation in case A1, COR does not remain high all the time. It is interesting to note that COR drops rapidly (below 0.9) when the NLC exceeds 0.5. The abrupt increase in the NLC around 34 min is quickly followed by significant increases in both EDISS and EDISP. This implies that errors between the TLM and NLP fields are mainly due to an abrupt increase in nonlinearity due to both the higher-order terms (mathematical) and the switching time variations (computational) in the system. Before this time, the storm is in its developing and early mature stages. Therefore, the switching effect might not yet be significant. It is interesting that, when the EDISP is very small, the trend in COR is inversely related to that of the NLC. When microphysical processes are involved, for example, during the middle and late stages of the storm, the variations in switching times due to a large perturbation (30%) can affect both the intensity and the phase (both in space and time) of the storm (see a more detailed discussion of this issue in section 6). This may lead to an increase in the EDISS and EDISP, after which the NLC varies little while the COR and error fields show large fluctuations. In particular, the behavior of COR is reflected by the EDISP.

One should note that COR can decrease significantly due to increases in nonlinearity (i.e., the NLC) even without a notable amount of EDISP, as demonstrated during the early storm stage. In fact, small COR with negligible errors occur only at few grid points near cloud top. For example, in Fig. 2, several grid points at cloud top have large values both in the TLM and NLP fields during the early sensitivity period. They are mostly related to the stepwise development at cloud top and are not physically meaningful.

In Fig. 5, we compare the evolution of the NLC and COR for the control and A1–A3 cases for a 1% initial updraft perturbation. This 1% error would seem to be small enough for the TLM to be valid in all cases. However, some NLCs still exceed unity. It is noteworthy that, with the same amount of error, each case demonstrates completely different behavior. After the NLC shows a sudden increase near unity, COR behaves similarly to the EDISP (not shown). When the NLC of case A2 peaks, EDISP is the largest among all cases by a factor of $10^3$–$10^6$, which explains the negative COR. Although the NLC of this case fluctuates around unity thereafter, the combined effect of the NLC and EDISP leads to an oscillation of COR at low values. The decrease in the NLC does not necessarily mean a decrease in total error (i.e., EDISS + EDISP). Even though the error between the TLM and NLP fields is quite large and thus the correlation is small, the NLC can be relatively small if the magnitude of the linear term (de-
nominator) is much larger than that of the higher-order terms (numerator) [see Eq. (10)].

It is notable that the behavior of the NLC for all cases is similar for both small and large perturbations (cf. Figs. 5a and Fig. 3). For example, the NLC of case A1 increases abruptly around 35 min and then becomes almost steady. This is a common feature in case A1 even with larger perturbations (see Fig. 3b). This implies that, for a given perturbation, the nonlinearity is inherently dependent upon the environmental sounding and advection scheme.

d. Perturbations in different model variables

Park et al. (1994) pointed out that the TLM is valid only for a specific range of perturbations, with this range a function of input parameters. We therefore investigate the validity of our TLM using ground rainfall intensity (GRI) as the predicted variable. Perturbations are added separately to \( w \), \( T \), and \( Q_n \) at all grid points for case A3.

In the unperturbed NLM run, GRI shows a smooth increase until about 53 min, followed by an abrupt increase from 53 to 57 min and then a sharp decrease until 62 min (not shown).

In Fig. 6, we show the ratio of the TLM results to the NLP fields [i.e., \( \frac{\partial \text{GRI} / \partial \mathbf{X} \cdot (\Delta \text{GRI} / \Delta \mathbf{X})}{1} \) where \( \mathbf{X} \) is an input vector]. Ideally, the ratio should be unity for identical TLM and NLP solutions. As one would expect, the validity of the TLM is a strong function of the variable being perturbed for two reasons. First, the model-dependent variables have quite different characteristic values, for example, order 10 m s\(^{-1}\) for vertical velocity and 300 K for temperature. Thus, each variable may have a different perturbation range for achieving the same TLM accuracy. For example, the TLM gives much better results with a 10% perturbation in \( w \) than with a 0.01% perturbation in \( T \) (cf. Figs. 6a and 6b).

Second, perturbations in each variable act on the model dynamics in different ways. For example, perturbations in thermodynamic and moisture variables may have a stronger and more direct influence on the switching time variation in microphysical processes than those applied to momentum variables. Therefore, the size of the \( Q_n \) perturbation may be more restrictive than that of \( w \) to obtain the same TLM accuracy (cf. Figs. 6a and 6c). In our case, the TLM solutions matched the NLP fields well for perturbations less than 1% in \( w \), 0.001% in \( T \), and 0.1% in \( Q_n \) (see solid lines in Fig. 6).

Considering that the ratio between the TLM and NLP fields in Fig. 6 is another measure of nonlinearity (i.e., highly nonlinear if the magnitude of the ratio is much larger or much less than unity), we note that the responses in GRI due to perturbations of 0.01% in \( T \) and 1.0% in \( Q_n \) are highly nonlinear during the early mature stage. However, the GRI responds most linearly to perturbations in \( w \), except near 57 min where the maximum in GRI is observed. Other variables also depict sudden changes around 57 min, apparently related to the abrupt (nearly discontinuous) change in GRI at this time. This provides a good example of how the nonlinear and discontinuous properties in model physics can affect the validity of the TLM for quite large perturbations. This issue is discussed in detail in terms of a switching process in the next section.

