Conservation and Linear Rossby-Mode Dispersion on the Spherical C Grid

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ABSTRACT

Discretizations of the linearized shallow-water equations on a spherical C grid are considered. Constraints on the schemes' coefficients that ensure conservation of mass, angular momentum, and energy are derived. These results generalize previously published results to the case of nonuniform and rotated grids (but are restricted to the linearized equations). Grids with v stored at the poles and grids with u and h stored at the poles are both considered. Energy conservation is shown to be problematic for grids with u and h at the poles.

It is also shown that an inappropriate averaging of the Coriolis terms leads to a misrepresentation of the Rossby modes with shortest meridional scale. The appropriate averaging is shown to be compatible with the constraints required for conservation, and, indeed, the energy-conserving averaging of the Coriolis terms improves the dispersion properties of Rossby modes.

1. Introduction

Arakawa and Lamb (1981) derived a spatial finite-difference scheme on the C grid for the shallow-water equations on the sphere that conserves finite-difference analogues of mass, energy, and potential enstrophy, and preserves an initially constant potential vorticity. Their scheme is second-order accurate. A version of the scheme described as fourth order was derived by Takano and Wurtele (1981) and is also presented by Arakawa (2000).

In considering the design of the Met Office Unified Model, particularly its recently implemented operational ''New Dynamics'' version (Cullen et al. 1997), which uses a C grid, some issues have arisen with regard to the horizontal discretization:

1) Is it better to place the v wind component at the poles, as in the Arakawa and Lamb (1981) scheme, or the u wind component and the geopotential or mass variable h, as in the current formulation of the New Dynamics?
2) Is it better to use the wind components u and v, or ''wind images'' $U = u \cos\phi$, $V = v \cos\phi$ (ϕ is latitude), or something else as prognostic wind variables?
3) Can any advantage be gained by using $\mu = \sin\phi$ rather than ϕ as the latitudinal coordinate?
4) The Arakawa and Lamb (1981) derivation focused on the case in which the latitudinal coordinate is ϕ, with constant latitudinal and longitudinal grid spacing $\Delta\phi$ and $\Delta\lambda$; does their derivation apply, or can it be extended, to other coordinates and to variable grid spacing (e.g., Côté et al. 1993; Fox-Rabinovitz et al. 1997; Krinner et al. 1997) while retaining desirable conservation properties?
5) In spherical geometry, care must be taken in the horizontal averaging of the Coriolis terms to avoid badly misrepresenting many of the Rossby modes (see section 6). Is the required averaging compatible with obtaining desirable conservation properties? Can higher-order accuracy be obtained for the Coriolis terms, for example, by analogy with the Takano and Wurtele (1981) scheme?

It is appropriate to revisit the Arakawa and Lamb (1981) derivation to attempt to address these issues.

The Arakawa and Lamb (1981) and Takano and Wurtele (1981) schemes are Eulerian. However, the stability criterion for explicit Eulerian schemes restricts the premissible time step, and the restriction is particularly tight on a longitude–latitude grid because of the convergence of the meridians at the poles. Consequently, the New Dynamics version of the Met Office Unified Model, in common with many current weather prediction models and some climate models, uses semi-La-

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grangian advection because it allows longer time steps with enhanced stability. Several conservative semi-Lagrangian schemes have been proposed recently (e.g., Leonard et al. 1996; Lin and Rood 1996; Nair and Machenhauer 2002; Zerroukat et al. 2002). It is not yet known whether schemes like these can be extended to solve the full dynamical equations while preserving the conservation properties considered desirable by Arakawa and Lamb (1981) and Takano and Wurtele (1981), but it is with this possibility in mind that we are motivated to address issues 1–5 above. If conservative semi-Lagrangian schemes can indeed be developed, then those schemes must reduce to conservative linear schemes when nonlinearity is weak. If, on the other hand, conservative semi-Lagrangian schemes cannot be developed, it is nevertheless still desirable to minimize inaccuracies in conservation by ensuring that the linear terms, including the dominant terms representing balance, are conservative. In this paper, therefore, the pragmatic step is taken of considering only the linearized shallow-water equations (section 2a).

The strategy of Arakawa and Lamb (1981) was to begin by writing down a rather general scheme for the shallow-water equations involving several unspecified coefficients. The various desired conservation properties then imply constraints on those coefficients. Here a similar strategy is followed, but with some differences in detail:

1) On a rotating sphere, the potential vorticity can be constant over some region only when the wind or depth perturbation are of finite amplitude. For the linearized system considered here, the background potential vorticity gradient always dominates the perturbation, so preservation of a constant potential vorticity cannot be considered.
2) Here the conservation of angular momentum is considered; it was not explicitly considered by Arakawa and Lamb (1981).
3) Here conservation of potential enstrophy is not considered. There is some debate about whether conservation of potential enstrophy really is beneficial in numerical models. Some authors argue that potential enstrophy is an invariant of the continuous governing equations, and that its conservation in numerical models helps to ensure realistic energy spectra and energy cascades (e.g., Arakawa and Lamb 1977, 1981). Other authors argue that potential enstrophy is not a robust invariant—it cascades to small scales and must ultimately be dissipated—and therefore numerical schemes that are inherently dissipative at small scales, such as semi-Lagrangian schemes based on interpolation, can in fact lead to correct energy spectra and energy cascades, provided that the cascade is driven by well-resolved, accurately represented flow (e.g., Arakawa and Hsu 1990; Thuburn 1995; Brown et al. 2000).
4) Here it is noted that an inappropriate averaging of the Coriolis terms in combination with spherical geometry can cause the higher meridional Rossby modes to be badly misrepresented. The question of whether the appropriate averaging is compatible with the constraints arising from the conservation requirements is investigated.

In summary, conservation of mass (section 3), conservation of angular momentum (section 4), conservation of energy (section 5), and the accurate representation of Rossby modes (section 6) are considered. For each conserved quantity, the case with v at the poles and two possible approaches to using u and h at the poles are considered.

2. Governing equations

a. Linearized shallow-water equations in general orthogonal curvilinear coordinates

Following Arakawa and Lamb (1981), let ξ and η be orthogonal curvilinear coordinates, and let m and n be map factors so that the distance elements in the ξ and η directions are

$$ds_\xi = \frac{1}{m}d\xi, \tag{2.1}$$

$$ds_\eta = \frac{1}{n}d\eta. \tag{2.2}$$

For example, if $\xi = \lambda$ (longitude) and $\eta = \phi$ (latitude), then $1/m = a\cos\phi$ and $1/n = a$, where a is the earth's radius, while if $\xi = \lambda$ and $\eta = \mu = \sin\phi$, then $1/m = a\cos\phi = a\sqrt{1 - \mu^2}$ and $1/n = a/\cos\phi = a/\sqrt{1 - \mu^2}$.

Let velocity components in the ξ and η directions be u and v, respectively, and let h be the depth perturbation and $\Phi = gh$ the geopotential perturbation. Then the shallow-water equations linearized about a state of rest and a mean fluid depth H are

$$\left(\frac{u}{m}\right)_t - \frac{fv}{m} + \Phi_\xi = 0, \tag{2.3}$$

$$\left(\frac{v}{n}\right)_t + \frac{fu}{n} + \Phi_\eta = 0, \tag{2.4}$$

$$\left(\frac{h}{mn}\right)_t + \left(\frac{uH}{n}\right)_\xi + \left(\frac{vH}{m}\right)_\eta = 0, \tag{2.5}$$

where f is the Coriolis parameter, and subscripts t, ξ, and η indicate partial derivatives.

b. Discrete equations

These equations are to be discretized on the C grid; Fig. 1 illustrates the spatial arrangement of the variables

