NOTES AND CORRESPONDENCE

Comparing Schemes for Integrating the Euler Equations

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ABSTRACT

Over the years there have been a number of studies comparing the relative merits of semi-Lagrangian and Eulerian schemes. These studies, which continue to appear in the literature up to the present, almost invariably conclude that semi-Lagrangian schemes are superior in accuracy, and produce less noise, than Eulerian schemes. It is argued in this note that such conclusions are not justified because they have compared semi-Lagrangian and Eulerian schemes of *different orders* of accuracy. Typically, the semi-Lagrangian schemes tested have employed cubic spatial interpolation (and therefore are third order) in space, whereas the Eulerian schemes have usually been second order (and sometimes fourth order) in space. It is shown here that when semi-Lagrangian and Eulerian schemes of the *same order* are applied to the test case, namely, that of ''warm bubble'' convection, there are almost indiscernible differences between the simulations. The contention presented here, therefore, is that it is the order of the scheme that is of primary importance, not whether it is semi-Lagrangian or Eulerian.

1. Introduction

A question that has been around for as long as researchers have been concerned with solving the atmospheric equations is whether there is a scheme that is clearly superior to the others. If the answer is yes, which one is it? In recent years the choice of schemes is becoming more clear-cut with, broadly speaking, atmospheric modelers currently facing a decision between two very different types of schemes, namely that between semi-Lagrangian and Eulerian techniques. There are many variants on both these approaches, but in this study we will look only at common forms of these schemes. In the case of the semi-Lagrangian schemes, the formulation of Robert (1993) can be taken as representative, while the Eulerian approach usually is represented by a scheme that is second order, third order (or higher) in space and second order in time (e.g., Clark 1977; Klemp and Wilhelmson 1978; Smolarkiewicz and Grabowski 1990). There are a growing number of studies in the literature that purport to provide a definitive answer to the question of how well the semi-Lagrangian and Eulerian approaches match up when applied to solving the Euler equations. The standard test used for the comparison has been the familiar ''warm bubble'' rising in an isentropic atmosphere, and the comparative studies seem to have overwhelmingly endorsed the semi-Lagrangian approach, suggesting that not only is it more efficient, but also that it is more accurate and less noisy. Two quotations serve to illustrate the point. When commenting on the bubble convection simulations of Smolarkiewicz and Grabowski (1990), who used a second-order Eulerian model, Robert (1993) contrasts his own semi-Lagrangian simulations and notes that, ''the results of Smolarkiewicz and Grabowski are noisy. This is not the case with the proposed model'' (Robert 1993, 1867). Even more recently, Pellerin et al. (1995, 3329) commented that, ''The semi-Lagrangian scheme suffers from little numerical dispersion, and the simulated results appear to have less noise than many Eulerian . . . schemes.''

The aim of this note is to demonstrate that there is an alternative, and more credible, explanation for the apparent superiority of the semi-Lagrangian schemes to the Eulerian alternative. Simply put, the semi-Lagrangian schemes in the above-mentioned numerical experiments were of a different formal *order* of accuracy (usually higher and odd) than that of the Eulerian schemes (usually lower and even). The thesis proposed here, therefore, is that the accuracy and level of noise in the numerical experiments has much to do with the accuracy of the scheme (higher-order schemes are gen-

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erally more accurate) and whether it is odd or even (odd-order schemes are generally less noisy than even-order ones) and little to do with whether it is semi-Lagrangian or Eulerian. Apart from the results presented in section 2, further support for this thesis comes from the work of Dietachmayer (1990) and Bates (1991), who have proven the formal equivalence of the semi-Lagrangian and Eulerian formulations, in specific applications.

2. The models

The model employed here is based on the so-called Euler equations, which, as mentioned in the introduction, are the complete equations governing motion in the atmosphere. As such, they allow sound waves because they avoid making either the hydrostatic or anelastic approximations invoked in many meteorological applications. However, the applications of numerical modeling in meteorology are extending rapidly and many centers are now using, or plan to use, the full Euler equations. In the atmospheric sciences there are at present two classes of schemes competing for favor, the semi-Lagrangian and the Eulerian schemes. These classes have a number of subsets, but the schemes described immediately below are among the most commonly used forms.