It is evident that the range of perturbations leading to a qualitatively “good” TLM solution is different for each physical variable. For a specific range of suffi-
ciently small perturbations, the TLM results describe the NLP fields almost perfectly in the simulated storm case. In the case of the GRI, our validation results are exact if we interpret them only for small perturbations.

6. Impact of moist physical processes

a. On–off switches

As a first step in testing the impact of moist physics on the validity of the TLM (items 1–4 in section 3), we discuss the effect of on–off switching time variations by evaluating the condensation process for several perturbations to the initial \( w \) field (0.1%, 5%, 10%, and 20%). Figure 7 depicts the NLP and TLM fields of condensation rate at \( z = 4.5 \) km for the given perturbations. Note that the switching time of the TLM is the same as in the unperturbed run, while that of the NLP is the same as in each perturbed run. For positive perturbations, we observe an earlier occurrence of the switching time with increasing perturbation size. For a 0.1% perturbation, the switching time and magnitude match very well in both the NLP and TLM fields (Fig. 7a). With a 5% perturbation (Fig. 7b) the switching time of the NLP fields occurs two time steps earlier than for the TLM field, and with a 10% perturbation (Fig. 7c) it occurs four time steps earlier. However, with a 20% perturbation, the switching time shows a very large difference of 115 time steps, which may seriously affect the validity of the TLM.

It is important to recognize that we use a “classic” TLM, similar to Zou et al. (1993), which does not consider the variation of switch time (as most other models do). Xu (1996a) points out that, when the perturbation field is computed in a classic TLM, the nonlinearity could be much stronger than the true nonlinearity caused by the discontinuous switch. However, our results show that the TLM validity due to the switching time variation is affected significantly only when a “large” perturbation is given (e.g., a 20% perturbation in the initial \( w \)).

b. Omission of variables

We now examine the impact of omitting some variables in the TLM. For example, we ran the NLM/TLM
system first without rain processes and then without both cloud and rain processes. When only $Q_c$ is eliminated and thus no sedimentation and rain evaporation occur, no downdraft develops and all fields become steady by 60 min. The maximum updraft is 21.8 m s$^{-1}$ at approximately 6.5 km, and cloud top rises continuously and extends to 9 km at 92 min. Due to an excess of latent heat from the lack of rain evaporation, this case shows more vigorous cloud development than the control case. The strong gradients and stepwise development along cloud top are still evident.

When both $Q_c$ and $Q_r$ are discarded, the NLM shows very weak updraft development and reaches a steady state as early as 33 min due to the absence of latent heating. The updraft is confined below 2 km with a maximum of 3.6 m s$^{-1}$. Instead of the stair-step pattern, only wiggles appear at cloud top due to gravitational oscillations.

We now compare the NLCs of the above two cases with the control case for a 20% error in the initial $w$ field. When both $Q_c$ and $Q_r$ are neglected, the NLCs remain well below unity (mostly less than 0.1; not shown) throughout the simulation. This is not unexpected because errors related to physical processes are excluded and the updraft development is very weak.

The NLC with only $Q_c$ excluded remains well above that of the control case (i.e., with all physics) after both show a sharp increase (over unity) at similar times ($\sim$23 min). Similar to the control case, the discrepancies occur mostly near cloud top. However, the duration of the discrepancy extends longer than in the control case due to the steady nature of the cloud. Although some physical processes (e.g., rain evaporation, cloud-to-rain conversion), which may include discontinuities and nonlinearity due to switching time variation, are not present, the TLM result is still no better than that of the control case. It is likely that the accuracy of TLM is dependent upon the inherent dynamics of the model. Although we admit that the TLM solutions are affected by discontinuous physical processes in microphysics, the validity of the TLM is more likely related to the dynamical characteristics of the simulated cloud imposed by microphysical processes than the discontinuous properties of the processes themselves.

c. Computational issues

To examine the impact on the TLM of exponential forcing terms [item 4 in section 3], we set an artificial lower bound on the amplitude of the moist variables to avoid artificially large magnitudes in derivative terms, that is, the minimum values for $Q_c$ and $Q_r$ are set to $1 \times 10^{-8}$ g g$^{-1}$ throughout the simulation as in Vukicevic and Errico (1993).

To illustrate the reasons for doing so, consider the expression

$$\frac{\partial Q}{\partial t} = AQ^\gamma,$$  \hspace{1cm} (14)

where $A$ is a constant and $\gamma < 1$. Terms of this form can be found in the parameterizations of terminal velocity and evaporation of cloud and rain (see the appendix). In the TLM, the rhs of (14) becomes

$$A\gamma Q^{\gamma-1} \delta Q = A\gamma \left(\frac{Q'}{Q}\right) \delta Q.$$  \hspace{1cm} (15)

If $Q$ is very small (e.g., $10^{-30}$ due to machine roundoff), (15) will be very large so that the linear approximation may not be valid. By imposing an artificial lower bound on $Q$, we can reduce this undesirable effect. As noted by Vukicevic and Errico (1993) and shown below, one can obtain useful results even though the exact linearization assumption has been violated.