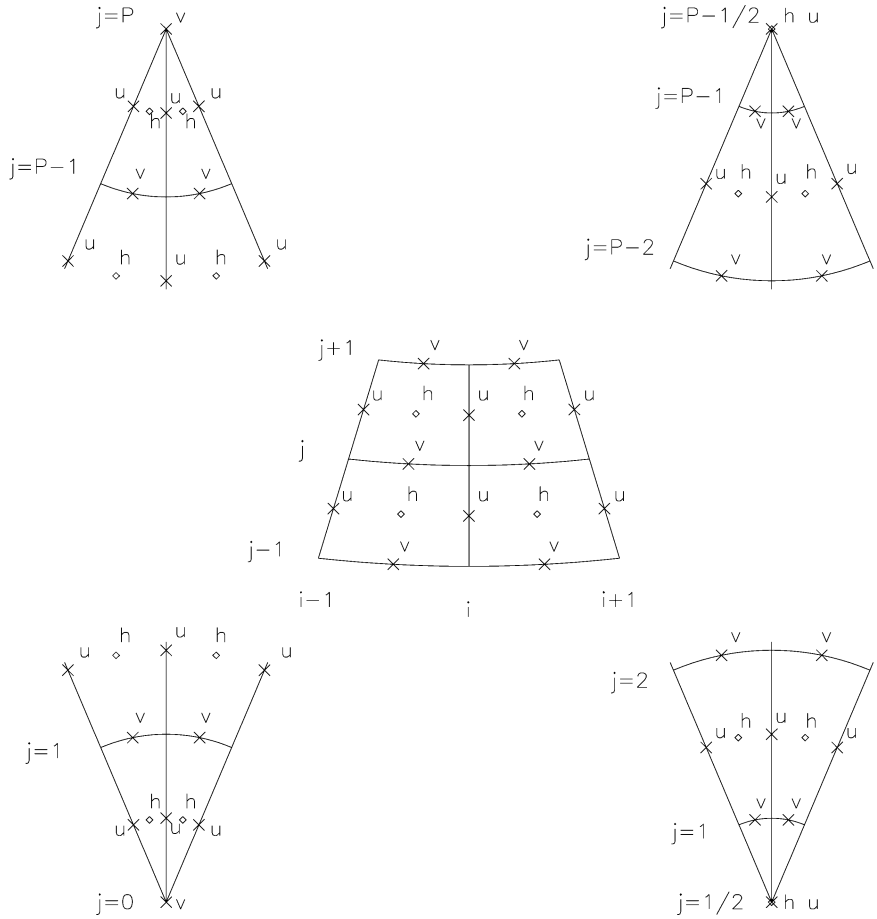

FIG. 1. Schematics showing the layout of variables on the C grid: The corner schematics show layouts near the poles when the poles are (left) v points and (right) u points.

and defines the convention for labeling grid locations. The discretization uses mass flux variables

$$u^*_{i,j+1/2} = \left(\frac{Hu\Delta\eta}{n}\right)_{i,j+1/2}, \tag{2.6}$$

$$v^*_{i+1/2,j} = \left(\frac{Hv\Delta\xi}{m}\right)_{i+1/2,j}, \tag{2.7}$$

and the grid cell mass

$$h^*_{i+1/2,j+1/2} = \left(\frac{\Delta\xi\Delta\eta h}{mn}\right)_{i+1/2,j+1/2}. \tag{2.8}$$

The following notes apply to these equations.

1) For the linearized system under consideration, one can simply take depth at u and v points to be H in (2.6) and (2.7). For the fully nonlinear system, however, some scheme for interpolating depth must be specified.
2) Arakawa and Lamb (1981) focused on the case of a nonrotated grid with constant $\Delta\xi$ and $\Delta\eta$. Here, as far as possible, these assumptions will be avoided. Of particular interest is the case of a grid uniformly spaced in latitude but with the coordinate itself given by $\eta = \mu$, so that $\Delta\eta$ is not constant.
3) It has not yet been specified how m, n, $\Delta\xi$, and $\Delta\eta$ are to be defined at the various grid locations. Clearly they must be approximations to the map factors and coordinate increments, and $(\Delta\xi\Delta\eta/mn)_{i+1/2,j+1/2}$, for example, must be an approximation to the area of cell $(i + 1/2, j + 1/2)$ (in fact, it could be exact if appropriately defined). But many choices are possible. For example, $\Delta\phi_{j+1/2}$ could be approximated as $\phi_{j+1} - \phi_j$ or as $(\sin\phi_{j+1} - \sin\phi_j)/\cos\phi_{j+1/2}$. The conservation requirements constrain the possible choices.

The linearized shallow-water equations may then be discretized in space as follows:

$$\frac{\partial}{\partial t}\left(\frac{u\Delta\xi}{m}\right)_{i,j+1/2} - \alpha_{i,j+1/2}v^*_{i+1/2,j+1} - \beta_{i,j+1/2}v^*_{i-1/2,j+1} - \gamma_{i,j+1/2}v^*_{i-1/2,j} - \delta_{i,j+1/2}v^*_{i+1/2,j} + \Phi_{i+1/2,j+1/2} - \Phi_{i-1/2,j+1/2} = 0, \tag{2.9}$$

$$\frac{\partial}{\partial t}\left(\frac{v\Delta\eta}{n}\right)_{i+1/2,j} + \gamma_{i+1,j+1/2}u^*_{i+1,j+1/2} + \delta_{i,j+1/2}u^*_{i,j+1/2} + \alpha_{i,j-1/2}u^*_{i,j-1/2} + \beta_{i+1,j-1/2}u^*_{i+1,j-1/2} + \Phi_{i+1/2,j+1/2} - \Phi_{i+1/2,j-1/2} = 0, \tag{2.10}$$

$$\frac{\partial}{\partial t}h^*_{i+1/2,j+1/2} + u^*_{i+1,j+1/2} - u^*_{i,j+1/2} + v^*_{i+1/2,j+1} - v^*_{i+1/2,j} = 0. \tag{2.11}$$

The following notes apply to the above equations.

1) Equations (2.9)–(2.11) may be regarded as a finite-volume discretization, though with (2.9) and (2.10) normalized by $\Delta\eta$ and $\Delta\xi$, respectively.
2) Coefficients α, β, γ, and δ define the horizontal averaging of the Coriolis terms. In particular, $\alpha + \beta + \gamma + \delta \approx f/H$ at any point for the linearized system under consideration. (For the nonlinear case α, β, γ, and δ will in general be functions of the flow field, in particular, the vorticity or potential vorticity.) For second-order accuracy the coefficients must satisfy

$$\alpha_{i,j+1/2}, \beta_{i,j+1/2}, \gamma_{i,j+1/2}, \delta_{i,j+1/2} = \frac{f_{i,j+1/2}}{4H} + O(\Delta\xi, \Delta\eta) \quad \text{and} \tag{2.12}$$

$$\alpha_{i,j+1/2} + \beta_{i,j+1/2} + \gamma_{i,j+1/2} + \delta_{i,j+1/2} = \frac{f_{i,j+1/2}}{H} + O(\Delta\xi^2, \Delta\eta^2) \tag{2.13}$$

for (2.9), and

$$\alpha_{i,j-1/2}, \beta_{i+1,j-1/2}, \gamma_{i+1,j+1/2}, \delta_{i,j+1/2} = \frac{f_{i+1/2,j}}{4H} + O(\Delta\xi, \Delta\eta) \quad \text{and} \tag{2.14}$$

$$\alpha_{i,j-1/2} + \beta_{i+1,j+1/2} + \gamma_{i+1,j+1/2} + \delta_{i,j+1/2} = \frac{f_{i+1/2,j}}{H} + O(\Delta\xi^2, \Delta\eta^2) \tag{2.15}$$

for (2.10).
3) The decision to express the Coriolis terms in terms of u^* and v^* might appear to limit the possible forms of averaging of the Coriolis terms under consideration. However, in fact any (linear) average of the four nearest v values in the u equation [including $1/4(v_{i+1/2,j+1} + v_{i-1/2,j+1} + v_{i-1/2,j} + v_{i+1/2,j})$] or the four nearest u values in the v equation can be obtained by a suitable definition of α, β, γ, and δ.
4) The averaging of v in the u equation is not independent of the averaging of u in the v equation because the same coefficients appear in the two equations. The appearance of the same coefficients in the two equations is essential to ensure that all Coriolis terms cancel in the energy equation, allowing energy conservation (section 5). The form given does not, therefore, involve any loss of generality once the desired conservation properties are taken into account.
5) Arakawa and Lamb (1981) and Arakawa (2000) consider more general schemes that involve u^* "Coriolis" terms in the u equation and v^* "Coriolis" terms in the v equation. These terms are in fact small—second order in the grid spacing—but are essential to obtain conservation of potential enstrophy. Because conservation of potential enstrophy is not considered here, and because of the somewhat unnatural appearance of these terms, they are not included here.

