Response Functions for Arbitrary Weight Functions and Data Distributions. Part I: Framework for Interpreting the Response Function

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ABSTRACT

The response function is a commonly used measure of analysis scheme properties. Its use in the interpretation of analyses of real-valued data, however, is unnecessarily complicated by the structure of the standard form of the Fourier transform. Specifically, interpretation using this form of the Fourier transform requires knowledge of the relationship between Fourier transform values that are symmetric about the origin. Here, these relationships are used to simplify the application of the response function to the interpretation of analysis scheme properties.

In doing so, Fourier transforms are used because they can be applied to studying effects that both data sampling *and* weight functions have upon analyses. A complication arises, however, in the treatment of constant and sinusoidal input since they do not have Fourier transforms in the traditional sense. To handle these highly useful forms, distribution theory is used to generalize Fourier transform theory. This extension enables Fourier transform theory to handle both functions that have Fourier transforms in the traditional sense and functions that can be represented using Fourier series.

The key step in simplifying the use of the response function is the expression of the inverse Fourier transform in a magnitude and phase form, which involves folding the integration domain onto itself so that integration is performed over only half of the domain. Once this is accomplished, interpretation of the response function is in terms of amplitude and phase modulations, which indicate how amplitudes and phases of input waves are affected by an analysis scheme. This interpretation is quite elegant since its formulation in terms of properties of input waves results in a one-to-one input-to-output wave interpretation of analysis scheme effects.

1. Introduction

The purpose of this investigation is twofold: 1) to illustrate a convenient means for using the response function to interpret the effects analysis schemes have upon input waves in real-valued data, and 2) to develop a new framework for computing response functions for arbitrary weight functions and data distributions. The former is considered herein while the latter is achieved in the second part of this investigation (Askelson et al. 2005, hereinafter Part II).

The response function is a commonly used measure of analysis scheme properties. As typically defined, the response function is the ratio of the Fourier transforms of the post- and preanalysis fields. Assuming fields are one-dimensional and have infinite, continuous domains, and denoting (direct) Fourier transforms with nonscript capital letters, the response function $\mathcal{R}(\nu)$ is

$$\mathcal{R}(\nu) = F_A(\nu)/F(\nu), \tag{1}$$

where $F_A(\nu)$ is the Fourier transform of the analysis field $f_A(x)$, $F(\nu)$ is the Fourier transform of the input

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field $f(x)$, and ν denotes frequency.[1] Herein, the one-dimensional Fourier transform pair for a function $f(x)$ that has an infinite, continuous domain is defined by

$$F(\nu) = \int_{x=-\infty}^{x=\infty} f(x) \exp(-j2\pi\nu x)\, dx = F_{\mathrm{Re}}(\nu) + jF_{\mathrm{Im}}(\nu), \tag{2a}$$

$$f(x) = \int_{\nu=-\infty}^{\nu=\infty} F(\nu) \exp(j2\pi\nu x)\, d\nu, \tag{2b}$$

where $j = \sqrt{-1}$ and $F_{\mathrm{Im}}(\nu)$ is defined here, and in Part II, such that it includes the negative sign. The first relation defines the (direct) Fourier transform of $f(x)$ while the second defines the Fourier transform of $F(\nu)$, which is also referred to as the inverse Fourier transform of $F(\nu)$ (e.g., Weaver 1983, p. 55; Cochran et al. 1987, p. 604). The Fourier transform $F(\nu)$ characterizes the sinusoidal waves that compose $f(x)$ (Weaver 1983, 57–62).

The response function is a generally complex-valued function that provides information concerning changes input waves undergo during analysis. To see this, consider that the complex-valued functions $\mathcal{R}(\nu)$ and $F(\nu)$ can be expressed in polar form. Using (1), this results in the following form for $F_A(\nu)$:

$$F_A(\nu) = F(\nu)\mathcal{R}(\nu) = |F(\nu)||\mathcal{R}(\nu)| \exp[j(\varphi_{F(\nu)} + \varphi_{\mathcal{R}(\nu)})]. \tag{3}$$

In (3), $|F(\nu)|$ and $|\mathcal{R}(\nu)|$ are the magnitudes of $F(\nu)$ and $\mathcal{R}(\nu)$ and $\varphi_{F(\nu)} = \tan^{-1}[F_{\mathrm{Im}}(\nu)/F_{\mathrm{Re}}(\nu)] + 2\pi n$ and $\varphi_{\mathcal{R}(\nu)} = \tan^{-1}[\mathcal{R}_{\mathrm{Im}}(\nu)/\mathcal{R}_{\mathrm{Re}}(\nu)] + 2\pi n$ are the phases, or arguments, of $F(\nu)$ and $\mathcal{R}(\nu)$, which include $2\pi n$ terms, n being an integer, because $\tan^{-1}(\ )$ is $2\pi n$ ambiguous. Because properties of input sinusoidal waves are described by $F(\nu)$ and because $F(\nu) = |F(\nu)| \exp[j\varphi_{F(\nu)}]$, $|F(\nu)|$ and $\varphi_{F(\nu)}$ must also describe properties of input sinusoidal waves [the relation of $|F(\nu)|$ and $\varphi_{F(\nu)}$ to properties of sinusoidal waves is considered in section 4]. The multiplication of $|F(\nu)|$ by $|\mathcal{R}(\nu)|$ and the addition of $\varphi_{\mathcal{R}(\nu)}$ to $\varphi_{F(\nu)}$, then, results in changes of the properties of input waves. These changes are imposed on the analysis field, as illustrated by the Fourier transform of $F_A(\nu)$

$$\begin{aligned} f_A(x) &= \int_{\nu=-\infty}^{\nu=\infty} F_A(\nu) \exp(j2\pi\nu x)\, d\nu \\ &= \int_{\nu=-\infty}^{\nu=\infty} |F(\nu)|\,|\mathcal{R}(\nu)| \exp[j(\varphi_{F(\nu)} \\ &\quad + \varphi_{\mathcal{R}(\nu)})] \exp(j2\pi\nu x)\, d\nu. \end{aligned} \tag{4}$$

Thus, the response function provides information concerning changes input waves undergo during analysis.

Interpretation of the impacts an analysis scheme has on real-valued data is clouded, however, by the presence of complex notation and negative frequencies in (4). For instance, if one is interested in the effects an analysis scheme has upon an input wave of a certain frequency, the use of (4) requires considering effects at multiple frequencies. When complex notation and negative frequencies are used for one-dimensional, real-valued data, input at a frequency ν_i shows up in the Fourier transform at both ν_i and $-\nu_i$ (Weaver 1983, 26–30). Consequently, using (4) to interpret how an analysis scheme affects input waves is not straightforward. It requires knowledge of the relationship between Fourier transform values that are symmetric about $\nu = 0$. The purpose herein is to utilize these relationships to produce a result similar to (4) that clearly indicates, through the response function, how an analysis scheme affects input waves. First, however, a discussion of the types of functions that can be analyzed using the method developed herein is required.

2. Generalized Fourier transform theory

Fourier transform theory (i.e., Fourier theory for continuous, infinite domains) is used in these studies to facilitate mathematical analysis. Within this framework, observation distributions can be represented using either functions, distributions,[2] or both, and powerful theorems like the product and convolution theorems can be utilized (see Part II). Together, these enable investigations of the effects of data sampling (e.g., Weaver 1983; Caracena et al. 1984; Pauley 1990). More-

[1] When applying Fourier analysis to spatial data, it is customary to use the term "frequency," which in this context is the reciprocal of the wavelength λ (e.g., Weaver 1983). Herein and in Part II, it is assumed that spatial data are being considered. If temporal data are being considered, then the term frequency means the reciprocal of the period, which is the definition to which many are accustomed. When data are both spatial and temporal, the term frequency should be restricted to mean the reciprocal of the period in order to avoid confusion.

[2] A distribution, or generalized function, is an abstraction of the concept of a function. Distributions are powerful in this context because they enable mathematical manipulation of impulsive behavior, such as that exhibited by an observation network in which a field is sampled at points instead of over regions. Aspects of distribution theory are briefly discussed in appendix B. For those interested, Bracewell (2000) provides a relatively straightforward introduction.

over, weight-function effects can also be analyzed with Fourier transforms by using the convolution theorem (Caracena et al. 1984; Pauley and Wu 1990; Askelson et al. 2000). The task in Part II is to develop a method for applying Fourier transform theory and the convolution theorem to the determination of the combined effects of weight functions and data distributions on response functions.

A complication that arises, however, is the consideration of Fourier transforms of functions like sin x, which is a very useful function owing to its simplicity but is not physically realizable because it produces nonzero values that extend over infinite distances. To handle such Fourier transforms, the generalization of "Fourier transforms in the limit" is required. With this generalization, Fourier transforms of very useful functions (e.g., sin x) that do not exist in the traditional sense (e.g., because $\int_{x=-\infty}^{x=\infty} |\sin x|\, dx$ does not exist) can be defined (Bracewell 2000, 8–11). This is accomplished by using distributions, which are discussed briefly in appendix B and more thoroughly in numerous texts (e.g., Jones 1982; Bracewell 2000). Hereinafter and in Part II, Fourier transforms in the limit will be used and will simply be called Fourier transforms. Use of the generalized theory will be indicated by the presence of distributions in equations involving Fourier transforms.