Figure 8a shows the NLM $Q_c$ field from an experiment identical to the control case, but with a lower bound of $1 \times 10^{-8}$ g g$^{-1}$ imposed on $Q_c$ and $Q_r$ (cf. Fig. 1b). With this lower bound, the AMQ occurs much earlier and thus the storm decays much earlier. Also, the local maxima in $Q_c$ at low levels during the developing stage (10–15 min) and the early mature stage (16–18 min) of the control case is not observed.

The corresponding TLM and NLP solutions are shown in Figs. 8b and 8c, respectively, for a 20% uniform positive bias in the initial updraft (compare with the control counterparts in Figs. 2c and 2d, respectively). Significant changes are evident in the TLM solution during the decaying stage and in the NLP field during both the mature and decaying stages. Otherwise, the agreement is excellent, especially near cloud top. The imposition of an artificial lower bound on liquid water does not necessarily suggest an improvement in the quality of the TLM solutions themselves. Rather, it acts on the NLP fields by suppressing the nonlinearity in the high-order terms for a given perturbation. As illustrated here and indicated by Vukicevic and Errico (1993), to simply perform an exact linearization on any NLM does not necessarily guarantee a useful TLM solution.

To summarize, our discussion of the accuracy of the TLM can be divided into the mature and the decaying stages of the storm. During the mature stage, the vigorous changes in dynamical (buoyancy, liquid water drag, advection, lateral mixing, entrainment, and rainwater flux) and microphysical (condensation, cloud water evaporation, and cloud-to-rain conversion) processes in the region near cloud top result in early failure of the TLM for large perturbations. During the decaying stage, where the contributions from most dynamical processes become less important and microphysics (especially evaporation of cloud water and rainwater) dominate, the TLM accuracy is affected by even small perturbations, though not significantly. The nonlinearity due to higher-order terms and the effect of switching
time variations in discontinuous physical processes, especially evaporation, play important roles in the validity of the TLM during this stage.

7. TLM behavior in the vicinity of a regime change

Nonlinear systems often exhibit abrupt changes from one physical state to another (e.g., the Lorenz attractor). In such cases, the model solution trajectory is very sensitive to perturbations, and a tiny difference in the initial perturbation may result in a completely different solution path. As noted by Vukicevic and Errico (1993), to have useful TLM/ADJM results with discontinuous forcing in the NLM, it is desirable either to keep the model trajectory within a single regime and away from transitions, or to have regime transitions that do not significantly affect the forecast.

In our model, a regime change occurs when the cloud-to-rain conversion factor $C_0$ lies somewhere between $0.004096 \text{ s}^{-1}$ and $0.004097 \text{ s}^{-1}$. When $C_0$ is sufficiently small ($\leq 4.096 \times 10^{-3} \text{ s}^{-1}$), the conversion time is long enough so that cloud droplets are carried by the updraft to upper levels before producing many raindrops. This leads to relatively small liquid water drag in the middle atmosphere and a steady state cloud. When the conversion factor is sufficiently large ($\leq 4.097 \times 10^{-3} \text{ s}^{-1}$), raindrops are produced rapidly and the drag due to large amounts of liquid water in the lower levels tends to overcome the buoyancy force, causing the cloud to decay. Consequently, the larger the conversion factor, the shorter the lifetime of the cloud (OT71).

For reasonably small changes in $C_0$ (on the order of $10^{-2}$–$10^{-3}$), the TLM might be expected to accurately describe the evolution of the changes, provided that one is not in the vicinity of a regime change from steady to nonsteady behavior. Since the TLM solutions are linearized about the trajectories of nonlinear solutions, they...
FIG. 9. The NLC of $Q_r$ field with (a) $C_0 = 0.003$, (b) $C_0 = 0.005$, and (c) $C_0 = 0.00409$ for various perturbations in $C_0$ (s$^{-1}$) in the control run.

preserve the characteristics of the nonlinear model states. Thus, the tangent linear approximation of the steady regime can never change to the nonsteady regime, even for reasonably small changes in $C_0$.

We have performed NLM/TLM experiments for three base values of $C_0$ onto each of which 10 perturbations are added ranging from $10^{-7}$ to $10^{-4}$ s$^{-1}$ (i.e., 30 experiments). The NLCs of the $Q_r$ fields for four perturbation cases are shown in Fig. 9. For $C_0 = 0.003$ s$^{-1}$, which is in the steady regime, the NLC remains well below unity throughout the simulation, indicating that the TLM is a good approximation to the difference between the control and perturbed NLM solutions (Fig. 9a). It is interesting to note that the NLCs for smaller errors (e.g., $4 \times 10^{-7}$ s$^{-1}$ or about 0.01%) are larger due to machine round-off error (Vukicevic and Errico 1993).