Having written the discretization (2.9)–(2.11), it is useful at this stage to summarize the freedom that remains available in the definition of the scheme. In the definition of the scheme itself one is still free to specify the exact definitions of $(\Delta\eta/n)_{i,j+1/2}$ [Eq. (2.6)], $(\Delta\xi/m)_{i+1/2,j}$ [Eq. (2.7)], $(\Delta\eta\Delta\xi/mn)_{i+1/2,j+1/2}$ [Eq. (2.8)], $(\Delta\xi/m)_{i,j+1/2}$ [Eq. (2.9)], and $(\Delta\eta/n)_{i+1/2,j}$ [Eq. (2.10)], and the coefficients α, β, γ, and δ. There is also some freedom in specifying the exact definitions of the quantities to be conserved, particularly angular momentum and energy.

c. *Discrete equations at the poles*

When the polar points are v points it is not actually necessary to store any values at the poles, and no prognostic equations are needed at the poles, for the linearized equations under consideration. This is because only v^* is needed to update the other equations, and v^* must vanish at the poles because $\Delta\xi/m$ must vanish there. It is enough, therefore, simply to substitute a value of 0 whenever the equations require a polar value of v^*.

When the polar points are u–h points, only a single value of h should be stored at each pole ($h_{1/2}$ and $h_{P-1/2}$; see Fig. 1), and prognostic equations are needed for these polar values. The mass in the circular south polar cap extending from the pole to latitude $j = 1$ can be defined to be $h^*_{1/2} = A_{1/2}h_{1/2}$, where $A_{1/2}$ is (an approximation to) the area of the polar cap, and similarly for the north polar cap. As with the other length and area elements, there is some freedom remaining in exactly how $A_{1/2}$ and $A_{P-1/2}$ are specified. Equations for the evolution of $h^*_{1/2}$ and $h^*_{P-1/2}$ (again, these may be regarded as finite-volume approximations) follow from the mass budgets of the polar caps

$$\frac{\partial}{\partial t}h^*_{1/2} + \sum_i v^*_{i+1/2,1} = 0, \tag{2.16}$$

$$\frac{\partial}{\partial t}h^*_{P-1/2} - \sum_i v^*_{i+1/2,P-1} = 0. \tag{2.17}$$

For the continuous equations, u is singular at the poles since it is multivalued. Two possible treatments for u at the poles are considered here; it will be seen that both of them are problematic. The first is the treatment used in the New Dynamics. The value of u is not predicted at the pole. Rather, using the fact that u must be given by a zonal wavenumber-1 Fourier component, it is diagnosed via a least squares fit to v at the v row nearest the pole [extending the method of McDonald and Bates (1989)]. For example, at the South Pole

$$u_{i,1/2} = V \sin(\lambda_i - \lambda_0), \qquad (2.18)$$

where

$$V = \frac{a \cos\lambda_0 + b \sin\lambda_0}{1 + c \cos(2\lambda_0) + d \sin(2\lambda_0)}, \qquad (2.19)$$

$$\lambda_0 = \tan^{-1}\left(\frac{b + bc - ad}{a - ac - bd}\right), \qquad (2.20)$$

$$a = \frac{1}{\pi} \sum_i \Delta\lambda_{i+1/2} v_{i+1/2,1} \cos\lambda_{i+1/2}, \qquad (2.21)$$

$$b = \frac{1}{\pi} \sum_i \Delta\lambda_{i+1/2} v_{i+1/2,1} \sin\lambda_{i+1/2}, \qquad (2.22)$$

$$c = \frac{1}{2\pi} \sum_i \Delta\lambda_{i+1/2} \cos(2\lambda_{i+1/2}), \qquad (2.23)$$

$$d = \frac{1}{2\pi} \sum_i \Delta\lambda_{i+1/2} \sin(2\lambda_{i+1/2}). \qquad (2.24)$$

In the special case in which the zonal grid spacing is uniform in λ, both c and d are 0 and the above expressions simplify.

The second possible treatment has a prognostic equation for a single value of u, obtained by considering a finite-volume discretization of the vorticity budget

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot (f\mathbf{u}) = 0 \qquad (2.25)$$

integrated over the polar cap within the nearest v row. The area integrals of both terms can be reexpressed as boundary line integrals. Then at the South Pole we have

$$\frac{\partial}{\partial t}(uD)_{1/2} - \sum_i (\alpha_{i,1/2} v^*_{i+1/2,1} + \beta_{i,1/2} v^*_{i-1/2,1}) = 0. \qquad (2.26)$$

Here $u_{1/2}$ is the average value of u at row $j = 1$, and $D_{1/2} = \Sigma_i (\Delta\xi/m)_{i+1/2,1}$ is the circumference of the polar cap region bounded by row $j = 1$. Similarly, for the North Pole,

$$\frac{\partial}{\partial t}(uD)_{P-1/2} - \sum_i (\gamma_{i,P-1/2} v^*_{i-1/2,P-1} + \delta_{i,1/2} v^*_{i+1/2,P-1}) = 0. \qquad (2.27)$$

Equations (2.26) and (2.27) are valid as written for both uniform- and variable-resolution grids.

The polar values of u^*, when they are needed, are obtained from the polar values of u by an equation just like (2.6). However, the interpretation of $(\Delta\eta/n)_{1/2}$ and $(\Delta\eta/n)_{P-1/2}$ is not clear-cut, though the requirement to satisfy conservation properties will constrain the allowed values.

Although $u_{1/2}$ is the average value of u at row $j = 1$, we will use it as if it were a polar value in the prognostic equation for v at row $j = 1$ and in checking whether the various conservation laws hold. A disadvantage of doing this is that the longitudinally varying polar values $u^*_{i,1/2}$ are approximated by $u^*_{1/2}$, which is longitudinally independent and small [the integral of the relative vorticity over the polar cap is $O(\Delta\eta^2)$ implying $u_{1/2} = O(\Delta\eta)$]; consequently, this scheme is not second-order accurate near the poles.

3. Conservation of mass

a. v at the poles

The total mass (perturbation) in the discretized shallow-water system is

$$W = \sum_{i,j} h^*_{i+1/2,j+1/2}, \qquad (3.1)$$

where j runs from 0 to $P - 1$ (Fig. 1). The rate of change of total mass is

$$\frac{dW}{dt} = \sum_{i,j} \frac{\partial}{\partial t} h^*_{i+1/2,j+1/2}. \qquad (3.2)$$

Substituting from (2.11) and canceling fluxes in the interior of the grid leaves

$$\frac{dW}{dt} = \sum_i v^*_{i+1/2,0} - \sum_i v^*_{i+1/2,P}. \qquad (3.3)$$

But, as noted in section 2c, v^* must vanish at the poles, so $dW/dt = 0$ and mass is conserved.

b. u and h at the poles

When the poles are u–h points, the total mass (perturbation) in the system is now

$$W = \sum_{i,j} h^*_{i+1/2,j+1/2} + h^*_{1/2} + h^*_{P-1/2}, \qquad (3.4)$$

where j runs from 1 to $P - 2$ (Fig. 1). It is clear, on substituting from (2.11), (2.16), and (2.17), that all contributions to the tendency of total mass cancel, and mass is conserved. Moreover, this result is independent of the formulation of the u equation at the poles.