Before proceeding, it is important to define the class of functions to which this generalized theory of Fourier transforms can be applied. In one dimension, this class is defined by functions $f(x)$ that can be expressed as

$$f(x) = g(x) + h(x), \tag{5}$$

where $g(x)$ is a function for which a Fourier transform exists in the traditional sense and $h(x) = C + s(x)$ is a periodic function that has a *Fourier series* representation, with C being the mean value of the function over one period.[3] Thus, this generalized theory extends traditional Fourier transform theory to handle constant signals over an infinite domain (when C is nonzero) and periodic signals over an infinite domain that have Fourier series representations. It does so by utilizing the Dirac distribution, which arises when transforming either a nonzero constant or a sinusoidal input that continues forever (Bracewell 2000, 105–108). In the case of a periodic function for which a Fourier series exists, that function can be represented as a sum of sinusoids. Consequently, the direct Fourier transform contains a Dirac distribution at each frequency at which the Fourier coefficient of the Fourier series is nonzero.

[3] It is noted that in contrast to Fourier transforms, Fourier series are computed over limited domains, with transformation equations given by $C_n = (1/\sqrt{2L}) \int_{x=-L}^{x=L} f(x) \exp[-j(2\pi/2L/n)x]\, dx$ and $f(x) = (1/\sqrt{2L}) \Sigma_{n=-\infty}^{\infty} C_n \exp[j(2\pi/2L/n)x]$, where C_n is the nth Fourier coefficient, L is the half-width of the periodic function, and n is an index that determines which frequencies are needed in the Fourier series representation of a periodic function.

3. The one-sided magnitude and phase form of the inverse Fourier transform for one-dimensional, real data

The two-sided inverse Fourier transform of one-dimensional, real-valued data can be collapsed into a one-sided Fourier transform that involves magnitudes and phases. The starting point for doing so is the inverse Fourier transform

$$f(x) = \int_{\nu=-\infty}^{\nu=\infty} F(\nu) \exp(j2\pi\nu x)\, d\nu. \tag{6}$$

As shown in appendix A, for functions that have Fourier transforms in the traditional sense, this can be expressed as

$$f(x) = \int_{\nu=0}^{\nu=\infty} 2|F(\nu)| \cos[2\pi\nu x + \varphi_{F(\nu)}]\, d\nu. \tag{7}$$

For functions that have Fourier transforms that involve distributions, however, (7) is incorrect. This problem with (7) arises in this case because to obtain (7) the integral in (6) is split at $\nu = 0$, and when $F(\nu)$ involves distributions, the value of the integrand at a single location (i.e., $\nu = 0$) can change the integral result. As shown in appendix A, this problem can be alleviated by altering (7) to the following form:

$$f(x) = \int_{\nu=0}^{\nu=\infty} \frac{2}{1 + \delta^0(\nu)} |F(\nu)| \cos[2\pi\nu x + \varphi_{F(\nu)}]\, d\nu, \tag{8}$$

where $\delta^0(x)$ is defined as in Bracewell (2000, p. 87) as

$$\delta^0(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}. \tag{9}$$

The fundamental result is that the real-valued function $f(x)$ can be expressed in terms of the magnitudes $|F(\nu)|$ and phases $\varphi_{F(\nu)}$ of its one-sided ($\nu \geq 0$) Fourier transform. This representation results in the removal of complex notation and negative frequencies in the inverse Fourier transform. As shown in section 5, this greatly simplifies interpretation of the response function.

4. The relationship of the magnitudes and phases of the one-sided Fourier transform to the amplitudes and phases of input sinusoids for one-dimensional, real data

The use of (8) in the interpretation of the response function is illustrated in section 5. To interpret the response function using (8), one must understand the relation of the magnitudes $|F(\nu)|$ and phases $\varphi_{F(\nu)} = \tan^{-1}[F_{\mathrm{Im}}(\nu)/F_{\mathrm{Re}}(\nu)] + 2\pi n$ of the Fourier transform $F(\nu)$ to the amplitudes and phases of input sinusoids. Because for periodic functions that have Fourier series representations this requires evaluation of the magnitude and phase of the Dirac distribution, this is not a trivial task.

To facilitate this analysis, it is noted that sinusoidal input at any frequency can be expressed in the form

$$f(x) = A\cos(2\pi\nu_i x + \varphi), \tag{10}$$

where A is the amplitude (defined here to be > 0), φ is the phase, and ν_i is defined to be nonnegative. This form is general because the phase can be used to adapt to any type of sinusoidal input at a given frequency; the cosine gives the "shape" and the phase provides the correct alignment. For instance, if the input wave is a sine wave and the input amplitudes and frequencies are positive, then the input sine wave is produced using the above form with $\varphi = -\pi/2$. If, instead, the input wave is a cosine wave and the frequency is positive but the amplitude is negative, the input wave can be replicated using the above form with $\varphi = \pm n\pi$, with n being a nonnegative integer. Other situations involving, for instance, negative frequencies are also handled easily with this form. It is noted that a sine wave could be used in the same manner as in (10) and that a sine wave representation could be used in (8) if desired.

Now, consider the Fourier transform of an input sinusoid represented using (10). From (2a), it is given by

$$\begin{aligned} F(\nu) = {} & A\cos\varphi \int_{x=-\infty}^{x=\infty} \cos(2\pi\nu_i x)\cos(2\pi\nu x)\,dx \\ & - A\sin\varphi \int_{x=-\infty}^{x=\infty} \sin(2\pi\nu_i x)\cos(2\pi\nu x)\,dx \\ & - jA\cos\varphi \int_{x=-\infty}^{x=\infty} \cos(2\pi\nu_i x)\sin(2\pi\nu x)\,dx \\ & + jA\sin\varphi \int_{x=-\infty}^{x=\infty} \sin(2\pi\nu_i x)\sin(2\pi\nu x)\,dx, \end{aligned} \tag{11}$$

where the identity $\cos(2\pi\nu_i x + \varphi) = \cos(2\pi\nu_i x)\cos\varphi - \sin(2\pi\nu_i x)\sin\varphi$ has been applied. But, because for all ν_i $\int_{x=-\infty}^{x=\infty} \sin(2\pi\nu_i x)\cos(2\pi\nu x)\,dx = \int_{x=-\infty}^{x=\infty} \cos(2\pi\nu_i x)\sin(2\pi\nu x)\,dx = 0$ owing to the integrands being either odd or zero,

$$\begin{aligned} F(\nu) = {} & A\cos\varphi \int_{x=-\infty}^{x=\infty} \cos(2\pi\nu_i x)\cos(2\pi\nu x)\,dx \\ & + jA\sin\varphi \int_{x=-\infty}^{x=\infty} \sin(2\pi\nu_i x)\sin(2\pi\nu x)\,dx. \end{aligned} \tag{12}$$

From the orthogonality relation for one-dimensional Fourier transforms (Jackson 1975, p. 68; Bracewell 2000, 105–108),

$$\int_{-\infty}^{\infty} \exp(j2\pi\nu_i x)\exp(-j2\pi\nu x)\,dx = \delta(\nu - \nu_i), \tag{13}$$

where $\delta(\nu)$ is the Dirac distribution, one can easily obtain

$$\int_{x=-\infty}^{x=\infty} \cos(2\pi\nu_i x)\cos(2\pi\nu x)\,dx = \frac{1}{2}\delta(\nu - \nu_i) + \frac{1}{2}\delta(\nu + \nu_i) \tag{14}$$

and

$$\int_{x=-\infty}^{x=\infty} \sin(2\pi\nu_i x)\sin(2\pi\nu x)\,dx = \frac{1}{2}\delta(\nu - \nu_i) - \frac{1}{2}\delta(\nu + \nu_i), \tag{15}$$

which are the Fourier transforms of $\cos(2\pi\nu_i x)$ and $j\sin(2\pi\nu_i x)$, respectively (Bracewell 2000, p. 108). Inserting these into (12) results in

$$\begin{aligned} F(\nu) = {} & \frac{A}{2}\delta(\nu - \nu_i)(\cos\varphi + j\sin\varphi) \\ & + \frac{A}{2}\delta(\nu + \nu_i)(\cos\varphi - j\sin\varphi). \end{aligned} \tag{16}$$

Here, however, one-sided spectra ($\nu \geq 0$) are used [with (8)] and thus (16) becomes

$$F(\nu) = \begin{cases} \frac{A}{2}\delta(\nu - \nu_i)(\cos\varphi + j\sin\varphi) & \nu_i > 0 \\ A\delta(\nu)\cos\varphi & \nu_i = 0 \end{cases}, \tag{17}$$

with the form for $\nu_i > 0$ resulting because $\nu + \nu_i = 0$ in the second term on the rhs of (16) is never satisfied. Thus, when one-sided spectra are used and the input is from a real-valued sinusoid, the Fourier transform is nonzero only at the frequency of the input sinusoid.