For $C_0 = 0.005$ s$^{-1}$, which is in the nonsteady regime, the NLCs abruptly rise for all cases at about 15 min, even above unity for changes larger than $7 \times 10^{-6}$ s$^{-1}$ (i.e., 0.14%; Fig. 9b). However, the NLCs quickly decrease and remain below unity thereafter. The correlation coefficients are mostly larger than 0.95 and the lowest one is 0.89 near 15 min for a change of $7 \times 10^{-6}$ s$^{-1}$. Note that this roughly corresponds to the time at which the low-level local maximum of $Q_r$ is observed and rainfall first reaches the ground (~16 min). For $C_0$ in the steady regime near the boundary of the regime change (i.e., at $C_0 = 0.00409$ s$^{-1}$), the NLC shows a very sharp increase above unity for errors larger than $7 \times 10^{-6}$ s$^{-1}$ (Fig. 9c). The larger error leads to an earlier occurrence of the precipitous increase in the NLC. It is evident from this quantitative analysis that there exists a specific range of perturbations for which the TLM is valid.

This experiment suggests that one should exercise caution in using gradient information from the TLM/ADJM system in optimization and parameter estimation, especially when the NLM solutions are near a regime change. Of course, we have no a priori knowledge of when or where this will occur. If the model state is in one regime but very near the boundary of a transition to another, and if our initial guess field in the model is in the other regime and close to the crossover point, we will never, using a TLM, cross from one regime to the other. Therefore, we propose that the validity of the TLM be established for various perturbation runs before proceeding to its application.

If the NLC shows an order of magnitude increase for specific ranges of perturbations that are similar to observational errors, we may anticipate failure of data assimilation starting from initial guess fields with such perturbations. We note that the gradients of the cost function computed from the linear ADJM are used to find the descent direction in the minimization algorithm of data assimilation. Therefore, if the cost function has only one global minimum, the minimization will find it regardless of the perturbation size although the convergence speed may vary for different perturbations (i.e., initial guess fields). However, if the cost function has a multiminima structure (Li 1994), it is possible that gradient information computed from an invalid ADJM (e.g., near a regime change) may induce convergence of the cost function to a different local minimum than desired.

8. Impact of basic-state update

Running the TLM/ADJM system requires using the NLM solutions as basic states. Ideally, the basic states
should be updated every model time step. Due to limited disk space in most computer systems, one is often forced to save the nonlinear fields at some appropriate time interval greater than the model time step; however, reasonable results can still be obtained (e.g., Errico et al. 1993) even though the basic states are varied discontinuously and are held constant between updates.

In mesoscale models, the dynamics of which are in many cases controlled by synoptic-scale forcing, the update frequency does not occasion any serious problem in the accuracy of a dry TLM if the update interval (or period) is sufficiently short (Errico and Vukicevic 1992; Errico et al. 1993). With a moist TLM, however, the update frequency is more restrictive, even for a mesoscale model. For example, Vukicevic and Errico (1993) updated the basic states every other time step while Zupanski (1993) did it every time step. In the latter, a considerable improvement in the accuracy of TLM was obtained and traced back to the cumulus convective adjustment scheme. However, neither paper discussed the impact of the update frequency on the validity of a moist TLM.

As noted in section 3 (item 5), this issue may lead to a serious degradation of the TLM in storm-scale prediction since the solution is so highly dependent upon moist processes. To investigate, we compare in Fig. 10 the correlation between the TLM and NLP fields of $w$ for various updating intervals in the control and A3 experiments. In the control case (Fig. 10a), where the basic states are updated every 2 s (i.e., $2\Delta t$), the correlation remains high for all time. With an update period of $3\Delta t$, the correlation drops to less than 0.5 as early as about 17 min and oscillates thereafter. For larger update intervals, a similar behavior occurs (cf. $3\Delta t$ and $10\Delta t$ cases).

In case A3 (Fig. 10b), the correlation is very high (>0.99) even with a $10\Delta t$ update period throughout the simulation. As the update interval becomes larger, the correlation decreases. However, it remains well above 0.9 even with an update of $30\Delta t$, except during the late decaying stage. A large decrease in the correlation coefficient is observed when strong gradients exist near cloud top during the mature stage. Another drop occurs during the entire decaying stage when all moist variables experience substantial changes in both time and space.

It is of interest to recognize that patterns of the CORs are related to those of the NLCs in both cases, especially during the mature stage—that is, the CORs decrease when the NLCs increase (cf. Fig. 10a with Fig. 3a and Fig. 10b with Fig. 3d). Consequently, an increase in the update period produces similar effects on the TLM validity as does an increase in the perturbation size.

A theoretical discussion of the TLM accuracy as it relates to the update period and perturbation magnitude is given by Errico et al. (1993). They point out that, for large update periods, the error between the TLM and NLP fields becomes linear in the perturbation magnitude. When the update period is small, the error is dominated by the perturbation magnitude. Although the error may decrease for smaller perturbations, the two fields may not be asymptotic due to the effect of the update frequency.

It is likely that, for convective storms, which are highly nonlinear and governed by discontinuous (in the numerical sense) processes, the TLM validity may be extremely sensitive to the basic-state update frequency.