a. Semi-Lagrangian scheme

The scheme used here is almost identical to that of Robert (1993) in that it is a semi-Lagrangian semi-implicit (SLSI) scheme. The temporal differencing is second-order centered, and the spatial differencing is optionally second (biquadratic) or third order (bicubic). It is perhaps worthwhile noting that, elsewhere, Leslie and Purser (1995) have described a later version of the SLSI model that employs a higher-order differencing scheme that is up to sixth order in both space and time. However, because the form of the SLSI model used here is that of Robert (1993), the reader is referred to the details contained within that reference.

b. Eulerian scheme

In the Eulerian scheme for this study, we use a split-explicit approach similar to that of Klemp and Wilhelmson (1978). Briefly, and again the reader is referred to the original reference for full details, the governing equations are integrated on two distinct timescales, one a slow (long) time step, and the other time step small enough to represent the fast (sound) waves. This approach is commonly known as the ''split-explicit'' method. The temporal differencing is second order, being Adams–Bashforth (Roache 1972) for the large time steps and forward–backward for the small time steps. In the present study, the Adams–Bashforth scheme, which is sufficiently stable (very weakly unstable), is used exclusively in the Eulerian model. It is noted also that the spatial differencing of the advective terms is on a staggered grid (the so-called Arakawa B grid) and allows for an Eulerian upwind scheme of arbitrary order from one up to six. However in this study, only results from spatial differencing of orders 2 and 3 are presented, as that is all that is required to support our arguments.

3. The warm-bubble experiments

The numerical experiments were, as stated above, modeling the evolution of a thermal, or warm bubble, rising in an isentropic atmosphere. The computational domain was taken to be 800 m in both breadth and depth, with a uniform grid spacing of 10 m. That is an 81 × 81 grid of points. The initial state is one of zero motion with a bubble of relatively hot air with its center located at 200 m above the ground and 400 m from both the left- and right-hand lateral boundaries. The bubble is 250 m in diameter and is shown in Figs. 1a and 1b. Unlike most of the warm-bubble experiments, it is not discontinuous at the edge, with the distribution of potential temperature having a constant interior value 0.5° in excess of the environment and an exponential decay away from the center. This warm bubble was suggested by Robert (1993, 1870). The four key experiments are described immediately below.

a. Experiment 1: Third-order-in-space semi-Lagrangian scheme

The first experiment is very similar to that performed by Robert (1993) in that we employ a second-order-in-time (three-time-level), third-order-in-space (bicubic interpolation) semi-Lagrangian scheme to predict the motion of the warm bubble out to 6 min. The evolution of the bubble at that time is shown in Fig. 2a and is consistent with the comments of Robert (1993) and Ostiguy and Laprise (1990) that the scheme appears to do an excellent job in capturing the fine structure of the thermal and exhibiting no sign of noise. Although it was not central to the aim of this study, a range of time steps was used, from the value of about 1-s value allowed by the CFL (Courant–Friedrichs–Lewy) criterion for an Eulerian model, up to 6 s. There were obvious signs of deterioration in the simulation with time steps greater than 4 s. The deterioration took the form of spurious extrema and broadening of transition zones between warm and cool fluid.

b. Experiment 2: Second-order-in-space Eulerian scheme

The next simulation employed a split-explicit Eulerian scheme, similar to that of Klemp and Wilhemson (1978). The differencing scheme was second order in time and second-order centered in space. The thermal is again shown at 6 min in Fig. 2b and is very poor, with a great deal of noise present. It is obvious im-