When one-sided spectra are used and the input is from a real-valued sinusoid, (17) shows that the Fourier transform is the product of either a real-valued distribution and a complex-valued function ($v_i > 0$) or a real-valued distribution and a real-valued function ($v_i = 0$). The magnitude of $F(v)$ is thus

$$|F(v)| = \begin{cases} \left|\frac{A}{2}\,\delta(v - v_i)\right| |\cos\varphi + j\sin\varphi| = \left|\frac{A}{2}\,\delta(v - v_i)\right| & v_i > 0 \\ |A\delta(v)|\,|\cos\varphi| & v_i = 0 \end{cases}. \tag{18}$$

As shown in appendix B, this is simply

$$|F(v)| = \begin{cases} \frac{A}{2}\,\delta(v - v_i) & v_i > 0 \\ A\delta(v)\,|\cos\varphi| & v_i = 0 \end{cases}, \tag{19}$$

which, as discussed in appendix B, holds because $A > 0$ and because of the form of the sequence that defines the Dirac distribution in this application. It is interesting that the magnitudes of Fourier transforms depend upon the amplitudes of the input sinusoids and upon a factor related to the size of the domain of the transform [since $\int_{x=-\infty}^{x=\infty} dx = \delta(0)$], which is similar to the case for Fourier series (e.g., Cochran et al. 1987, section7.4).

From (17), for $v_i > 0$ $F(v)$ is the product a real-valued distribution and a complex-valued function. In this case, the phase of $F(v)$ is the phase of $(A/2)\ \delta(v - v_i)$ added to the phase of $\cos\varphi + j\sin\varphi$. The phase of $(A/2)\ \delta(v - v_i)$ is indeterminate where $(A/2)\ \delta(v - v_i)$ is zero and is zero where $(A/2)\ \delta(v - v_i)$ is nonzero because $A > 0$ and the defining sequence for the Dirac distribution in this case is always greater than zero. For $v_i = 0$, $F(v)$ in (17) is real valued. In this case, the phase of $F(v)$ is indeterminate where $\delta(v)$ is zero and depends only on $\cos\varphi$ when $\delta(x)$ is not zero since $A > 0$ and the defining sequence for the Dirac distribution in this case is always greater than zero. Thus, the phase of $F(v)$ is given by

$$\varphi_{F(v)} = \begin{cases} \varphi & v_i > 0, v = v_i \\ 0 & v_i = 0, v = v_i, \text{ and } \cos\varphi > 0 \\ \pi & v_i = 0, v = v_i, \text{ and } \cos\varphi < 0 \\ \text{indeterminate} & \text{otherwise.} \end{cases} \tag{20}$$

5. Interpretation of the response function

The re-expression of (2b) in terms of the magnitudes and phases of the one-sided Fourier transform (8) can be used to illustrate the utility of the response function in describing the impact an analysis scheme has upon input waves. Consider an analysis field $f_A(x)$ that has the Fourier transform $F_A(v)$ that is related to the Fourier transform of the input field $F(v)$ through the response function $\mathcal{R}(v)$ by $F_A(\mathrm{v}) = F(v)\mathcal{R}(v)$. From (8),

$$\begin{aligned} f_A(x) &= \int_{v=0}^{v=\infty} \frac{2}{1 + \delta^0(v)}\,|F_A(v)| \cos[2\pi vx + \varphi_{F_A(v)}]\,dv \\ &= \int_{v=0}^{v=\infty} \frac{2}{1 + \delta^0(v)}\,|F(v)|\,|\mathcal{R}(v)| \cos[2\pi vx + \varphi_{F(v)} \\ &\quad + \varphi_{\mathcal{R}(v)}]\,dv, \end{aligned} \tag{21}$$

which follows from (3). For an input field described by (10) with $v_i > 0$, (19) inserted into (21) produces

$$\begin{aligned} f_A(x) = \int_{v=0}^{v=\infty} \frac{2}{1 + \delta^0(v)}\,\frac{A}{2}\,\delta(v - v_i)|\mathcal{R}(v)| \cos[2\pi vx \\ + \varphi_{F(v)} + \varphi_{\mathcal{R}(v)}]\,dv. \end{aligned} \tag{22}$$

Utilizing the sifting property of the Dirac distribution, $\int_{x=-\infty}^{x=\infty} \delta(x - a)f(x)\,dx = f(a)$ (Bracewell 2000, p. 79), and (20), this simplifies to

$$f_A(x) = A|\mathcal{R}(v_i)| \cos(2\pi v_i x + \varphi + \varphi_{\mathcal{R}(v_i)}). \tag{23}$$

This is equivalent to the input field (10) except for the amplitude modulation $|\mathcal{R}(v_i)|$ and the phase shift $\varphi_{\mathcal{R}(v_i)}$. Thus, the removal of complex notation and negative frequencies in the inverse Fourier transform leads to the following straightforward interpretation of the response function: *The amplitude modulation $|\mathcal{R}(v_i)|$ indicates how the amplitude of the input wave is changed during an analysis while the phase modulation $\varphi_{\mathcal{R}(v_i)}$ indicates how the phase of an input wave is changed during an analysis.*

For an input field described by (10) with $v_i = 0$, (19) and (20) inserted into (21) result in

$$f_A(x) = A|\cos\varphi||\mathcal{R}(0)|\cos(\varphi_{F(0)} + \varphi_{\mathcal{R}(0)})$$
$$= A|\cos\varphi||\mathcal{R}(0)|\{\cos[\varphi_{F(0)}]\cos[\varphi_{\mathcal{R}(0)}]$$
$$- \sin[\varphi_{F(0)}]\sin[\varphi_{\mathcal{R}(0)}]\}, \quad (24)$$

where

$$\varphi_{F(0)} = \begin{cases} 0 & \cos\varphi > 0 \\ \pi & \cos\varphi < 0 \end{cases}. \quad (25)$$

The $\varphi_{\mathcal{R}(0)}$ term is quite interesting. Help interpreting $\varphi_{\mathcal{R}(0)}$ can be gained by "jumping ahead" and using the result of Part II that the response function arises from the complex conjugate of the Fourier transform of the "effective" weight function. At this point, the details of this result are not critical. What is important is that the response function comes from the Fourier transform of, in this context, a real-valued function. This results in the imaginary part of $\mathcal{R}(0)$ being zero because it involves terms that include $\sin(2\pi\nu x)$. Thus, $\varphi_{\mathcal{R}(0)}$ is either 0 or π, depending on whether the real part of $\mathcal{R}(0)$ is positive or negative, respectively. Since $\varphi_{\mathcal{R}(0)}$ is either 0 or π, (24) simplifies to

$$f_A(x) = A|\cos\varphi|\cos[\varphi_{F(0)}]|\mathcal{R}(0)|\cos[\varphi_{\mathcal{R}(0)}]. \quad (26)$$

From (25) and (26), it is apparent that the $f_A(x) = A|\cos\varphi|\cos[\varphi_{F(0)}]$ in (26) reproduces the input field (10) while the $|\mathcal{R}(0)|$ and $\cos[\varphi_{\mathcal{R}(0)}]$ encapsulate the changes to the input field resulting from the analysis scheme. First, consider $\cos[\varphi_{\mathcal{R}(0)}]$. Since $\varphi_{\mathcal{R}(0)}$ is either 0 or π, its value is either +1 or −1, respectively. Thus, $\cos[\varphi_{\mathcal{R}(0)}]$ is essentially the sgn function, being +1 if the real part of $\mathcal{R}(0)$ is positive and −1 if the real part of $\mathcal{R}(0)$ is negative. From Part II, the real part of $\mathcal{R}(0)$ is the sum of the weights used in the analysis if the data are discrete and is the integral of the weights used in the analysis if the data are continuous. Thus, if the sum or integral of the weights is greater than zero, the sign of the zeroth harmonic is unchanged by the analysis. If the sum or integral of the weights is less than zero, the sign of the zeroth harmonic is changed. This plays a role that is analogous to that of the phase change $\varphi_{\mathcal{R}(\nu_i)}$ for the case of $\nu_i > 0$. There, a phase shift of π radians causes a sign change in the field. Because for $\nu_i > 0$ the input field is in the form of a sinusoid, however, a variety of phase shift values are possible. For the zeroth harmonic, "phase shifts" of 0 and π radians are the only ones that are possible since the input field is a constant. As in the case of $\nu_i > 0$, the $|\mathcal{R}(0)|$ in (26) is the amplitude modulation of the input sinusoid.

For simplicity, the input to this point has been considered to be composed of one sinusoid. The above analysis, however, holds for input composed of numerous sinusoids, with (23) and (26) applying to each sinusoid as appropriate.

Owing to their relative difficulty, attention has, to this point, been focused on functions that are periodic and that have Fourier series representations. For other functions of interest here, those that have Fourier transforms in the traditional sense, the interpretation of the response function is the same. For those functions, $F(\nu)$, and, thus, $|F(\nu)|$ and $\varphi_{F(\nu)}$, are finite. From (8), $|F(\nu)|$ plays the role of sinusoid amplitude. It is the amplitude at a given wavelength that is required to reproduce the input function. Moreover, $\varphi_{F(\nu)}$ is the phase at a given wavelength that is required to reproduce the input function. From (21), then, $|\mathcal{R}(\nu)|$ is the amplitude modulation and $\varphi_{\mathcal{R}(\nu)}$ is the phase modulation of an analysis scheme at a given wavelength. Note that this method of interpretation, where $|F(\nu)|$ and $\varphi_{F(\nu)}$ are just those that are needed to reproduce the input field, also applies to functions that are periodic and that have Fourier series representations. In that case, however, the presence of Dirac distributions necessitates the preceding examination.