9. Impact of parameterizations

Because the uncertainties discussed in section 3 are basically related to physical processes, and because the TLM contains the first derivatives of those processes, their representation in the NLM may be crucial to the success of the TLM. Here, we investigate changes in the general properties of the TLM solutions due to changes in the formulation of both the microphysical parameterization as well as the representation of a dynamical process. The former is tested by varying the formulation of the terminal velocity, while the latter involves prescribed variations in the entrainment coefficient.
TABLE 1. Description of experiments with parameterizations of terminal velocity.

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Terminal velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>V1</td>
<td>$V_1 = 3120(\rho Q)^{0.125}$</td>
</tr>
<tr>
<td>V2</td>
<td>$V_2 = 3634(\rho Q)^{0.164}$</td>
</tr>
<tr>
<td>V3</td>
<td>$V_3 = 2118(\rho Q)^{0.2}$</td>
</tr>
<tr>
<td>V4</td>
<td>$V_4 = 2910(\rho Q)^{0.125}$</td>
</tr>
</tbody>
</table>

A 10% perturbation is used in all experiments since, in our control run, the TLM solutions (i.e., linear perturbation) are in good agreement with the NLP for this size of uniform error.

a. Terminal velocity formulation

The terminal velocity $V$ plays an important role in redistributing rainwater throughout the cloud. It is parameterized in the form

$$V = \alpha(\rho Q)^{\beta}$$

where $\rho$ is the air density and $\alpha$ and $\beta$ are dimensionless coefficients.

Table 1 summarizes the terminal velocity experiments, where $V_i$ is expressed in centimeters per second, $\rho$ in grams per cubic centimeter and $Q$ in grams per gram. Here, $V_1$ represents a mean volume-weighted $V$ (OT71), while $V_2$ is derived from an empirical formula relating rainwater content and rainfall intensity (Soong and Ogura 1973). Here, $V_3$ describes a mean water content-weighted $V$ (Liu and Orville 1969), and $V_4$ is the terminal velocity for a median volume diameter (Kessler 1969). Note that the air density is implicit in $V_3$. For $\rho = 10^{-3}$ g cm$^{-3}$ and $Q = 10^{-3}$ g g$^{-1}$, we obtain $V_1 = 5.55$, $V_2 = 5.52$, $V_3 = 5.32$, and $V_4 = 5.17$ m s$^{-1}$. In the control run, $\alpha$ and $\beta$ are set to 3124 and 0.125, respectively, which results in $V = 5.56$ m s$^{-1}$. Thus, only small numerical variations occur in $V$ among all parameterizations.

Except for $V4$, all experiments show good agreement between the NLP and corresponding TLM results. To illustrate, we show in Fig. 11 the $Q$ field for case V4. The NLP field increases during the late mature stage and decreases during the decaying stage (Fig. 11a). This indicates that, in the perturbation NLM run, a 10% increase in the initial updraft leads to an earlier occurrence of the AMQ, and hence earlier decay of the storm. In both the NLP and TLM fields, as $\alpha$ and the parameterized $V$ become smaller, the onset of the decaying stage (zero contour) occurs later due to the extension of the storm. This is physically plausible because the updraft is less hindered when $V$ is smaller.

In general, the TLM solutions describe the pattern of the NLP very well (experiment V1 shows the best agreement), especially the locations of local extrema and the sign change across the zero contour (i.e., transition from the mature stage to the decaying stage). Furthermore, for each experiment, the zero contour of the TLM solution during the decaying stage exactly matches the AMQ in the NLM run (not shown), which indicates that the TLM provides the correct derivative information. However, the TLM solutions are very different in amplitude, especially during the late mature and the decaying stages, where strong temporal and spatial gradients are observed in all fields.

Note the tremendous difference in experiment V4 (cf. Figs. 11a and 11b), where the TLM fails to describe the magnitude of the NLP even at 40 min, or early in the mature stage. The TLM solutions are up to five times larger than the corresponding NLP fields during the late mature stage and four times larger during the decaying stage. It is evident that the characteristics of TLM solutions depend significantly on the manner in which...
microphysical processes are parameterized. For all these experiments, the characteristics of the TLM and NLP fields do not change with small perturbations; that is, when the initial perturbation is decreased, the disagreement between the two fields also decreases, though they are never asymptotic. For example, the qualitative performance of the TLM in V4 did not improve even when the error was reduced to 0.1%.

Considering the similarities in the formulation of experiments V1 and V4, the large difference in the TLM solutions is quite surprising. In Fig. 12, we compare the behavior of the TLM, NLP and high-order terms (HOT; i.e., NLP minus TLM) of $Q_r$ at $z = 3$ km for two experiments. Note that the ordinate of case V1 is scaled by a factor of 0.2 to that of case V4; therefore, the NLP fields depict a similar magnitude and pattern for both experiments except for the time shift near the decaying stage.

The points where the difference curves cross zero reflect the transition from the mature to the decaying stage. In case V1 (Fig. 12a), the Taylor expansion is well approximated; that is, the total error (NLP) is closely mimicked by the first-order term (TLM), and the HOT stay smaller than the TLM throughout the simulation. All fields change in the same direction and cross zero at the same time. Thus, the transition from the mature to the decaying stage in the NLP is accurately represented by the TLM.