4. Conservation of angular momentum

The derivations in this section assume that u is the eastward component of the velocity and therefore depend on the grid being nonrotated.

a. v at the poles

Define the total angular momentum (perturbation) of the discrete linearized system to be

$$M = \sum_{i,j} H\left(\frac{\Delta\xi\Delta\eta}{mn}uC\right)_{i,j+1/2} + \sum_{i,j} \Omega(h^*G)_{i+1/2,j+1/2}, \tag{4.1}$$

where $C_{i,j+1/2}$ is an approximation to $(a\cos\phi)_{i,j+1/2}$, $G_{i+1/2,j+1/2}$ is an approximation to $(a^2\cos^2\phi)_{i+1/2,j+1/2}$, and j runs from 0 to $P - 1$. For the moment there is some freedom available in exactly how C and G are chosen, but the requirement to conserve angular momentum will constrain that choice. The area element $(\Delta\xi\Delta\eta/mn)_{i,j+1/2}$ in (4.1) must be consistent with the length elements $(\Delta\eta/n)_{i,j+1/2}$ in (2.6) and $(\Delta\xi/m)_{i,j+1/2}$ in (2.9). Taking the time derivative of (4.1) and substituting from (2.9) and (2.11) implies that the rate of change of total angular momentum is

$$\frac{dM}{dt} = \sum_{i,j} H(\alpha_{i,j+1/2}v^*_{i+1/2,j+1} + \beta_{i,j+1/2}v^*_{i-1/2,j+1} + \gamma_{i,j+1/2}v^*_{i-1/2,j} + \delta_{i,j+1/2}v^*_{i+1/2,j})\left(\frac{\Delta\eta}{n}C\right)_{i,j+1/2} - \sum_{i,j}(v^*_{i+1/2,j+1} - v^*_{i+1/2,j})\Omega G_{i+1/2,j+1/2}. \tag{4.2}$$

The tendency of total angular momentum will vanish provided that the coefficient of each $v^*_{i+1/2,j}$ in (4.2) vanishes, that is, provided that

$$\left(\alpha\frac{\Delta\eta}{n}C\right)_{i,j-1/2} + \left(\beta\frac{\Delta\eta}{n}C\right)_{i+1,j-1/2} + \left(\gamma\frac{\Delta\eta}{n}C\right)_{i+1,j+1/2} + \left(\delta\frac{\Delta\eta}{n}C\right)_{i,j+1/2} + \frac{\Omega}{H}(G_{i+1/2,j+1/2} - G_{i+1/2,j-1/2}) = 0 \tag{4.3}$$

for $j = 1, \ldots, P - 1$. This constraint can be satisfied by a suitable choice of the coefficients C and G appearing in the definition of angular momentum. Examples are given in section 8.

b. u and h at the poles—New Dynamics formulation

Away from the poles ($j = 2, \ldots, P - 2$), it is clear that (4.3) must hold for this case too. But now the poles themselves require special consideration. Let $C_{1/2}$ be an approximation to $a\cos\phi$ and let $G_{1/2}$ be an approximation to $a^2\cos^2\phi$ in the south polar cap region, so that the contribution to the total angular momentum from the south polar cap is

$$M_{1/2} = H\sum_i\left(\frac{\Delta\xi\Delta\eta}{mn}u\right)_{i,1/2}C_{1/2} + \Omega(h^*G)_{1/2}. \tag{4.4}$$

To begin with, assume that the polar area elements are independent of longitude, so that the first term on the right-hand side will sum to 0, leaving

$$M_{1/2} = \Omega(h^*G)_{1/2}. \tag{4.5}$$

The tendency of this contribution is obtained by substituting from (2.16), giving

$$\frac{dM_{1/2}}{dt} = -\sum_i v^*_{i+1/2,1}\Omega G_{1/2}. \tag{4.6}$$

Combining this with the angular momentum budget for latitude row 3/2 and requiring that the coefficient of $v^*_{i+1/2,1}$ should vanish shows that (4.3) must hold for $j = 1$ too, except that the terms involving α and β at row 1/2 are omitted. Similarly, (4.3) must hold for $j = P - 1$, except that the terms involving γ and δ at row $P - 1/2$ are omitted. These requirements are equivalent to requiring that (4.3) should hold at all latitudes with $C_{1/2} = C_{P-1/2} = 0$. Finally, note that when this constraint on the polar values of C holds, the first term on the right-hand side of (4.4) always vanishes even without the assumption that polar area elements are independent of longitude, so angular momentum will be conserved even on a stretched grid.

c. u and h at the poles—Vorticity budget formulation

Again, (4.3) must hold away from the poles ($j = 2, \ldots, P - 2$). As before, let $C_{1/2}$ be an approximation to $a\cos\phi$ and let $G_{1/2}$ be an approximation to $a^2\cos^2\phi$ in the south polar cap region. Define the contribution to the total angular momentum from the south polar cap to be

$$M_{1/2} = H\left(\frac{D\Delta\eta}{n}uC\right)_{1/2} + \Omega(h^*G)_{1/2}. \tag{4.7}$$

The tendency of this contribution is obtained by substituting from (2.16) and (2.26), giving

$$\frac{dM_{1/2}}{dt} = \sum_i H(\alpha_{i,1/2}v^*_{i+1/2,1} + \beta_{i,1/2}v^*_{i-1/2,1})\left(\frac{\Delta\eta}{n}C\right)_{1/2} - \sum_i v^*_{i+1/2,1}\Omega G_{1/2}. \tag{4.8}$$

Combining this with the angular momentum budget for latitude row 3/2 and requiring that the coefficient of $v^*_{i+1/2,1}$ should vanish shows that (4.3) must hold for $j = 1$ too, with the only difference being that $[(\Delta\eta/n)C]_{1/2}$ and $G_{1/2}$ are independent of i. Similarly, consideration of the angular momentum budget for the north polar cap shows that (4.3) must hold for $j = P - 1$, with the only difference being that $[(\Delta\eta/n)C]_{P-1/2}$ and $G_{P-1/2}$ are independent of i.

5. Conservation of energy

a. v at the poles

Define the total energy E of the discretized linear system by

$$2E = \sum_i \sum_{j=0}^{P-1} H\left(\frac{\Delta\xi\Delta\eta}{mn}u^2\right)_{i,j+1/2} + \sum_i \sum_{j=1}^{P-1} H\left(\frac{\Delta\xi\Delta\eta}{mn}\upsilon^2\right)_{i+1/2,j} + \sum_i \sum_{j=0}^{P-1} (h^*\Phi)_{i+1/2,j+1/2}. \quad (5.1)$$

The tendency of total energy is given by taking $u^*_{i,j+1/2}$ times (2.9) plus $\upsilon^*_{i+1/2,j}$ times (2.10) plus $\Phi_{i+1/2,j+1/2}$ times (2.11), each summed over the appropriate range of indices. The analysis is straightforward, but care must be taken to keep track of the range of the index j in the various sums that appear:

$$\begin{aligned}\frac{dE}{dt} &+ \sum_i \sum_{j=1}^{P-1} \alpha_{i,j-1/2}u^*_{i,j-1/2}\upsilon^*_{i+1/2,j} \\ &- \sum_i \sum_{j=0}^{P-1} \alpha_{i,j+1/2}u^*_{i,j+1/2}\upsilon^*_{i+1/2,j+1} \\ &+ \sum_i \sum_{j=1}^{P-1} \beta_{i+1,j-1/2}u^*_{i+1,j-1/2}\upsilon^*_{i+1/2,j} \\ &- \sum_i \sum_{j=0}^{P-1} \beta_{i,j+1/2}u^*_{i,j+1/2}\upsilon^*_{i-1/2,j+1} \\ &+ \sum_i \sum_{j=1}^{P-1} \gamma_{i+1,j+1/2}u^*_{i+1,j+1/2}\upsilon^*_{i+1/2,j} \\ &- \sum_i \sum_{j=0}^{P-1} \gamma_{i,j+1/2}u^*_{i,j+1/2}\upsilon^*_{i-1/2,j} \\ &+ \sum_i \sum_{j=1}^{P-1} \delta_{i,j+1/2}u^*_{i,j+1/2}\upsilon^*_{i+1/2,j} \\ &- \sum_i \sum_{j=0}^{P-1} \delta_{i,j+1/2}u^*_{i,j+1/2}\upsilon^*_{i+1/2,j} \\ &+ \sum_i \sum_{j=0}^{P-1} u^*_{i,j+1/2}(\Phi_{i+1/2,j+1/2} - \Phi_{i-1/2,j+1/2}) \\ &+ \sum_i \sum_{j=1}^{P-1} \upsilon^*_{i+1/2,j}(\Phi_{i+1/2,j+1/2} - \Phi_{i+1/2,j-1/2}) \\ &+ \sum_i \sum_{j=0}^{P-1} \Phi_{i+1/2,j+1/2}(u^*_{i+1,j+1/2} - u^*_{i,j+1/2}) \\ &+ \sum_i \sum_{j=0}^{P-1} \Phi_{i+1/2,j+1/2}(\upsilon^*_{i+1/2,j+1} - \upsilon^*_{i+1/2,j}) = 0. \quad (5.2)\end{aligned}$$

By shifting indices and using $\upsilon^*_{i+1/2,0} = 0$ and $\upsilon^*_{i+1/2,P} = 0$, it may be verified that all of the Coriolis terms cancel. Also, by shifting the i index where necessary, it may be verified that all of the $u\Phi$ terms cancel. Finally, by shifting the j index where necessary and again using $\upsilon^*_{i+1/2,0} = 0$ and $\upsilon^*_{i+1/2,P} = 0$, it may be verified that all of the $\upsilon\Phi$ terms cancel. So $dE/dt = 0$ and total energy is conserved.