6. Discussion

Other methods for interpreting the response function have been proposed. Achtemeier (1986) used a method wherein an input sine wave resulted in two waves upon output. The first was a sine wave having the same frequency but generally different amplitude relative to the input wave, and the second was a cosine wave having the same frequency as the input wave. Because the input wave that was used was a sine wave, the second wave arises only if the imaginary part of the response function is nonzero (Pauley 1990). Pauley (1990) also used this interpretation of the response function and termed the ratio of the amplitude of the second wave to the amplitude of the input wave, which Pauley (1990) states is the imaginary component of $\mathcal{R}(\nu)$, the "phase-shifted response." This approach is equivalent to that presented herein in that a wave given by (10) can be expressed as the sum of a sine and a cosine wave. However, the amplitude and phase interpretation used herein seems to be more convenient than that used by Achtemeier (1986) and Pauley (1990) because it does not require bookkeeping for two output waves (resulting from one input wave) and because it clearly relates the properties of the output wave to those of the input wave. For instance, a phase shift means that the input wave has been moved, with the amount and direction of displacement given by the magnitude and sign of the phase shift. The Achtemeier–Pauley interpretation, on the other hand, does not provide such insight.

With the amplitude and phase modulation approach used herein, the effects of an analysis scheme are described quite simply by how the analysis scheme changes the intensity of input waves (amplitude modulation) and by how the analysis scheme moves input waves around (phase modulation). Because in real analyses the inhomogeneous distribution of observations generally results in phase shifts (Pauley 1990; Buzzi et al. 1991), phase modulation information should be included in addition to amplitude modulation information in the evaluation of response functions. Unfortunately, as Buzzi et al. (1991) indicate, phase modulation has received less scrutiny than it would seemingly warrant. Its importance has been increasingly noted, however, in numerous analyses and discussions, including Achtemeier (1986), Pauley (1990), Buzzi et al. (1991), Carr et al. (1995), and Askelson et al. (2000). In fact, Buzzi et al. (1991) used the amplitude and phase approach outlined herein, but did not illustrate in detail how input fields can be interpreted in terms of amplitudes and phases or how the magnitudes and phases of response functions impact input fields.

Significant effort is expended in this study in handling input fields that are composed of physically unrealizable functions like $\sin x$. While this effort may seem to have been unnecessary in that no real input field could have nonzero values over an infinite domain, the reality is that input can look just like a simple wave, like a sine wave, over a limited domain and thus can be well represented by a simple waveform. In that case, the use of a simple waveform will greatly simplify analyses. Simple waveforms, for instance, can be very useful in analytic tests, like the response function tests in Part II.

For simplicity, the above analysis has been performed for one-dimensional data. Data are often multidimensional, however, and so extension to multiple dimensions is important. Extension to two dimensions is provided in appendix C. Except for some extra housekeeping, many of the concepts are the same. By using the definition of the response function for two-dimensional data $F_A(u, v) = F(u, v)\,\mathcal{R}(u, v)$ and polar representations of $F(u, v)$ and $\mathcal{R}(u, v)$, it is easily seen from (C32) that, in a manner similar to that for one dimension, $|\mathcal{R}(u, v)|$ and $\varphi_{\mathcal{R}(u,v)}$ are the amplitude and phase modulations experienced by input waves during an analysis.

7. Conclusions

The following list summarizes the results of this work:

1) Inverse Fourier transforms for real-valued functions that have Fourier transforms in the traditional sense, that are constant, or that are sinusoidal can be expressed in a "one-sided" magnitude and phase form.
2) The one-sided magnitude and phase form of the inverse Fourier transform for real-valued functions simplifies interpretation of the effects analyses have on input by removing the need to track effects at two frequencies for one input sinusoid.
3) The magnitude of the Fourier transform of a real-valued sinusoid is dependent upon the amplitude of that sinusoid and upon a factor related to the size of the domain of the Fourier transform. The phase of the Fourier transform of a real-valued sinusoid is directly related to the phase of that sinusoid.
4) Response functions can be eloquently described in terms of amplitude and phase modulations of input waves.
5) The framework developed herein can be extended to multiple dimensions.

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APPENDIX A

Derivation of the One-Sided Magnitude and Phase Form of the Inverse Fourier Transform for One-Dimensional, Real Data

The starting point for this derivation is the inverse Fourier transform

$$f(x) = \int_{\nu=-\infty}^{\nu=\infty} F(\nu)\exp(j2\pi\nu x)\,d\nu, \qquad \text{(A1)}$$

where

$$F(\nu) = \int_{x=-\infty}^{x=\infty} f(x)\exp(-j2\pi\nu x)\,dx = F_{\mathrm{Re}}(\nu) + jF_{\mathrm{Im}}(\nu). \tag{A2}$$

An important property of $F(\nu)$, for $f(x)$ real, is that it is Hermitian (Bracewell 2000, 13–14), which means that

$$F_{\mathrm{Re}}(\nu) = F_{\mathrm{Re}}(-\nu)$$
$$F_{\mathrm{Im}}(\nu) = -F_{\mathrm{Im}}(-\nu). \tag{A3}$$

Under the appropriate conditions (discussed at the end of this derivation), the integral in (A1) can be split at $\nu = 0$ to produce

$$f(x) = \int_{\nu=-\infty}^{\nu=0} F(\nu)\exp(j2\pi\nu x)\,d\nu + \int_{\nu=0}^{\nu=\infty} F(\nu)\exp(j2\pi\nu x)\,d\nu. \tag{A4}$$

By considering the integrand in the first integral of (A4) it is obvious that (A4) can be written as

$$f(x) = \int_{\nu=0}^{\nu=\infty} F(\nu)\exp(j2\pi\nu x) + F(-\nu)\exp(-j2\pi\nu x)\,d\nu; \tag{A5}$$

a result that can be obtained mathematically through the substitution $u = -\nu$ in the first integral in (A4). Moreover, $F(\nu)$ and $F(-\nu)$ can be expressed in polar form as

$$F(\nu) = |F(\nu)|\exp[j\varphi_{F(\nu)}] \tag{A6}$$

and

$$F(-\nu) = |F(-\nu)|\exp[j\varphi_{F(-\nu)}]. \tag{A7}$$

From (A3), $|F(-\nu)| = F(\nu)$, $\varphi_{F(-\nu)} = \tan^{-1}[-F_{\mathrm{Im}}(\nu)/F_{\mathrm{Re}}(\nu)] + 2\pi n$, and, thus, $\varphi_{F(-\nu)} = -\varphi_{F(\nu)}$.[A1] Thus, (A5) can be written, using (A6) and (A7), as

$$f(x) = \int_{\nu=0}^{\nu=\infty} |F(\nu)|\{\exp[j(2\pi\nu x + \varphi_{F(\nu)})] + \exp[-j(2\pi\nu x + \varphi_{F(\nu)})]\}\,d\nu. \tag{A8}$$

Since $e^{j\theta} + e^{-j\theta} = 2\cos\theta$, (A8) simplifies to

$$f(x) = \int_{\nu=0}^{\nu=\infty} 2|F(\nu)|\cos[2\pi\nu x + \varphi_{F(\nu)}]\,d\nu. \tag{A9}$$

This result is not valid when C in $h(x)$ in (5) is not zero. In that case, the Fourier transform of $f(x)$ [from (A1)] is

$$F(\nu) = G(\nu) + C\delta(\nu) + S(\nu), \tag{A10}$$

where the Dirac distribution $\delta(\nu)$ arises because $\delta(\nu) = \int_{x=-\infty}^{x=\infty}\exp(-j2\pi\nu x)\,dx$, which is the Fourier transform of 1, and $S(\nu)$ also contains Dirac distributions since $S(x)$ is composed of sine and cosine waves (Bracewell 2000, 105–108). Insertion of (A10) into (A1) produces $f(x) = g(x) + C + s(x)$, where the sifting property of the Dirac distribution, $\int_{x=-\infty}^{x=\infty}\delta(x-a)f(x)\,dx = f(a)$ (Bracewell 2000, p. 79), has been applied. Insertion of (A10) into (A9), on the other hand, produces $f(x) = g(x) + 2C + s(x)$. To see this, consider that (A9) can be applied to each component of (A10) because insertion of (A10) into (A1) results in individual Fourier transforms for $G(\nu)$, $C\delta(\nu)$, and $S(\nu)$. The steps that produced (A9) from (A1) can then be replicated for each of the components of (A10). A difficulty, however, arises in this process owing to the need to evaluate the magnitude and phase of the Dirac distribution [associated with $C\delta(\nu)$ and $S(\nu)$], which is not trivial since distributions do not interact with operators in the same manner as functions. Their evaluation is provided in section 4, where these issues are addressed owing to the central role they play there. From the results of that section [(10), (19), and (20)] and the fact that $S(0) = 0$ because $s(x)$ is composed only of sine and cosine waves, $\int_{\nu=0}^{\nu=\infty} 2|C\delta(\nu)|\cos[2\pi\nu x + \varphi_{C\delta(\nu)}]\,d\nu = 2C$ and $\int_{\nu=0}^{\nu=\infty} 2|S(\nu)|\cos[2\pi\nu x + \varphi_{S(\nu)}]\,d\nu = s(x)$. Consequently, when C is not zero, the inverse Fourier transform Eq. (A1) produces the original function $f(x) = g(x) + C + s(x)$ whereas the amplitude and phase form of the inverse Fourier transform Eq. (A9) produces the incorrect result $f(x) = g(x) + 2C + s(x)$.