In experiment V4 (Fig. 12b), the behavior of each field is quite different, though the zero crossing occurs at the same time. During the late mature and decaying stages, the first-order term is exceptionally large and the magnitude of the HOT are comparable to that of the latter but in the opposite direction. They compensate each other to produce NLP fields similar to those in case V1. The nonlinearity does not increase significantly due to the large magnitude of the TLM (actually the NLCs of V4 at the latter stages were approximately 0.8 and lower than those of V2 and V3). The correlation coefficients are also relatively high (greater than 0.98 after 70 min) during the latter stages since most errors are in the form of amplitude (dissipation), not phase (dispersion).

We also investigate the behavior of the HOT for various choices of $\alpha$ [see Eq. (16)], ranging from 3124 to 2910 with $\beta = 0.125$. In Fig. 13, the HOT in the $Q_r$ field at 3 km are shown for selected values of $\alpha$. As $\alpha$ becomes smaller, the zero crossing, that is, the transition from the mature to decaying stage, occurs later, and hence the mature stage lasts longer. Except for V4 ($\alpha = 2910$), all experiments show an increase in the difference fields during the late mature stage followed by a decrease during the early decaying stage, indicating that the TLM generally underestimates the absolute values of the NLP. Experiment V4 shows a remarkably different behavior in both amplitude and pattern.

We also examined the same field for values of $\alpha$ rang-
ing from 2912 to 2908 in one-unit decrements, which corresponds to dimensional decrements of approximately 0.002 m s\(^{-1}\) in \(V\) (not shown). Extraordinary behavior was observed only when \(\alpha = 2910\). For very small changes in \(\alpha\) (e.g., around 2910), we observed a chaotic change in the behavior of the TLM solutions even though the corresponding NLM solutions did not show any substantial variability. It is possible that \(\alpha = 2910\) induces a more prominent discontinuity in the NLM solutions compared to other values of \(\alpha\) such that both the first-order term (i.e., TLM) and higher-order terms have large magnitudes (see Fig. 12b).

The sensitivity of the TLM solution to variations in parameters such as \(\alpha\) should not be unexpected. The highly nonlinear and often discontinuous nature of the parameterization associated with microphysical processes is not unlike well-known chaotic problems, e.g., the logistic map. The extent to which this sensitivity impacts parameter estimation and optimization is not completely known and will remain the subject of further studies.

b. Lateral eddy mixing

As explained in section 2a, entrainment in our model is composed of two parts; *lateral eddy mixing* and *dynamical entrainment*. The lateral eddy mixing has the same form as that for the entrainment in other models, with the entrainment rate given by

\[
\mu = \frac{b}{R} |w|,
\]

(17)

where \(b\) is a dimensionless coefficient and \(R\) is the radius of the cloud. The value of \(b\) varies by a factor of three or more for a given cloud size depending on the environmental conditions (Cotton 1975) and is set to 0.2 in our control run. Ferrier and Houze (1989) demonstrated that including the lateral mixing term can induce a significant change in convection due to the rapid mixing of the environmental air into the convective cell. A cloud with larger value of \(b\) has a lower cloud top and smaller maximum updraft (Cotton 1975). In our experiments, we do not change the form of the dynamical entrainment term, but vary only the value of \(b\) in the lateral mixing term. The experiment with \(b = 0.4\) is denoted E1, while that with \(b = 0.6\) is denoted E2. Perturbation runs were performed for each case with a 10% perturbation in the initial \(w\) field.

We first ran several experiments neglecting the dynamical entrainment term entirely and using various values for \(b\) in the lateral mixing term. In them, the NLM runs developed a steady-state cloud as described in Currie and Jance (1989). In addition, for a given value of \(b\), the perturbation NLM runs showed little difference from their control counterparts, and the NLP and TLM fields were in good agreement. In our model, the turbulent mixing term drives the model to steady state and the dynamical term, which has a much greater influence, induces a variety of complications in cloud development. Our discussion will thus focus on the cases where both terms are included.

The major differences in the unperturbed NLM runs for experiments E1 and E2 are 1) the maximum updraft in E1 (13.8 m s\(^{-1}\)) is larger than in E2 (12.6 m s\(^{-1}\)); 2) the cloud top in E1 is higher than in E2; and 3) the lifetime of the storm in E2 exceeds that in E1 by about 4 min. In experiment E1, the NLP field is described well by the TLM field in both magnitude and pattern. In this case, the perturbation run showed no change in the cloud-top height. However, in experiment E2, a noticeable change in both the NLP and TLM fields occurs, especially near cloud top during the mature stage. In contrast to experiment E1, a 10% increase in the initial updraft in E2 induces lifting of cloud top during the entire mature stage, resulting in a significant change in the NLP. However, when we reduced the magnitude of the initial error to 5%, this disagreement disappeared and the NLP fields were accurately described by the TLM.

Finally, in contrast to experiments in which the terminal velocity formulation was varied, the lateral mixing formulation tests show that the TLM solutions approach the NLP solutions asymptotically as the perturbation size is reduced. This suggests that greater solution sensitivity occurs when perturbations are applied to fields directly associated with on-off switches (i.e., microphysical quantities or parameters in terms describing microphysical processes).