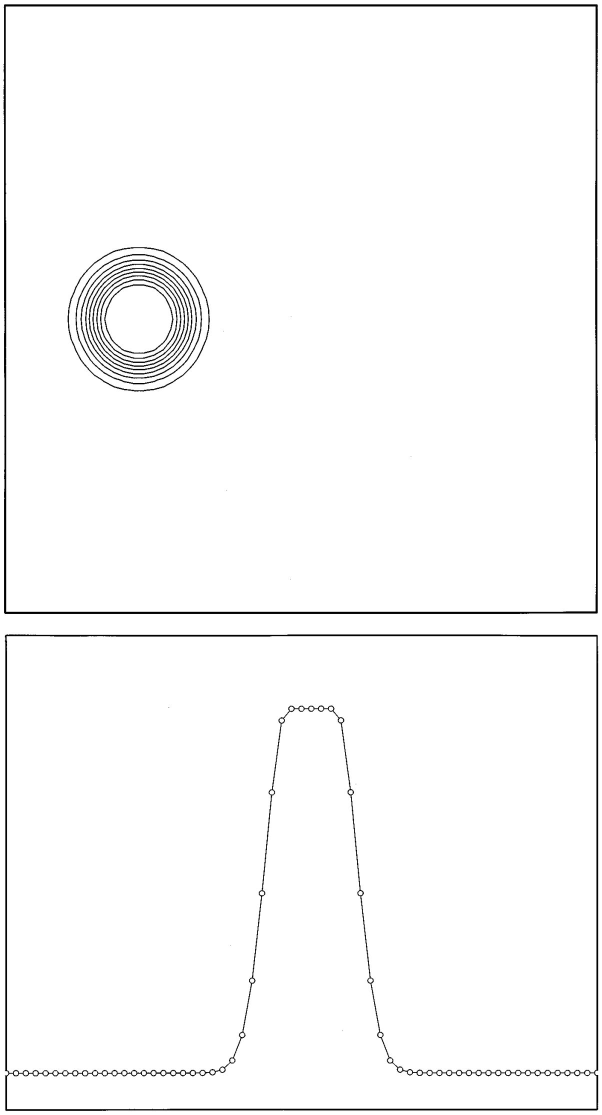

FIG. 1. (a) The warm bubble at time $t = 0$ s. (b) A section through the warm bubble showing the constant temperature inner core and the exponential decay to the environment.

mediately from the simulation why such an Eulerian scheme has received such adverse comments as those of Ostiguy and Laprise (1990), Robert (1993), and Pellerin et al. (1995), among others. Compared with the simulation in experiment 1, the simulation in experiment 2 is indeed vastly inferior.

c. Experiment 3: Third-order-in-space Eulerian scheme

In this numerical experiment the order of the advective terms in the Eulerian scheme of experiment 2 was increased from second-order centered to third-order upwind. The upwind approach has not been widely used in the meteorological modeling community but has found favor in other disciplines because of its excellent phase and amplitude properties. There is almost no numerical dispersion, and the dissipation comes in through the leading truncation error term, which is a ∇^4 operator, and as such is no more severe than that introduced by common filtering attempts to remove the two-grid-length noise generated by finite-difference algorithms. Note that the thermal at 6 min (see Fig. 2c) is so close to that of the semi-Lagrangian simulation of experiment 1 that it takes some effort to detect any differences, and these differences are not worth noting.

d. Experiment 4: Second-order-in-space semi-Lagrangian scheme

The fourth and final experiment in the series is to reduce the spatial differencing in the semi-Lagrangian scheme from third order down to second order (biquadratic interpolation) and time. As such, it possesses the same formal order of accuracy as the scheme in experiment 2, namely a second-order Eulerian scheme. The warm bubble is again shown in Fig. 2d at 6 min into the integration period. The similarity with the Eulerian simulation in experiment 2 (Fig. 2b) is as striking as the similarity between experiments 1 and 3, but in the opposite sense that whereas experiments 1 and 3 are excellent simulations, both experiments 2 and 4 are unacceptable. There is very little discernible difference between experiments 2 and 4, with both simulations being poor and dominated by noise. The loss of accuracy results from the decrease in spatial order of accuracy from third to second, and the noise has been introduced by the fact that the spatial differencing is now even order, so that the leading error terms are now third order and do not act to dampen the noise, unlike the even-order leading truncation error terms associated with odd-order spatial differencing.

4. Summary and discussion

This study had as its principal aim to show that previous work comparing the relative merits of semi-Lagrangian and Eulerian integration schemes have wrongly concluded that the semi-Lagrangian schemes are *intrinsically* superior to the Eulerian schemes. It has been argued here that these studies were not "clean" experiments in that they compared semi-Lagrangian and Eulerian schemes of different orders of accuracy. The conclusions drawn were that semi-Lagrangian schemes are demonstrably superior. In this study we showed that not only are the earlier results easily reproduced, but more significantly that when the orders of the semi-Lagrangian and Eulerian schemes were reversed, the semi-Lagrangian schemes were then clearly inferior to the Eulerian schemes. Finally, when the semi-Lagrangian and Eulerian schemes were of the same order, almost indistinguishable results were obtained. There are a number