The reason for this discrepancy is the split of the inverse Fourier transform at $\nu = 0$. Normally, this would have no impact since the value of the integrand at one location usually does not alter the value of the integral (differences must normally be spread out over

[A1] Note that $|F(\nu)| = |F(-\nu)|$ can be understood by using (A3) and the relation $|F(\nu)| = [F_{\mathrm{Re}}(\nu)^2 + F_{\mathrm{Im}}(\nu)^2]^{1/2}$. In the case of sinusoidal input, however, the Dirac distribution arises in $F(\nu)$. Because the square of the Dirac distribution does not exist (Mikusiński 1966), the relation $|F(\nu)| = [F_{\mathrm{Re}}(\nu)^2 + F_{\mathrm{Im}}(\nu)^2]^{1/2}$ cannot be used in this case. Regardless, by repeating the analysis of section 4 for the second function on the rhs of (16) to determine $F(-\nu)$, it is apparent that when sinusoidal input is used, $|F(\nu)| = |F(-\nu)|$. Moreover, this exercise also illustrates that when sinusoidal input is used, $\varphi_{F(-\nu)} = -\varphi_{F(\nu)}$. Thus, the results $|F(\nu)| = |F(-\nu)|$ and $\varphi_{F(-\nu)} = -\varphi_{F(\nu)}$ are quite general.

a finite range). Here, however, Fourier transforms of sinusoids and constant functions [which are simply sinusoids having zero frequency (10)] are of interest. To deal with these Fourier transforms, distributions like the Dirac distribution are needed. When the Dirac distribution is utilized, its sifting property can result in differences in integrands at individual points being of consequence. This is what occurs at $\nu = 0$ above. To correct for the above problem, the magnitude in (A9) is altered by the appropriate factor, resulting in

$$f(x) = \int_{\nu=0}^{\nu=\infty} \frac{2}{1 + \delta^0(\nu)} |F(\nu)| \cos[2\pi\nu x + \varphi_{F(\nu)}]\, d\nu, \tag{A11}$$

where $\delta^0(x)$ is defined as in Bracewell (2000, p. 87) as

$$\delta^0(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}. \tag{A12}$$

It is noted that the adjustment in (A11) could be rendered unnecessary by adapting the convention $\int_{x=0}^{x=\infty} \delta(x)\, dx = ½$, which is mentioned by Bracewell (2000, p. 104). Herein, the convention $\int_{x=0}^{x=\infty} \delta(x)\, dx = 1$ is used. It is also noted that (A11) is the Fourier transform analog of the shifted cosine form of Fourier series (e.g., Cochran et al. 1987, section 7.5).

APPENDIX B

Justification of $|A\delta(x)| = A\delta(x)$ for $A > 0$ and for the Sequence Defining $\delta(x)$ for Fourier Transforms of Sinusoids

In one dimension, a distribution is defined by a sequence of functions $p_\tau(x)$ for which $\lim_{\tau\to 0} \int_{x=-\infty}^{x=\infty} p_\tau(x) f_p(x)\, dx$ exists and is finite for $p_\tau(x)$ and any $f_p(x)$ that belong to the class of particularly well behaved functions, which contains functions that have derivatives of all orders and that die off at least as rapidly as $|x|^{-N}$ as $|x| \to \infty$, no matter the magnitude of N (Jones 1982, p. 53; Bracewell 2000, 94–96). The Dirac distribution $\delta(x)$, for instance, is defined by the sequence $\tau^{-1} \exp[-\pi x^2/\tau^2]$, which, as $\tau \to 0$, approaches zero everywhere except at $x = 0$, where it approaches an infinitely short and infinitely strong pulse, and has the property

$$\lim_{\tau\to 0} \int_{x=-\infty}^{x=\infty} \tau^{-1} \exp[-\pi x^2/\tau^2] f_p(x)\, dx = f_p(0) \tag{B1}$$

(Jones 1982, p. 216; Bracewell 2000, 74–79).[B1] As discussed by Bracewell (2000, chapter 5), sequences that do not belong to the class of particularly well behaved functions can also be used to define an "impulse symbol" that has the property (B1). An example is the sequence of rectangular pulses $\tau^{-1}\Pi(x/\tau)$, where

$$\Pi(x) = \begin{cases} 0 & |x| > \frac{1}{2} \\ \frac{1}{2} & |x| = \frac{1}{2} \\ 1 & |x| < \frac{1}{2} \end{cases}. \tag{B2}$$

The problem at hand is to show that $|(A/2)\, \delta(\nu - \nu_i)| = (A/2)\, \delta(\nu - \nu_i)$ when $A > 0$ and the Dirac distribution arises from the Fourier transform of a sinusoid. This can be accomplished by considering the sequence of functions that define the Dirac distribution in this case, which is $p_\tau(\nu) = (\sqrt{\pi}/\tau) \exp[-\pi^2(\nu - \nu_i)^2/\tau^2]$ (Bracewell 2000, p. 108). Because $A > 0$ and $(\sqrt{\pi}/\tau) \exp[-\pi^2(\nu - \nu_i)^2/\tau^2] > 0$ for all ν,

$$\left| A \frac{\sqrt{\pi}}{\tau} \exp[-\pi^2(\nu - \nu_i)^2/\tau^2] \right| = A \frac{\sqrt{\pi}}{\tau} \exp[-\pi^2(\nu - \nu_i)^2/\tau^2], \tag{B3}$$

and thus $|(A/2)\, \delta(\nu - \nu_i)| = (A/2)\, \delta(\nu - \nu_i)$ in this case.

Note that there are sequences that define impulse symbols that are not equal to their absolute value. Bracewell (2000, p. 520) provides an example of such a sequence. For these, a relation like (B3) does not hold and consequently their absolute value inserted into an integral like that in (B1) does not produce $f_p(0)$. It is thus reiterated that the relation $|\delta(x)| = \delta(x)$ is valid only in those situations where the defining sequence is equal to its absolute value.

APPENDIX C

Extension to Two Dimensions

Data are often multidimensional. Most meteorological fields, for instance, are distributed in three spatial

[B1] The Dirac distribution can be succinctly expressed as

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases},$$

with the condition that $\int_{x=-\infty}^{x=\infty} \delta(x) f_p(x)\, dx = f_p(0)$.

dimensions and in time. The purpose here is to illustrate how concepts developed for one-dimensional data can be extended to multidimensional data.

a. Class of functions

As in the one-dimensional case, "Fourier transforms in the limit" will be used so that simple functions like $\sin(x + y)$ can be handled. The class of functions that will be considered is

$$f(x, y) = g(x, y) + h(x, y), \tag{C1}$$

where $g(x, y)$ is a function for which a Fourier transform exists in the traditional sense and $h(x, y) = C + s(x, y)$ is a periodic function that has a *Fourier series* representation, with C being the mean value of the function over one period.

b. Fourier transforms of two-dimensional sinusoids

To handle all of the functions in the function class (C1), one must be able to determine Fourier transforms of sinusoidal functions that are nonzero over infinite domains. These Fourier transforms can be obtained by considering that

$$\begin{aligned}&\int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty}\exp[-(ax)^2-(ay)^2]\exp[j2\pi(u_ix+v_iy)]\exp[-j2\pi(ux+vy)]\,dx\,dy\\&=\int_{x=-\infty}^{x=\infty}\exp[-(ax)^2]\exp(j2\pi u_ix)\exp(-j2\pi ux)\,dx\int_{y=-\infty}^{y=\infty}\exp[-(ay)^2]\exp(j2\pi v_iy)\exp(-j2\pi vy)\,dy\\&=\left\{\frac{\sqrt{\pi}}{a}\exp[-\pi^2(u-u_i)^2/a^2]\right\}\left\{\frac{\sqrt{\pi}}{a}\exp[-\pi^2(v-v_i)^2/a^2]\right\},\end{aligned} \tag{C2}$$

where the final result follows from

$$\int_{x=-\infty}^{x=\infty}\exp[-(ax)^2]\exp(j2\pi s_ix)\exp(-j2\pi sx)\,dx=\frac{\sqrt{\pi}}{a}\exp[-\pi^2(s-s_i)^2/a^2], \tag{C3}$$

which is the equation on the top of p. 108 of Bracewell (2000) generalized so that s_i can be variable instead of the fixed value of ½ as in Bracewell (2000).[C1] Applying $\lim_{a\to 0}$ to (C2) produces