10. Summary and discussion

We used a 1D nonlinear Eulerian cloud model (NLM) to investigate the validity of the associated tangent linear model (TLM) for deep moist convection. The discontinuous and nonlinear properties of solutions produced by such a model may have a deleterious impact on the results of the associated adjoint model (ADJM) (which is based on the tangent linear approximation of the complete model) and hence may produce inaccurate results when applied to sensitivity analysis, parameter estimation, and data assimilation. The dynamical framework of our model is simple enough to make the investigation computationally tractable, yet the physical processes are sufficiently complex to allow for a realistic assessment of TLM validity.

The behavior of the nonlinear perturbation (NLP) (the difference between a control and perturbed forecast from the full nonlinear model) was found to be very sensitive to the magnitude of initial errors applied to the vertical velocity, and thus the linear approximation was valid only for a specific range of small perturbations. The largest disagreement between the TLM and the NLP fields occurred in a thin layer near the cloud top during the mature stage, where unrealistically strong gradients in all fields were observed.

The TLM validity was strongly affected by nonlin-
earity. A precipitous increase in the nonlinearity coefficient (NLC) was followed by increases in both the dissipation and dispersion errors between the TLM and NLP fields. The latter increase was mainly ascribed to the change in cloud dynamics, especially near cloud top, for a given amount of error, and became a critical factor for determining the validity of the TLM in our experiments. The variation of the correlation between the TLM and NLP fields was found to be related to that of the NLC during the early storm stage and then to both the NLC and the dispersion error (EDISP) after a notable amount of EDISP was present.

The activation times of on–off switches in microphysical processes may vary considerably for a large perturbation, and this variation can affect the accuracy of TLM. Experiments with variations applied to liquid water variables demonstrated that the validity of the TLM is more likely impacted by the dynamical characteristics of the simulated cloud, possibly forced by moist physical processes, than the discontinuous properties associated with microphysical parameterizations themselves. Discontinuities and strong gradients in the latter, especially evaporation, play an important role in TLM behavior during the decaying stage in an unsteady cloud. When a lower bound is given on the magnitude of the water variables, significant changes are noticed in the NLM, TLM, and NLP solutions, but with excellent agreement between the TLM and NLP fields. This is possibly due to the suppression of nonlinearity in high-order terms.

The TLM results based on the control simulation appeared to be extremely sensitive to the frequency with which the nonlinear basic states were updated. The correlation between the NLP and TLM fields decayed rapidly even with an update period as small as 3τt. However, this result depends on the environmental sounding and advection scheme. For example, with a more stable sounding in the low levels and a 1-2-1 averaged advection velocity (case A3), the correlation exceeded 0.99 for all time even with a 10Δτ update period.

The validity of TLM also critically depended on the parameterization of both microphysical and dynamical processes. However, the nature of TLM solution variability is significantly different for each process. For a microphysical process, a small change in the coefficient of a physical parameter may result in erratic behavior of the TLM solution, and this behavior cannot be mitigated (in an asymptotic sense) by decreasing the magnitude of the initial error. On the other hand, for a dynamical process, the TLM and NLP solutions approached each other asymptotically when the magnitude of the initial error was reduced. This implies that, when uncertainty is imposed on the parameterization of a microphysical process, the TLM may be strongly affected by the sources of uncertainty inherent in moist physical processes such as discontinuities, on–off switches, and nonlinearities. For a specific coefficient in the physical parameterization, those properties might dominate the model solution and lead to incorrect derivative information in the TLM. Dynamical parameters may also impact microphysical processes through nonlinear interaction; however, they may have less of an effect on the TLM solution since they do not involve direct differentiation of moist variables.

In our model, a single control parameter in the conversion process from cloud to rain was found to play a vital role in a regime transition between steady and nonsteady cloud behavior. The TLM solution, which follows the trajectory of the NLM solution, is unable to describe this regime change. One thus needs to exercise caution when using gradient information near such transitions for data assimilation and parameter estimation.

The regime change or bifurcation similar to that observed in our model also occurs in many other simple dynamical systems such as the logistic equation and the Lorenz system (e.g., Thompson and Stewart 1986). However, because those systems have only a few control parameters and variables, the dynamical behavior of the solution is highly dependent on changes in control parameters or initial conditions. Because a 1D cloud model has very simple dynamics with a relatively small number of variables compared to those in 3D cloud models or in nature, its solutions may be more strongly controlled by microphysical processes. That is, the results from simple dynamical systems often present an overly pessimistic outlook with respect to sensitivity or predictability.

To overcome the shortcomings inherent in our 1D model, we have extended our investigation of the TLM validity to a fully 3D cloud model (Park 1996). Those results will be reported in a subsequent paper.

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APPENDIX

The 1D Eulerian Cloud Model

The 1D cloud model is based on that of OT71 except for differences in some physical processes, including: 1) absence of ice physics; 2) use of the Soong and Ogura (1973) saturation technique for computing the condensation rate; 3) advection velocity and initial sounding (see their section 4); 4) terminal velocity.