Note that the cancellation of the Coriolis terms in the energy budget depends crucially on the correspondence between the α, β, γ, and δ terms in (2.9) and (2.10) that was imposed from the start. Apart from this, energy conservation imposes no new constraints on the scheme except that the area elements $(\Delta\xi\Delta\eta/mn)_{i,j+1/2}$ and $(\Delta\xi\Delta\eta/mn)_{i+1/2,j}$ appearing in the definition of the energy must be consistent with the length elements $(\Delta\xi/m)$ and $(\Delta\eta/n)$ appearing in (2.6), (2.7), (2.9), and (2.10).

Note also, however, that although the global integral of energy is conserved, it is not possible to write a discrete local flux-form energy equation because the potential energy and two kinetic energy contributions are defined at different points; in particular, the Coriolis term contributions to the kinetic energy tendency cannot be made to cancel except in a global sum.

b. u and h at the poles

The definition of total energy must now include contributions from the polar caps. However, it is not obvious how to define those polar contributions, and different definitions may be appropriate for different discretization schemes. Therefore the definitions are left open for the moment, and relations will be derived that must be satisfied by the polar contributions in order to have energy conservation. Let

$$2E = \sum_i \sum_{j=1}^{P-2} H\left(\frac{\Delta\xi\Delta\eta}{mn}u^2\right)_{i,j+1/2} + \sum_i \sum_{j=1}^{P-1} H\left(\frac{\Delta\xi\Delta\eta}{mn}\upsilon^2\right)_{i+1/2,j} + \sum_i \sum_{j=1}^{P-2} (h^*\Phi)_{i+1/2,j+1/2} + 2E_{\mathrm{SP}} + 2E_{\mathrm{NP}}, \quad (5.3)$$

where E_{SP} and E_{NP} are the energy contributions from the southern and northern polar caps.

Upon taking the time derivative and substituting from (2.9)–(2.11), almost all terms cancel except for contributions from near the poles:

$$\begin{aligned}\frac{dE}{dt} - \frac{dE_{\mathrm{SP}}}{dt} - \frac{dE_{\mathrm{NP}}}{dt} &= -\sum_i \alpha_{i,1/2}u^*_{i,1/2}\upsilon^*_{i+1/2,1} - \sum_i \beta_{i,1/2}u^*_{i,1/2}\upsilon^*_{i-1/2,1} \\ &- \sum_i \gamma_{i,P-1/2}u^*_{i,P-1/2}\upsilon^*_{i-1/2,P-1} \\ &- \sum_i \delta_{i,P-1/2}u^*_{i,P-1/2}\upsilon^*_{i+1/2,P-1} + \sum_i \upsilon^*_{i+1/2,1}\Phi_{i+1/2,1/2} \\ &- \sum_i \upsilon^*_{i+1/2,P-1}\Phi_{i+1/2,P-1/2}. \quad (5.4)\end{aligned}$$

For conservation of energy we require the tendencies of the polar energy contributions to cancel the terms on the right-hand side of (5.4).

The obvious choice for the polar potential energy is

$$2(\mathrm{PE})_{\mathrm{SP}} = (h^*\Phi)_{1/2}, \qquad (5.5)$$

with an analogous expression for the north pole. Their tendencies, obtained by substituting from (2.16) and (2.17), exactly cancel the $\upsilon^*\Phi$ terms in (5.4). Therefore, conservation of total energy requires the polar kinetic energy contributions to satisfy

$$\frac{d(\mathrm{KE})_{\mathrm{SP}}}{dt} = \sum_i \alpha_{i,1/2} u^*_{i,1/2} \upsilon^*_{i+1/2,1} + \sum_i \beta_{i,1/2} u^*_{i,1/2} \upsilon^*_{i-1/2,1},$$

$$\frac{d(\mathrm{KE})_{\mathrm{NP}}}{dt} = \sum_i \gamma_{i,P-1/2} u^*_{i,P-1/2} \upsilon^*_{i-1/2,P-1} + \sum_i \delta_{i,P-1/2} u^*_{i,P-1/2} \upsilon^*_{i+1/2,P-1}. \qquad (5.6)$$

1) NEW DYNAMICS FORMULATION

Under the New Dynamics formulation (2.18)–(2.24), the tendency of u at the South Pole is determined by the tendencies of υ at row $j = 1$, which in turn depend on the values of u and Φ at row $j = 3/2$. It is clear, therefore, that no simple expression for the polar kinetic energy involving only local values of the velocity can satisfy (5.6). It is also clear that including Arakawa and Lamb's (1981) extra u Coriolis terms in the u equation and υ Coriolis terms in the υ equation will not help to satisfy (5.6).

2) VORTICITY BUDGET FORMULATION

Under the vorticity budget formulation (2.26)–(2.27), the south polar kinetic energy contribution may be defined as

$$2(\mathrm{KE})_{\mathrm{SP}} = H\left(\frac{D\Delta\eta}{n}u^2\right)_{1/2}, \qquad (5.7)$$

with an analogous expression for the North Pole. Taking the time derivative and subsitituting from (2.26)–(2.27) shows that (5.6) is satisfied and hence energy is conserved for this formulation.

6. Accurate representation of Rossby modes

The dispersion properties of simulated Rossby modes for a variety of numerical schemes and grids have been studied by Wajsowicz (1986) and Neta and Williams (1989). Both of those studies were restricted to beta-plane geometry. In this section we highlight a possible problem with the numerical simulation of Rossby modes that arises specifically through the spherical geometry in combination with the latitudinal variation of the Coriolis parameter.

Thuburn et al. (2002) describe numerical calculations of normal modes of a deep compressible rotating spherical atmosphere. At an early stage of that work the results included, in addition to the expected acoustic, inertia–gravity, and Rossby modes, a number of spurious modes. These spurious modes were slowly eastward propagating and had features in their latitudinal structure at the scale of the computational grid, indicating that they must be numerical artefacts. The discretization used was based on standard centered differences on a latitude–height version of the C grid with u and h at the poles, similar to the New Dynamics, raising concerns that similar problems might affect the New Dynamics. It was therefore important to understand the origin of the spurious modes and, if possible, find a way to eliminate them.

The problem, and its solution, can be understood in the context of the shallow-water equations, so only that case is considered here. Geostrophically balanced flows, in particular Rossby modes, have small divergence. However, the divergence is nonzero when the Coriolis parameter is a function of position. The geostrophic wind is defined as

$$\upsilon_g = \frac{m}{f}\Phi_\xi, \qquad u_g = -\frac{n}{f}\Phi_\eta. \qquad (6.1)$$

Its divergence is given by

$$mn\left[\left(\frac{u_g}{n}\right)_\xi + \left(\frac{\upsilon_g}{m}\right)_\eta\right] = -\frac{\mathbf{u}_g \cdot \boldsymbol{\nabla} f}{f}. \qquad (6.2)$$

The divergence of the geostrophic wind is a small residual (of the order of the Rossby number) of two much larger quantities. Therefore, any discretization errors in the computation of the geostrophic wind or its spatial derivatives will be greatly amplified in the divergence. Although the divergence of the geostrophic wind is small, it nevertheless plays an important role in determining the structures and frequencies of Rossby modes. It is therefore important that the finite-difference approximation for the divergence of the geostrophic wind should be a good approximation to the right-hand side of (6.2).