$$\begin{aligned}\mathrm{FT}[e^{j2\pi(u_ix+v_iy)}]&=\int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty}\exp[j2\pi(u_ix+v_iy)]\exp[-j2\pi(ux+vy)]\,dx\,dy\\&=\lim_{a\to 0}\left\{\frac{\sqrt{\pi}}{a}\exp[-\pi^2(u-u_i)^2/a^2]\right\}\left\{\frac{\sqrt{\pi}}{a}\exp[-\pi^2(v-v_i)^2/a^2]\right\}=\delta(u-u_i)\delta(v-v_i),\end{aligned} \tag{C4}$$

where FT[ ] denotes the (direct) Fourier transform and the final relation results because $\lim_{a\to 0}(\sqrt{\pi}/a)\exp[-\pi^2(s-s_i)^2/a^2]$ is a defining sequence for $\delta(s - s_i)$ (Bracewell 2000, p. 108). This can be expressed, using the two-dimensional Dirac distribution ${}^2\delta(x, y)$ and the relation ${}^2\delta(x, y) = \delta(x)\delta(y)$ (Bracewell 2000, p. 89),[C2] as

$$\mathrm{FT}[e^{j2\pi(u_ix+v_iy)}] = {}^2\delta(u - u_i, v - v_i). \tag{C5}$$

The relation (C5) is the orthogonality relation for

[C1] This generalized form can be derived from the Fourier transform of a Gaussian function, $\int_{x=-\infty}^{x=\infty}\exp(-\pi x^2)\exp(-j2\pi sx)\,dx = \exp(-\pi s^2)$ (Bracewell 2000, p. 105), through a series of variable substitutions.

[C2] The two-dimensional Dirac distribution can be expressed as

$${}^2\delta(x,y)=\begin{cases}0 & x^2+y^2\neq 0\\ \infty & x^2+y^2=0\end{cases},$$

with the condition that $\int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty}{}^2\delta(x,y)f(x,y)\,dx\,dy=f(0,0)$ (Bracewell 2000, 89–90).

two-dimensional Fourier transforms. From it, Fourier transforms of two-dimensional sinusoids can be determined. To do so, consider that (C5) can be expressed, using Euler's formula, as

$$
\begin{aligned}
{}^2\delta(u - u_i, v - v_i) = &\int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} \cos[2\pi(u_i x + v_i y)]\cos[2\pi(ux + vy)]\,dx\,dy \\
&+ \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} \sin[2\pi(u_i x + v_i y)]\sin[2\pi(ux + vy)]\,dx\,dy \\
&+ j\int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} \sin[2\pi(u_i x + v_i y)]\cos[2\pi(ux + vy)]\,dx\,dy \\
&- j\int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} \cos[2\pi(u_i x + v_i y)]\sin[2\pi(ux + vy)]\,dx\,dy. \qquad \text{(C6)}
\end{aligned}
$$

Consider the third integral on the rhs of (C6). Using the trigonometric identities $\sin(a + b) = \sin a\cos b + \cos a\sin b$ and $\cos(a + b) = \cos a\cos b - \sin a\sin b$, its integrand can be expressed as

$$
\begin{aligned}
&\sin[2\pi(u_i x + v_i y)]\cos[2\pi(ux + vy)] \\
&\quad = \sin(2\pi u_i x)\cos(2\pi v_i y)\cos(2\pi ux)\cos(2\pi vy) \\
&\qquad + \cos(2\pi u_i x)\sin(2\pi v_i y)\cos(2\pi ux)\cos(2\pi vy) \\
&\qquad - \sin(2\pi u_i x)\cos(2\pi v_i y)\sin(2\pi ux)\sin(2\pi vy) \\
&\qquad - \cos(2\pi u_i x)\sin(2\pi v_i y)\sin(2\pi ux)\sin(2\pi vy). \qquad \text{(C7)}
\end{aligned}
$$

Because terms 1 and 4 on the rhs of (C7) are odd in x and terms 2 and 3 are odd in y, the third integral on the rhs of (C6) is zero. The same argument applies to the fourth integral on the rhs of (C6). Thus, one of the relations that results from (C5) is

$$
\int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} \sin[2\pi(u_i x + v_i y)]\cos[2\pi(ux + vy)]\,dx\,dy = 0, \qquad \text{(C8)}
$$

with (C6) simplifying to

$$
{}^2\delta(u - u_i, v - v_i) = \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} \cos[2\pi(u_i x + v_i y)]\cos[2\pi(ux + vy)]\,dx\,dy + \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} \sin[2\pi(u_i x + v_i y)]\sin[2\pi(ux + vy)]\,dx\,dy. \qquad \text{(C9)}
$$

Two other important relations are obtained from (C5) by splitting the Dirac distribution on the lhs of (C9) into two terms and by considering the resulting dependence of those two terms on the independent variables u and v. The Dirac distribution on the lhs of (C9) can be written as

$$
\begin{aligned}
&{}^2\delta(u - u_i, v - v_i) \\
&\quad = \overbrace{\frac{1}{2}\,{}^2\delta(u - u_i, v - v_i) + \frac{1}{2}\,{}^2\delta(u + u_i, v + v_i)}^{f_1(u,v)} \\
&\qquad \overbrace{- \frac{1}{2}\,{}^2\delta(u + u_i, v + v_i) + \frac{1}{2}\,{}^2\delta(u - u_i, v - v_i)}^{f_2(u,v)},
\end{aligned}
$$

with $f_1(u, v) = f_1(-u, -v)$ and $f_2(u, v) = -f_2(-u, -v)$. Because the first integral on the rhs of (C9) has the same dependence on u and v as $f_1(u, v)$ and the second integral has the same functional dependence as $f_2(u, v)$,

$$
\int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} \cos[2\pi(u_i x + v_i y)]\cos[2\pi(ux + vy)]\,dx\,dy = \frac{1}{2}\,{}^2\delta(u - u_i, v - v_i) + \frac{1}{2}\,{}^2\delta(u + u_i, v + v_i) \qquad \text{(C10)}
$$

and

$$
\int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} \sin[2\pi(u_i x + v_i y)]\sin[2\pi(ux + vy)]\,dx\,dy = \frac{1}{2}\,{}^2\delta(u - u_i, v - v_i) - \frac{1}{2}\,{}^2\delta(u + u_i, v + v_i). \qquad \text{(C11)}
$$

Given that the (direct) two-dimensional Fourier transform is given by

$$\mathrm{FT}[f(x, y)] = F(u, v) = \int_{y=-\infty}^{y=\infty} \int_{x=-\infty}^{x=\infty} f(x, y) \exp[-j2\pi(ux + vy)]\, dx\, dy, \tag{C12}$$

it is apparent from (C8) and (C10) that

$$\mathrm{FT}\{\cos[2\pi(u_i x + v_i y)]\} = \frac{1}{2}\,{}^2\delta(u - u_i, v - v_i) + \frac{1}{2}\,{}^2\delta(u + u_i, v + v_i) \tag{C13}$$

and, from (C8) and (C11), that

$$\mathrm{FT}\{\sin[2\pi(u_i x + v_i y)]\} = j\left[\frac{1}{2}\,{}^2\delta(u + u_i, v + v_i) - \frac{1}{2}\,{}^2\delta(u - u_i, v - v_i)\right]. \tag{C14}$$

These are the desired Fourier transforms of sinusoidal functions.

To facilitate subsequent analysis, it is noted that *any* two-dimensional sinusoid can be expressed in the form

$$f(x, y) = A \cos[2\pi(u_i x + v_i y) + \varphi], \tag{C15}$$

where A is the amplitude (defined here to be > 0), φ is the phase, and v_i is defined to be nonnegative. The cosine gives the wave the right shape, the u_i and v_i values provide the orientation and wavelength of the wave, and φ dictates the locations of troughs (or any phase value) relative to the origin.

By utilizing the identity $\cos(a + b) = \cos a \cos b - \sin a \sin b$, the Fourier transform of (C15) can be expressed as

$$\begin{aligned} F(u, v) = {} & A \cos\varphi \int_{y=-\infty}^{y=\infty} \int_{x=-\infty}^{x=\infty} \cos[2\pi(u_i x + v_i y)] \cos[2\pi(ux + vy)]\, dx\, dy \\ & - A \sin\varphi \int_{y=-\infty}^{y=\infty} \int_{x=-\infty}^{x=\infty} \sin[2\pi(u_i x + v_i y)] \cos[2\pi(ux + vy)]\, dx\, dy \\ & - A \cos\varphi j \int_{y=-\infty}^{y=\infty} \int_{x=-\infty}^{x=\infty} \cos[2\pi(u_i x + v_i y)] \sin[2\pi(ux + vy)]\, dx\, dy \\ & + A \sin\varphi j \int_{y=-\infty}^{y=\infty} \int_{x=-\infty}^{x=\infty} \sin[2\pi(u_i x + v_i y)] \sin[2\pi(ux + vy)]\, dx\, dy. \end{aligned} \tag{C16}$$

The second and third integrals on the rhs of (C16) are zero owing to (C8) while the first and last integrals on the rhs can be substituted for using (C10) and (C11), resulting in

$$F(u, v) = \frac{A}{2}\,{}^2\delta(u - u_i, v - v_i)[\cos\varphi + j \sin\varphi] + \frac{A}{2}\,{}^2\delta(u + u_i, v + v_i)[\cos\varphi - j \sin\varphi]. \tag{C17}$$

c. Magnitudes and phases of (direct) two-dimensional Fourier transforms

Attention is focused upon Fourier transforms of two-dimensional sinusoids since they contain Dirac distributions, which complicate the problem. The magnitudes and phases for functions $g(x, y)$ that have Fourier transforms in the traditional sense $G(u, v) = G_{\mathrm{Re}}(u, v) + jG_{\mathrm{Im}}(u, v)$ are given by $|G(u, v)| = [G_{\mathrm{Re}}(u, v)^2 + G_{\mathrm{Im}}(u, v)^2]^{1/2}$ and $\varphi_{G(u,v)} = \tan^{-1}[G_{\mathrm{Im}}(u, v)/G_{\mathrm{Re}}(u, v)] + 2\pi n$, where n is an integer.