In the following equations, variables with a “zero” subscript denote environmental quantities, and those with a tilde (~) denote values at the perimeter of the cloud.
The prognostic variables are vertical velocity \(w\) (m s\(^{-1}\)), temperature \(T\) (K), and mixing ratios (g g\(^{-1}\)) of water vapor \((Q_v)\), cloud water \((Q_c)\), and rainwater \((Q_r)\). Since we use notation similar to OT71, the reader is referred to OT71 for an explanation of each term and for details regarding the computation of the tilde variables.

The equation for vertical velocity is given by

\[
\frac{\partial w}{\partial t} = - w \frac{\partial v}{\partial z} \frac{\alpha}{R} [w] w + \bar{u} (w - \bar{w}) + \frac{T}{T_{\text{ref}}} - \frac{T_v}{T_{\text{ref}}} - g (Q_c + Q_r), \tag{A1}
\]

where \(t\) is time, \(z\) is height, \(\alpha\) is the nondimensional coefficient for lateral eddy mixing at the perimeter of the cloud, \(u\) is the radius-weighted radial velocity, \(g\) is the acceleration due to gravity, and \(T_v\) is the virtual temperature. In (A1), \(\alpha\) and \(R\) are set to 0.2 and 3 km, respectively, with \(\bar{u}\) determined by mass continuity for dry air:

\[
\bar{u} + \frac{1}{\rho_0} \frac{\partial (\rho_0 w)}{\partial z} = 0, \tag{A2}
\]

where \(\rho_0\) is the environmental air density (kg m\(^{-3}\)).

The equations for temperature, water vapor, cloud, and rainwater are, respectively,

\[
\frac{\partial T}{\partial t} = - w \frac{\partial T}{\partial z} (\Gamma_d) - \frac{\alpha}{R} [w] (T - T_{\text{ref}}) + \bar{u} (T - \bar{T}) + \frac{L_v}{C_p} (P_1 - P_2 - P_3) \tag{A3}
\]

\[
\frac{\partial Q_v}{\partial t} = - w \frac{\partial Q_v}{\partial z} - \frac{\alpha}{R} [w] (Q_v - Q_{v,0}) + \bar{u} (Q_v - \bar{Q}_v) - P_1 + P_2 + P_3 \tag{A4}
\]

\[
\frac{\partial Q_c}{\partial t} = - w \frac{\partial Q_c}{\partial z} - \frac{\alpha}{R} [w] (Q_c - Q_{c,0}) + \bar{u} (Q_c - \bar{Q}_c) - P_1 - P_2 - P_4 \tag{A5}
\]

\[
\frac{\partial Q_r}{\partial t} = - (w - V) \frac{\partial Q_r}{\partial z} + \frac{Q_r \partial (\rho_0 V)}{\rho_0 \partial z} - \frac{\alpha}{R} [w] (Q_r - Q_{r,0}) + \bar{u} (Q_r - \bar{Q}_r) - P_1 + P_4, \tag{A6}
\]

where \(C_v\) is the specific heat of dry air, \(\Gamma_d\) the dry adiabatic lapse rate, \(L_v\) the latent heat of vaporization, and \(V\) the terminal velocity, given by

\[
V = 3124 \times (\rho_0 Q_v)^{0.125} \text{ cm s}^{-1} \tag{A7}
\]

for \(\rho_0\) in units of grams per cubic centimeter [cf. with Eq. (14) of OT71]. Note that \(V\) and \(\rho_0\) in (A6) are in meters per second and kilograms per cubic meter, respectively. Here, \(P_1\) represents the condensation rate, \(P_2\) the evaporation rate of cloud water, \(P_3\) the evaporation rate of rainwater, and \(P_4\) the conversion rate from cloud droplets to raindrops:

\[
P_1 = r_i (Q_v^* - Q_v) \tag{A8}
\]

\[
P_2 = - \frac{1}{\rho_0} \frac{5.4 \times 10^4}{4.1 \times 10^7} \frac{Q_v}{Q_{v,0} - 1} (\rho_0 Q_v)^{0.525} \tag{A9}
\]

\[
P_3 = - \frac{1}{\rho_0} \frac{5.4 \times 10^4}{4.1 \times 10^7} \frac{Q_c}{Q_{c,0} - 1} (\rho_0 Q_c)^{0.525} \tag{A10}
\]

\[
P_4 = C_0 Q_r, \tag{A11}
\]

where the superscript asterisk [as in (A8)] represents the value after the dynamical terms in the model have been computed, \(Q_v^*\) is the saturation mixing ratio, \(r_i\) is the coefficient calculated from the saturation technique (Soong and Ogura 1973; cf. with \(P_1\) in OT71), \(e_w\) is the saturation water vapor pressure, \(C_0\) is the cloud-to-rain conversion rate, and \(C\) is the ventilation coefficient, given by

\[
C = 1.6 + 0.57 \times 10^{-3} V^{1.5}. \tag{A12}
\]

A typical value for \(C_0\) is 0.005 s\(^{-1}\), which is used in our study and corresponds to approximately 3.3 min.

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