To simplify the discussion, restrict attention to a purely latitudinal discretization and assume a modal structure proportional to $e^{ik\lambda}$ in the zonal direction. Also take $\alpha_{i,j+1/2} = \beta_{i,j+1/2} = \alpha_{j+1/2}$ and $\gamma_{i,j+1/2} = \delta_{i,j+1/2} = \gamma_{j+1/2}$. The dominant terms in (2.9) and (2.10) define the geostrophic wind for the discretization:

$$-2\alpha_{j+1/2}\upsilon^*_{j+1} - 2\gamma_{j+1/2}\upsilon^*_j + ik\Delta\lambda\Phi_{j+1/2} = 0, \qquad (6.3)$$

$$2\alpha_{j-1/2}u^*_{j-1/2} + 2\gamma_{j+1/2}u^*_{j+1/2} + \Phi_{j+1/2} - \Phi_{j-1/2} = 0. \qquad (6.4)$$

(For clarity the subscript g is omitted in the rest of this section.)

Eliminating Φ and rearranging gives

$$\gamma_{j+1/2}(ik\Delta\lambda u^*_{j+1/2} + \upsilon^*_{j+1} - \upsilon^*_j)$$

$$+ \alpha_{j-1/2}(ik\Delta\lambda u_{j-1/2} + v_j^* - v_{j-1}^*)$$
$$+ v_{j+1}^*(\alpha_{j+1/2} - \gamma_{j+1/2}) + 2v_j^*(\gamma_{j+1/2} - \alpha_{j-1/2})$$
$$+ v_{j-1}^*(\alpha_{j-1/2} - \gamma_{j-1/2}) = 0. \quad (6.5)$$

The first two terms in (6.5) are proportional to the divergence of the geostrophic flow at latitudes $j + 1/2$ and $j - 1/2$, respectively. The equation will therefore be a good approximation to (6.2) provided the last three terms are a good approximation to $v^*\Delta f/2H$ at latitude j. Whether this holds will depend on the detailed definitions of α and γ. It is possible that small errors or inconsistencies in α and γ can lead to order 1 errors in differences between them, making (6.5) a poor approximation to (6.2). For example, consider the following obvious latitudinal discretization of the linearized shallow-water equations:

$$\frac{\partial u_{j+1/2}}{\partial t} - f_{j+1/2}\frac{(v_j + v_{j+1})}{2} + \frac{ik\Phi_{j+1/2}}{a\cos\phi_{j+1/2}} = 0, \quad (6.6)$$

$$\frac{\partial v_j}{\partial t} + \frac{(f_{j-1/2}u_{j-1/2} + f_{j+1/2}u_{j+1/2})}{2} + \frac{(\Phi_{j+1/2} - \Phi_{j-1/2})}{a\Delta\phi} = 0, \quad (6.7)$$

$$\frac{\partial h_{j+1/2}}{\partial t} + \frac{H}{a\cos\phi_{j+1/2}}\left[iku_{j+1/2} + \frac{(v_{j+1}\cos\phi_{j+1} - v_j\cos\phi_j)}{\Delta\phi}\right] = 0. \quad (6.8)$$

This uses a simple centered averaging of v in the Coriolis terms, which is equivalent to using the Arakawa–Lamb conservative form of the Coriolis terms with

$$\alpha_{j+1/2} = \frac{f_{j+1/2}}{4H}\frac{\cos\phi_{j+1/2}}{\cos\phi_{j+1}}, \quad (6.9)$$

$$\gamma_{j+1/2} = \frac{f_{j+1/2}}{4H}\frac{\cos\phi_{j+1/2}}{\cos\phi_j}. \quad (6.10)$$

These satisfy (2.12)–(2.15), so the scheme is second-order accurate. However, the variations in the geometrical $\cos\phi$ factors make unphysical contributions to the last three terms in (6.5) that are comparable to the physical contributions coming from variations in f. The errors are particularly bad when v varies rapidly with latitude. Consequently, balanced flows are poorly represented. Figure 2 shows an example of gravity- and Rossby-normal-mode frequencies computed using this discretization on a grid uniformly spaced in ϕ and with a v-at-pole staggering. Almost half of the Rossby modes are missing and have been replaced by spurious eastward propagating modes.

For comparison, Fig. 3 shows the gravity- and Rossby-mode frequencies when α and γ are defined by

$$\alpha_{j+1/2} = [\sigma f_{j+1/2} + (1 - \sigma)f_{j+1}]/4H, \quad (6.11)$$

$$\gamma_{j+1/2} = [\sigma f_{j+1/2} + (1 - \sigma)f_j]/4H, \quad (6.12)$$

with $\sigma = 1/2$. The full spectrum of Rossby modes is present, and their frequencies are captured accurately. Results for other values of σ between 0 and 1 are almost identical.

There are 89 degrees of freedom in the discretization, and there are 89 numerical normal modes, none of which are computational modes with zero frequency. Nevertheless, eight modes, with the highest latitudinal structures, are imperfectly captured: two eastward gravity modes appear with identical frequencies with latitudinal index 2; two other eastward gravity modes appear with latitudinal indices 20 and 21; two westward gravity modes appear with identical frequencies with latitudinal index 2; and two Rossby modes appear with identical frequencies with latitudinal index 6. In all cases the mode frequencies are close to the correct values, but

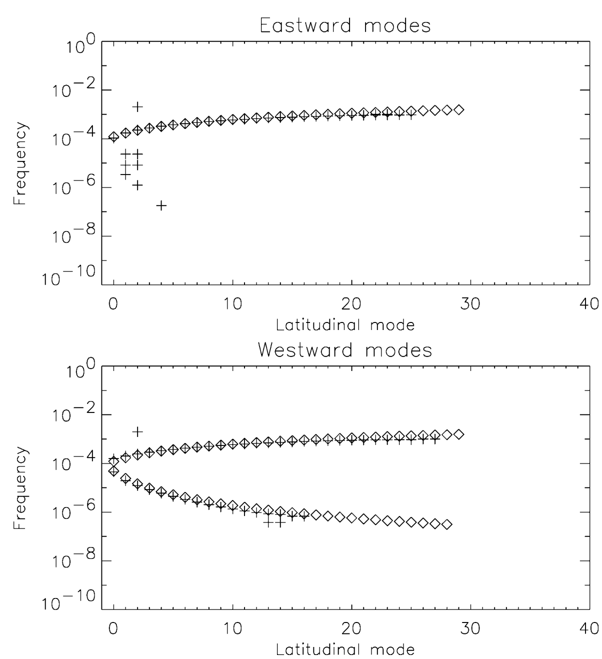

FIG. 2. Crosses show frequencies of gravity and Rossby normal modes computed using (6.6)–(6.8). The mean depth is 10^4 m, the zonal wavenumber is 2, and $P = 30$ latitudes are used with the boundary condition (for zonal wavenumber >1) $v = 0$ at the poles. The diamonds show the analytic approximations to the mode frequencies for the limit of large Froude number $gH/a^2\Omega^2$ (Hough 1897), which is a good approximation here.

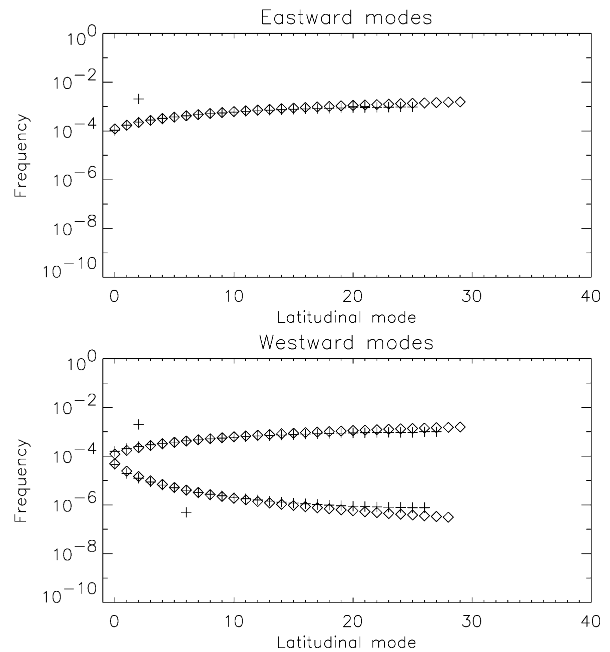

FIG. 3. Frequencies of gravity and Rossby normal modes computed using a version of (6.6)–(6.8) modified so that the implied values of α and γ are given by (6.11) and (6.12) with $\sigma = 1/2$ (cf. Fig. 2).