The task is to determine the magnitude and phase of (C17), which is the Fourier transform of a two-dimensional sinusoid. From the definition of ${}^2\delta(x, y)$, (C17) is nonzero only when $u = u_i$, $v = v_i$ and $u = -u_i$, $v = -v_i$. The magnitude of (C17), therefore, is nonzero only for these conditions. The first condition produces a nonzero value in the first term on the rhs of (C17) $(A/2)\,{}^2\delta(u - u_i, v - v_i)\,[\cos\varphi + j \sin\varphi]$. Because this is the product of a real-valued distribution $(A/2)\,{}^2\delta(u - u_i, v - v_i)$ and a complex-valued function $\cos\varphi + j\sin\varphi$, its magnitude is the product of the magnitudes of these and is thus $|(A/2)\,{}^2\delta(u - u_i, v - v_i)|$. From (C4), the defining sequence for ${}^2\delta(u - u_i, v - v_i)$ is $\{(\sqrt{\pi}/a) \exp[-\pi^2(u - u_i)^2/a^2]\}\{(\sqrt{\pi}/a) \exp[-\pi^2(v - v_i)^2/a^2]\}$. Because $A > 0$ and the defining sequence for ${}^2\delta(u - u_i, v - v_i)$ is always greater than zero, $|(A/2)\,{}^2\delta(u - u_i, v - v_i)| = (A/2)\,{}^2\delta(u - u_i, v - v_i)$. A similar argument applies to the second term on the rhs of (C17), resulting in

$$|F(u,v)| = \begin{cases} \frac{A}{2}\,{}^2\delta(u-u_i, v-v_i) + \frac{A}{2}\,{}^2\delta(u+u_i, v+v_i) & u_i^2+v_i^2 \neq 0 \\ A^2\delta(u,v)|\cos\varphi| & u_i^2+v_i^2 = 0, \end{cases} \quad \text{(C18)}$$

with the $u_i^2 + v_i^2 = 0$ result arising from inserting $u_i = v_i = 0$ into (C17) and applying a similar line of reasoning.

Because (C17) is nonzero only when $u = u_i$, $v = v_i$ and $u = -u_i$, $v = -v_i$, its phase is indeterminate except at these points. The first condition produces a nonzero value in the first term on the rhs of (C17) $(A/2)\,{}^2\delta(u - u_i, v - v_i)[\cos\varphi + j\sin\varphi]$. Because this is the product of a real-valued distribution $(A/2)\,{}^2\delta(u - u_i, v - v_i)$ and a complex-valued function $\cos\varphi + j\sin\varphi$, its phase is the sum of the phases of $(A/2)\,{}^2\delta(u - u_i, v - v_i)$ and $\cos\varphi + j\sin\varphi$. At $(u = u_i, v = v_i)$, the phase of $(A/2)\,{}^2\delta(u - u_i, v - v_i)$ is zero because $A > 0$ and the defining sequence for ${}^2\delta(u - u_i, v - v_i)$ is always greater than zero. The phase of $(A/2)\,{}^2\delta(u - u_i, v - v_i)\,[\cos\varphi + j\sin\varphi]$ at $(u = u_i, v = v_i)$, then, is the phase of $\cos\varphi + j\sin\varphi$, which is φ. The phase of $(A/2)\,{}^2\delta(u + u_i, v + v_i)\,[\cos\varphi - j\sin\varphi]$ at $(u = -u_i, v = -v_i)$, by similar reasoning, is $-\varphi$. The phase of (C17) can thus be expressed as

$$\varphi_{F(u,v)} = \begin{cases} \varphi & u_i^2+v_i^2 \neq 0, \quad u = u_i, \quad v = v_i \\ -\varphi & u_i^2+v_i^2 \neq 0, \quad u = -u_i, \quad v = -v_i \\ 0 & u_i^2+v_i^2 = 0, \quad u = v = 0, \quad \cos\varphi > 0\,, \\ \pi & u_i^2+v_i^2 = 0, \quad u = v = 0, \quad \cos\varphi < 0 \\ \text{indeterminate} & \text{otherwise} \end{cases} \quad \text{(C19)}$$

where the $u_i^2 + v_i^2 = 0$ results arise from inserting $u_i = v_i = 0$ into (C17) and applying similar reasoning.

d. The half-plane magnitude and phase form of the inverse Fourier transform of two-dimensional, real data

Here the inverse Fourier transform a two-dimensional, real-valued function $f(x, y)$ is expressed in terms of magnitudes and phases. The starting point for this derivation is the inverse Fourier transform

$$f(x,y) = \int_{v=-\infty}^{v=\infty}\int_{u=-\infty}^{u=\infty} F(u,v)\exp[j2\pi(ux+vy)]\,du\,dv, \quad \text{(C20)}$$

where

$$F(u,v) = \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} f(x,y)\exp[-j2\pi(ux+vy)]\,dx\,dy = F_{\mathrm{Re}}(u,v) + jF_{\mathrm{Im}}(u,v). \quad \text{(C21)}$$

The Hermitian property described by Bracewell (2000, 13–14) for one-dimensional Fourier transforms extends to two-dimensional Fourier transforms. This means that for $f(x, y)$ real,

$$\begin{aligned} F_{\mathrm{Re}}(u,v) &= F_{\mathrm{Re}}(-u,-v), \\ F_{\mathrm{Re}}(-u,v) &= F_{\mathrm{Re}}(u,-v), \\ F_{\mathrm{Im}}(u,v) &= -F_{\mathrm{Im}}(-u,-v), \\ F_{\mathrm{Im}}(-u,v) &= -F_{\mathrm{Im}}(u,-v), \end{aligned} \quad \text{(C22)}$$

which are easily verified using (C21).

The integral in (C20) can be split to produce

$$\begin{aligned} f(x,y) = &\int_{v=0}^{v=\infty}\int_{u=-\infty}^{u=0} F(u,v)\exp[j2\pi(ux+vy)]\,du\,dv + \int_{v=0}^{v=\infty}\int_{u=0}^{u=\infty} F(u,v)\exp[j2\pi(ux+vy)]\,du\,dv \\ &+ \int_{v=-\infty}^{v=0}\int_{u=-\infty}^{u=0} F(u,v)\exp[j2\pi(ux+vy)]\,du\,dv + \int_{v=-\infty}^{v=0}\int_{u=0}^{u=\infty} F(u,v)\exp[j2\pi(ux+vy)]\,du\,dv. \end{aligned} \quad \text{(C23)}$$

By applying the change of variables $u = -s$, $v = -t$ to the third and fourth double integrals in (C4) and changing the directions of integration in the resultant integrals, the third double integral can be combined with the second and the fourth double integral can be combined with the first to produce

$$f(x, y) = \int_{v=0}^{v=\infty} \int_{u=0}^{u=\infty} F(u, v) \exp[j2\pi(ux + vy)] + F(-u, -v) \exp[-j2\pi(ux + vy)]\, du\, dv + \int_{v=0}^{v=\infty} \int_{u=-\infty}^{u=0} F(u, v) \exp[j2\pi(ux + vy)] + F(-u, -v) \exp[-j2\pi(ux + vy)]\, du\, dv, \quad \text{(C24)}$$

which can be written as

$$f(x, y) = \int_{v=0}^{v=\infty} \int_{u=-\infty}^{u=\infty} F(u, v) \exp[j2\pi(ux + vy)] + F(-u, -v) \exp[-j2\pi(ux + vy)]\, du\, dv. \quad \text{(C25)}$$