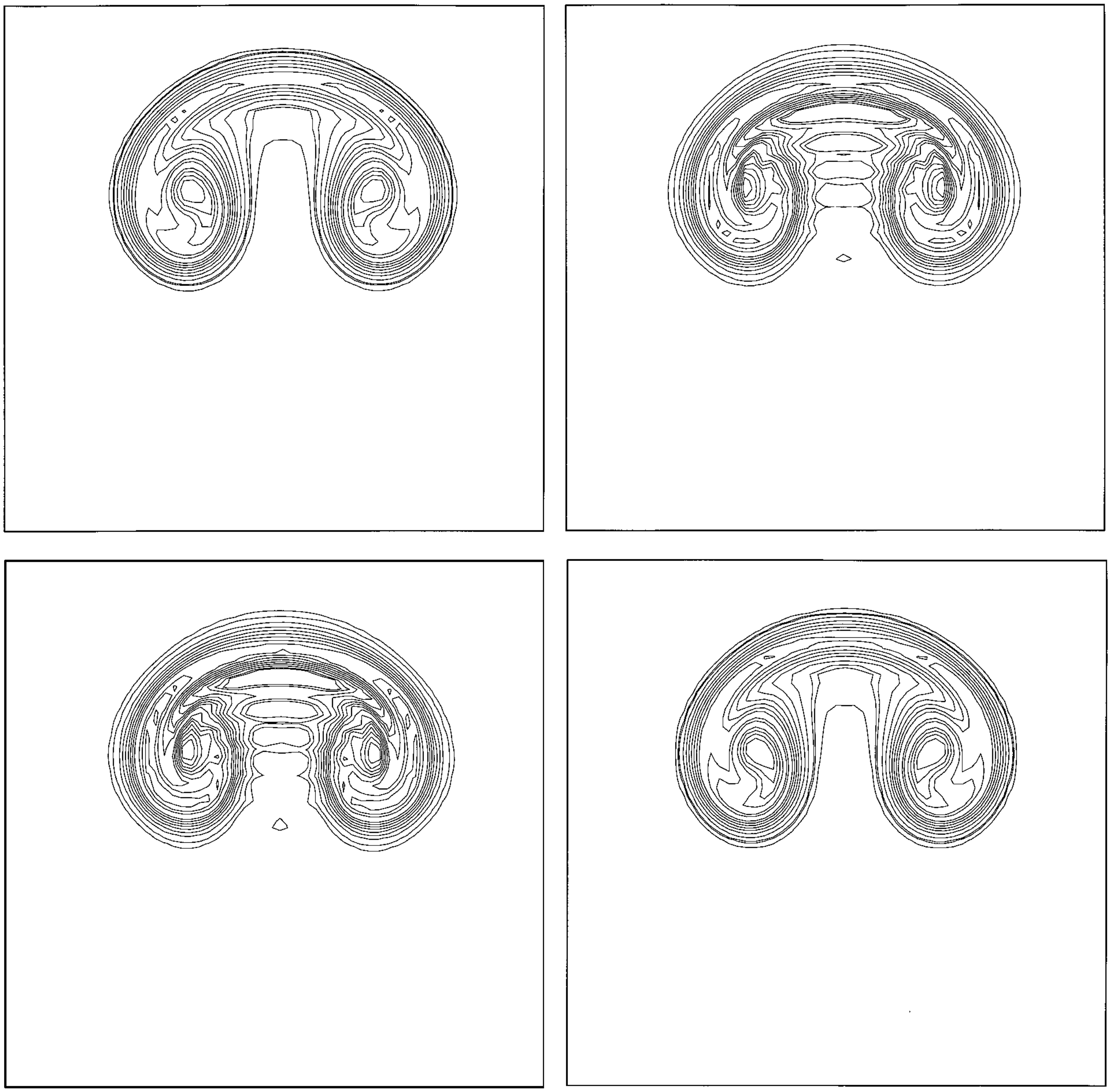

FIG. 2. (a) Contour plot of the temperature field at $t = 6$ min for the warm-bubble convection experiment, using a semi-Lagrangian scheme with third-order (bicubic) spatial interpolation. (b) As in (a) except for the Eulerian scheme with second-order centered spatial differencing. (c) As in (b) except with third-order upwind spatial differencing. (d) As in (a) except for second-order (biquadratic) spatial interpolation.

of other questions relating to the importance of odd versus even differencing, and the relative efficiency of the semi-Lagrangian and Eulerian approaches, but they are not central to the aims of this study.

In summary, it is argued here that the dispersion and dissipation properties of the semi-Lagrangian schemes when compared with Eulerian schemes has little to do with whether the scheme is semi-Lagrangian or Eulerian, but rather it is the *order* of the algorithm that determines the properties of the algorithm. When semi-Lagrangian schemes of a given order are compared with Eulerian schemes of the same order, the numerical simulations are very difficult to distinguish from each other. On the other hand, when schemes of different order are compared, misleading conclusions have been made about the relative merits of the two different approaches.

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