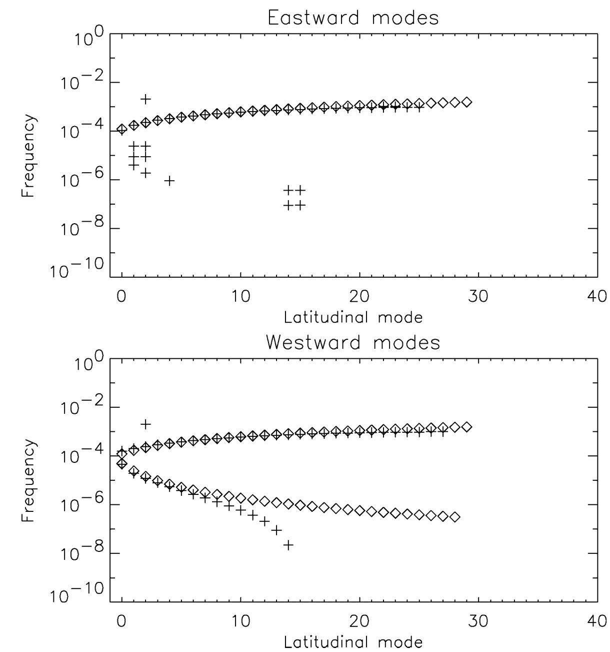

FIG. 4. As in Fig. 2, but with the Coriolis term in (6.7) replaced by (6.13).

their latitudinal structures are somewhat distorted, becoming localized near one or other pole, so that automatic identification of the modes by counting zeros becomes problematic. The disruption of these modes appears to be associated with the polar singularities. When the calculation is repeated on a spherical channel with boundaries at $\phi = \pm\phi_b$, for ϕ_b up to 80° all modes appear in their expected positions; as ϕ_b is increased further, pairs of modes are disrupted in turn until a total of eight are disrupted for $\phi_b = 90°$.

For completeness, note that the conservative forms of the Coriolis terms given by Arakawa and Lamb (1981) help to obtain accurate Rossby-mode phase speeds. For example, if the Coriolis term in (6.7) is replaced by the apparently simpler but nonconservative expression

$$f_j \frac{u_{j-1/2} + u_{j+1/2}}{2} \qquad (6.13)$$

then not only are many of the Rossby modes replaced by spurious eastward propagating modes, but the remaining Rossby modes are artificially slowed (Fig. 4). In this case the problem with the geostrophic divergence, and hence the spurious modes, can be fixed (without restoring conservation) by replacing the Coriolis term in (6.6) by

$$f_{j+1/2} \frac{\upsilon_j \cos\phi_j + \upsilon_{j+1} \cos\phi_{j+1}}{\cos\phi_{j+1/2}}, \qquad (6.14)$$

but the problem of artificial slowing remains (not shown).

7. Summary of constraints

In the integration scheme for the shallow-water equations, before any constraints implied by conservation properties are imposed, there is freedom to choose the exact definitions of $(\Delta\eta/n)_{i,j+1/2}$ [Eq. (2.6)], $(\Delta\xi/m)_{i+1/2,j}$ [Eq. (2.7)], $(\Delta\xi\Delta\eta/mn)_{i+1/2,j+1/2}$ [Eq. (2.8)], $(\Delta\xi/m)_{i,j+1/2}$ [Eq. (2.9)], and $(\Delta\eta/n)_{i+1/2,j}$ [Eq. (2.10)], and the coefficients $\alpha_{i,j+1/2}$, $\beta_{i,j+1/2}$, $\gamma_{i,j+1/2,}$ and $\delta_{i,j+1/2}$. When the poles are u–h points, there are polar cap areas $A_{1/2}$ and $A_{P-1/2}$ instead of $(\Delta\xi\Delta\eta/mn)_{i+1/2,1/2}$ and $(\Delta\xi\Delta\eta/mn)_{i+1/2,P-1/2}$ in the definition of the polar cap h*. There is also freedom to choose the coefficients $C_{i,j+1/2}$ and $G_{i,+1/2,j+1/2}$, which appear in the definition of angular momentum.

Equation (2.11) automatically gives local and global mass conservation without imposing any additional constraints on the coefficients in the scheme.

Angular momentum will be conserved provided C and G are related to α, β, γ, and δ by (4.3). In the New Dynamics formulation, the polar values of C must vanish. It is also necessary that the area element $(\Delta\xi\Delta\eta/mn)_{i,j+1/2}$ used in defining angular momentum [Eq. (4.1)] be given by the product of the length elements $(\Delta\eta/n)_{i,j+1/2}$ and $(\Delta\xi/m)_{i,j+1/2}$ appearing in (2.6) and (2.9).

The integration scheme automatically gives conservation of energy without imposing any additional constraints on the coefficients in the scheme. Note, however, that α, β, γ, and δ must appear as in (2.9) and (2.10) to ensure that all contributions from Coriolis terms cancel. It is also necessary that the area element $(\Delta\xi\Delta\eta/mn)_{i,j+1/2}$ used in defining energy [e.g., (5.1)] must be given by the product of the length elements

$(\Delta\eta/n)_{i,j+1/2}$ and $(\Delta\xi/m)_{i,j+1/2}$ appearing in (2.6) and (2.9), and the area element $(\Delta\xi\Delta\eta/mn)_{i+1/2,j}$ used in defining energy must be given by the product of the length elements $(\Delta\eta/n)_{i+1/2,j}$ and $(\Delta\xi/m)_{i+1/2,j}$ appearing in (2.10) and (2.7).

The requirements for conservation of mass, angular momentum, and energy do not appear to introduce any constraints on the h cell areas $(\Delta\xi\Delta\eta/mn)_{i+1/2,j+1/2}$ (and $A_{1/2}$ and $A_{P-1/2}$ when the poles are h points) beyond the need for second-order accuracy. However, for a zonally uniform grid it is natural to make the h cell areas equal to the u cell areas at the same latitude.

Accurate representation of balanced flows, including Rossby modes, depends on (6.5) being an accurate approximation to (6.2). There is no simple and concise way to express this requirement as a constraint on the coefficients in the scheme, but it may be noted that (6.11) and (6.12) define a family of viable coefficients for the Coriolis terms, giving schemes that are found to work well in normal-mode calculations.

8. Examples

Two possible schemes are presented here satisfying the constraints derived above. Note, however, that considerable freedom remains in the choice of coefficients, and that it might be possible to exploit this freedom to satisfy additional requirements.

a. Longitude–latitude coordinates

Let $\xi = \lambda$ and $\eta = \phi$, the usual longitude and latitude coordinates. Analytically, the map factors are $1/m = a\cos\phi$, $1/n = a$. Choose constant grid spacing $\Delta\lambda$ and $\Delta\phi$, so that all coefficients are independent of longitude. Let

$$(\Delta\xi/m)_{j+1/2} = a\Delta\lambda\cos\phi_{j+1/2}, \tag{8.1}$$

$$(\Delta\eta/n)_{j+1/2} = a\Delta\phi, \tag{8.2}$$

so that

$$(\Delta\xi\Delta\eta/mn)_{j+1/2} = a^2\Delta\lambda\Delta\phi\cos\phi_{j+1/2}. \tag{8.3}$$

Now choose

$$\alpha_{j-1/2} = \beta_{j-1/2} = \gamma_{j+1/2} = \delta_{j+1/2} = \frac{\Omega}{2H}\sin\phi_j, \tag{8.4}$$

consistent with (6.11) and (6.12), with $\sigma = 0$. It is reasonable to take the v-cell areas to be the mean of the two overlapping h-cell areas, as Arakawa and Lamb (1981) do (though this is not the only possible choice). This can be satisfied by choosing

$$\left(\frac{\Delta\xi}{m}\right)_j = \frac{1}{2}a\Delta\lambda(\cos\phi_{j-1/2} + \cos\phi_{j+1/2}) = a\Delta\lambda\cos\phi_j\cos(\Delta\phi/2), \tag{8.5}$$

$$(\Delta\eta/n)_j = a\Delta\phi, \tag{8.6}$$

so that

$$(\Delta\xi\Delta\eta/mn)_j = a^2\Delta\lambda\Delta\phi\cos\phi_j\cos(\Delta\phi/2). \tag{8.7}$$