Moreover, $F(u, v)$ and $F(-u, -v)$ can be expressed in polar form as

$$F(u, v) = |F(u, v)| \exp[j\varphi_{F(u,v)}] \quad \text{(C26)}$$

and

$$F(-u, -v) = |F(u, v)| \exp[j\varphi_{F(-u,-v)}], \quad \text{(C27)}$$

where the fact that $|F(-u, -v)| = F(u, v)$ has been utilized.[C3] For functions that have Fourier transforms in the traditional sense, (C22) means that $\varphi_{F(-u,-v)} = \tan^{-1}[F_{\mathrm{Im}}(-u, -v)/F_{\mathrm{Re}}(-u, -v)] + 2\pi n = \tan^{-1}[-F_{\mathrm{Im}}(u, v)/F_{\mathrm{Re}}(u, v)] + 2\pi n$ and thus that $\varphi_{F(-u,-v)} = -\varphi_{F(u,v)}$. For two-dimensional sinusoids, (C19) provides the same result. Thus, (C27) can be written as

$$F(-u, -v) = |F(u, v)| \exp[-j\varphi_{F(u,v)}]. \quad \text{(C28)}$$

Inserting (C26) and (C28) into (C25) produces

$$f(x, y) = \int_{v=0}^{v=\infty} \int_{u=-\infty}^{u=\infty} |F(u, v)|(\exp\{j[2\pi(ux + vy) + \varphi_{F(u,v)}]\} + \exp\{-j[2\pi(ux + vy) + \varphi_{F(u,v)}]\})\, du\, dv. \quad \text{(C29)}$$

Since $e^{j\theta} + e^{-j\theta} = 2\cos\theta$, this simplifies to

$$f(x, y) = \int_{v=0}^{v=\infty} \int_{u=-\infty}^{u=\infty} 2|F(u, v)| \cos[2\pi(ux + vy) + \varphi_{F(u,v)}]\, du\, dv. \quad \text{(C30)}$$

As in the one-dimensional case, an adjustment to (C30) is required. However, in contrast to the one-dimensional case where the adjustment was required only at a point, the adjustment here is required along the line $v = 0$, along which the inverse Fourier transform (C20) was "folded over." To illustrate this, $h(x, y)$ in (C1) is reexpressed in the form $h(x, y) = C + s_1(x) + s_2(x, y)$, where $s_1(x)$ varies only in x and thus, from (C15), $v_i = 0$. It is noted that $s_1(x)$ could be composed of a sum of sinusoids having the form (C15), but for simplicity is considered in the following to be composed of only one sinusoid (this does not alter the validity of the result). From (C21), the Fourier transform of a function in this function class $f(x, y) = g(x, y) + C + s_1(x) + s_2(x, y)$ is

$$F(u, v) = G(u, v) + C^2\delta(u, v) + \frac{A}{2}\,{}^2\delta(u - u_i, v) \times [\cos\varphi + j\sin\varphi] + \frac{A}{2}\,{}^2\delta(u + u_i, v) \times [\cos\varphi - j\sin\varphi] + S_2(u, v), \quad \text{(C31)}$$

which follows from (C17). Inserting (C31) into (C20) produces the original function $f(x, y) = g(x, y) + C + s_1(x) + s_2(x, y)$, where the sifting property of ${}^2\delta(x, y)$, $\int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} {}^2\delta(x - a, y - b) f(x, y)\, dx\, dy = f(a, b)$, has been applied.[C4] However, insertion of (C31) into (C30)

[C3] For functions that have Fourier transforms in the traditional sense, $|F(-u, -v)| = |F(u, v)|$ can be understood by using (C22) and the relation $|F(u, v)| = [F_{\mathrm{Re}}(u, v)^2 + F_{\mathrm{Im}}(u, v)^2]^{1/2}$. For two-dimensional sinusoidal functions, $|F(-u, -v)| = |F(u, v)|$ is apparent from (C18).

[C4] Note that $\int_{v=-\infty}^{v=\infty}\int_{u=-\infty}^{u=\infty} \{(A/2)\,{}^2\delta(u - u_i, v)[\cos\varphi + j\sin\varphi] + (A/2)\,{}^2\delta(u + u_i, v)[\cos\varphi - j\sin\varphi]\} \exp[j2\pi(ux + vy)]\, du\, dv = (A/2)\, e^{j(2\pi u_i x + \varphi)} + (A/2)\, e^{-j(2\pi u_i x + \varphi)} = A\cos(2\pi u_i x + \varphi)$.

produces $f(x, y) = g(x, y) + 2C + 2s_1(x) + s_2(x, y)$. To see this, consider that (C30) can be applied to each of the functions in (C31) because the steps that produced (C30) from (C20) can be replicated for each of the functions in (C31). From the reasoning applied in the derivation of (C18), insertion of $C^2\delta(u, v)$ into (C30) results in $2C$. Similarly, insertion of $(A/2)\,^2\delta(u - u_i, v)$ $[\cos\varphi + j \sin\varphi]$ into (C30) results in $A\cos(2\pi u_i x + \varphi)$, where (C19) has been applied. Insertion of $(A/2)\,^2\delta(u + u_i, v)$ $[\cos\varphi - j \sin\varphi]$ into (C30) results in $A\cos(-2\pi u_i x - \varphi) = A\cos(2\pi u_i x + \varphi)$, where again (C19) has been applied. Thus, the insertion of $(A/2)\,^2\delta(u - u_i, v)$ $[\cos\varphi + j \sin\varphi]$ and $(A/2)\,^2\delta(u + u_i, v)$ $[\cos\varphi - j \sin\varphi]$ into (C30) results in $2A\cos(2\pi u_i x + \varphi) = 2s_1(x)$. From (C18) and (C19), insertion of $S_2(u, v)$ into (C30) produces $s_2(x, y)$ because from (C15) v_i is restricted to be nonnegative and thus only the $(u = u_i, v = v_i)$ values from (C18) and (C19) contribute to the inverse Fourier transform. Thus, when C and $s_1(x)$ are not zero, the inverse Fourier transform (C20) produces the original function $f(x, y) = g(x, y) + C + s_1(x) + s_2(x, y)$ whereas the amplitude and phase form of the inverse Fourier transform (C30) produces the incorrect result $f(x, y) = g(x, y) + 2C + 2s_1(x) + s_2(x, y)$.

An adjusted form of (C30) that produces results consistent with (C20) is

$$f(x, y) = \int_{v=0}^{v=\infty} \int_{u=-\infty}^{u=\infty} \frac{2}{1 + \delta^0(v)} |F(u, v)| \cos[2\pi(ux + vy) + \varphi_{F(u,v)}]\, du\, dv, \qquad \text{(C32)}$$

where $\delta^0(x)$ is defined as in Bracewell (2000, p. 87)

$$\delta^0(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}. \qquad \text{(C33)}$$

The fundamental result of this analysis is that the real function $f(x, y)$ can be expressed in terms of the magnitudes $|F(u, v)|$ and phases $\varphi_{F(u,v)}$ of the half-plane $v \geq 0$ Fourier transform.

REFERENCES

Achtemeier, G. L., 1986: The impact of data boundaries upon a successive corrections objective analysis of limited-area datasets. *Mon. Wea. Rev.,* **114,** 40–49.

Askelson, M. A., J.-P. Aubagnac, and J. M. Straka, 2000: An adaptation of the Barnes filter applied to the objective analysis of radar data. *Mon. Wea. Rev.,* **128,** 3050–3082.

——, P. M. Pauley, and J. M. Straka, 2005: Response functions for arbitrary weight functions and data distributions. Part II: Response function derivation and verification. *Mon. Wea. Rev.,* **133,** 2132–2147.

Bracewell, R. N., 2000: *The Fourier Transform and Its Applications*. 3d ed. McGraw Hill Series in Electrical and Computer Engineering, McGraw Hill, 616 pp.

Buzzi, A., D. Gomis, M. A. Pedder, and S. Alonso, 1991: A method to reduce the adverse impact that inhomogeneous station distributions have on spatial interpolation. *Mon. Wea. Rev.,* **119,** 2465–2491.

Caracena, F., S. L. Barnes, and C. A. Doswell III, 1984: Weighting function parameters for objective interpolation of meteorological data. Preprints, *10th Conf. on Weather Forecasting and Analysis*, Clearwater Beach, FL, Amer. Meteor. Soc., 109–116.

Carr, F. H., P. L. Spencer, C. A. Doswell III, and J. D. Powell, 1995: A comparison of two objective analysis techniques for profiler time–height data. *Mon. Wea. Rev.,* **123,** 2165–2180.

Cochran, J. A., H. C. Wiser, and B. J. Rice, 1987: *Advanced Engineering Mathematics*. 2d ed. Brooks/Cole, 659 pp.

Jackson, J. D., 1975: *Classical Electrodynamics*. Wiley & Sons, 848 pp.

Jones, D. S., 1982: *The Theory of Generalised Functions*. 2d ed. Cambridge University Press, 539 pp.

Mikusiński, J., 1966: On the square of the Dirac delta-distribution. *Bull. Acad. Pol. Sci., Ser. Sci. Math. Astron. Phys.,* **14,** 511–513.

Pauley, P. M., 1990: On the evaluation of boundary errors in the Barnes objective analysis scheme. *Mon. Wea. Rev.,* **118,** 1203–1210.

——, and X. Wu, 1990: The theoretical, discrete, and actual response of the Barnes objective analysis scheme for one- and two-dimensional fields. *Mon. Wea. Rev.,* **118,** 1145–1163.

Weaver, H. J., 1983: *Applications of Discrete and Continuous Fourier Analysis*. Wiley & Sons, 375 pp.