Using these values, (4.3) implies that for angular momentum conservation C and G must be related by

$$\sin\phi_j(C_{j-1/2} + C_{j+1/2})a\Delta\phi = -(G_{j+1/2} - G_{j-1/2}). \tag{8.8}$$

For example, let

$$G_{j+1/2} = a^2(\cos\phi_{j+1/2})^2. \tag{8.9}$$

Then the right-hand side of (8.8) can be written

$$\begin{aligned} &-a^2[(\cos\phi_{j+1/2})^2 - (\cos\phi_{j-1/2})^2] \\ &= -a^2(\cos\phi_{j+1/2} + \cos\phi_{j-1/2})(\cos\phi_{j+1/2} - \cos\phi_{j-1/2}) \\ &= a^2(\cos\phi_{j+1/2} + \cos\phi_{j-1/2})2\sin\phi_j\sin(\Delta\phi/2). \end{aligned} \tag{8.10}$$

Clearly, (8.8) can then be satisfied by choosing

$$C_{j+1/2} = \frac{2\sin(\Delta\phi/2)}{\Delta\phi}a\cos\phi_{j+1/2} \tag{8.11}$$

(but not the more obvious $C_{j+1/2} = a\cos\phi_{j+1/2}$).

b. Longitude–sine–latitude coordinates

Let $\xi = \lambda$ and $\eta = \mu = \sin\phi$. Analytically, the map factors are $1/m = a\cos\phi$, $1/n = a/\cos\phi$. Choose the grid to have constant spacing $\Delta\lambda$ and $\Delta\phi$, implying $\Delta\mu_{j+1/2} = \sin\phi_{j+1} - \sin\phi_j$ is not constant. Again all coefficients are independent of longitude. Let

$$\left(\frac{\Delta\xi}{m}\right)_{j+1/2} = a\Delta\lambda\cos\phi_{j+1/2}, \tag{8.12}$$

$$\left(\frac{\Delta\eta}{n}\right)_{j+1/2} = \frac{a\Delta\mu_{j+1/2}}{\cos\phi_{j+1/2}} = 2a\sin\left(\frac{\Delta\phi}{2}\right), \tag{8.13}$$

so that

$$(\Delta\xi\Delta\eta/mn)_{j+1/2} = a^2\Delta\lambda\Delta\mu_{j+1/2}. \tag{8.14}$$

Note that this formula for the area elements is exact. Now choose

$$\alpha_{j-1/2} = \beta_{j-1/2} = \gamma_{j+1/2} = \delta_{j+1/2} = \frac{\Omega}{2H}\mu_j, \tag{8.15}$$

consistent with (6.11) and (6.12), with $\sigma = 0$. Again, take the v-cell areas to be the mean of the two overlapping h-cell areas, which can be satisfied by choosing

$$(\Delta\xi/m)_j = a\Delta\lambda\cos\phi_j, \tag{8.16}$$

$$(\Delta\eta/n)_j = a\sin\Delta\phi, \tag{8.17}$$

so that

$$(\Delta\xi\Delta\eta/mn)_j = a^2\Delta\lambda\cos\phi_j\sin\Delta\phi. \tag{8.18}$$

Using these values, (4.3) implies that for angular momentum conservation C and G must be related by

$$a \sin\phi_j\left[\left(\frac{\Delta\mu C}{\cos\phi}\right)_{j-1/2} + \left(\frac{\Delta\mu C}{\cos\phi}\right)_{j+1/2}\right] = -(G_{j+1/2} - G_{j-1/2}). \quad (8.19)$$

For example, let

$$C_{j+1/2} = a \cos\phi_{j+1/2}. \quad (8.20)$$

Then the left-hand side of (8.19) can be written

$$a^2 \sin\phi_j(\Delta\mu_{j-1/2} + \Delta\mu_{j+1/2}) = a^2\mu_j(\mu_{j+1} - \mu_{j-1}). \quad (8.21)$$

Equation (8.19) can then be satisfied by choosing

$$G_{j+1/2} = a^2(1 - \mu_j\mu_{j+1}) \quad (8.22)$$

[but not the more obvious $G_{j+1/2} = a^2(\cos\phi_{j+1/2})^2$].

9. Discussion

The analysis in sections 3–6 allows us to address the questions raised in the introduction.

1) There appear to be significant advantages to making the poles v points rather than u–h points. When the poles are v points, it is sufficient to assign 0 to the polar values of v^* whenever they are needed in the linear prognostic equations. It is then possible to choose coefficients in the scheme to obtain conservation of mass, angular momentum, and energy. On the other hand, when the poles are u–h points, u is singular at the pole, and u^* cannot be taken to be 0 there. A polar value of u (defined to be a zonal wavenumber-1 function of longitude) can be diagnosed using (2.18)–(2.24) as in the New Dynamics. By a suitable choice of coefficients, the resulting scheme can be made to conserve mass and angular momentum, but not energy. [Burridge and Haseler (1977) also found that energy conservation was problematic with u and h at the poles.] An alternative polar formulation, derived from the polar vorticity budget, is given in section 2c. With this vorticity budget formulation, the scheme can be made to conserve mass, angular momentum, and energy. However, this scheme is not second-order accurate near the poles.
2) There is no real issue over whether wind components or ''wind images'' are used as prognostic variables since it is trivial to convert between them at any stage. (A possible exception is at the poles where u and v are singular but U and V go to 0; however, using V is not necessary when the poles are v points, while using U would not solve the existing problems when the poles are u–h points.) More important is the question of which variables are used in the spatial finite-difference formulas and, in particular, in the horizontal averaging to form the Coriolis terms. The above analysis shows that various conservation properties can be obtained by using the forms (2.9)–(2.11), where u^* and v^* appear on the right-hand sides of the equations, and that misrepresentation of Rossby modes can be eliminated by carefully choosing the Coriolis averaging coefficients α, β, γ, and δ to accurately capture the divergence of the geostrophic wind.
3) Schemes that conserve mass, angular momentum, and energy are available with either $\mu = \sin\phi$ or ϕ itself as the latitudinal coordinate. A small advantage of using μ is that simple exact expressions (rather than second-order accurate expressions) are available for the grid cell areas. Hence the surface area of the earth, and hence the global integral of any constant field, are also given exactly. When ϕ is used the errors are small: the maximum relative error in any cell area is of the order 10^{-5} when $P = 100$.
4) The constraints derived above for conservation of mass and energy do not depend on the grid being nonrotated or uniformly spaced in either longitude or latitude. Conservation of angular momentum does depend on the grid being nonrotated, that is, ξ must be a function of longitude λ, and η must be a function of latitude ϕ. However, the coordinates may be stretched, and the grid need not be uniformly spaced. Thus, for the linearized equations, this work extends that of Arakawa and Lamb (1981) to the case of nonuniform and (for mass and energy) rotated grids.
5) Schemes can be derived that have various conservation properties *and* have accurate Rossby wave dispersion properties without spurious eastward propagating modes; the two sets of requirements lead to compatible constraints. Indeed using Arakawa and Lamb's energy-conserving form of the Coriolis terms actually helps to obtain accurate Rossby-mode dispersion properties. The results shown in section 6 (Fig. 3) suggest that if the second-order scheme is properly implemented then there is little requirement for a higher-order scheme for the Coriolis terms since there is little room for improvement.

The results presented here are for the linearized shallow-water equations, whereas a practical numerical weather prediction scheme must solve the nonlinear three-dimensional equations. A conservative fully nonlinear scheme must reduce to a conservative linear scheme in the limit of weak disturbances, so the constraints derived here are necessary, though not sufficient, for a fully nonlinear scheme. The work of Arakawa and Lamb (1977, 1981) and Burridge and Haseler (1977), for example, suggests one route by which these constraints could be extended to the nonlinear three-dimensional case using Eulerian centered-difference schemes, but those schemes suffer from a tight restriction on time step size for stability, exacerbated by the convergence of meridians near the poles. It is an open research question whether an alternative route, with a much weaker time step restriction, might be possible using conservative semi-Lagrangian schemes.

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