Unification of the Anelastic and Quasi-Hydrostatic Systems of Equations

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ABSTRACT

A system of equations is presented that unifies the nonhydrostatic anelastic system and the quasi-hydrostatic compressible system for use in global cloud-resolving models. By using a properly defined quasi-hydrostatic density in the continuity equation, the system is fully compressible for quasi-hydrostatic motion and anelastic for purely nonhydrostatic motion. In this way, the system can cover a wide range of horizontal scales from turbulence to planetary waves while filtering vertically propagating sound waves of all scales. The continuity equation is primarily diagnostic because the time derivative of density is calculated from the thermodynamic (and surface pressure tendency) equations as a correction to the anelastic continuity equation. No reference state is used and no approximations are made in the momentum and thermodynamic equations. An equation that governs the time change of total energy is also derived. Normal-mode analysis on an *f* plane without the quasigeostrophic approximation and on a midlatitude β plane with the quasigeostrophic approximation is performed to compare the unified system with other systems. It is shown that the unified system reduces the westward retrogression speed of the ultra-long barotropic Rossby waves through the inclusion of horizontal divergence due to compressibility.

1. Introduction

The nonhydrostatic anelastic system of equations (called the anelastic system in this paper) is widely used in theoretical and numerical studies of small-scale nonacoustic motions, such as turbulence and convection, while most large-scale models use the compressible quasi-hydrostatic system (the "primitive equations," called the quasi-hydrostatic system in this paper) as the dynamics core. Both of these systems filter vertically propagating sound waves, but they do so in quite different ways.

In the anelastic system (e.g., Ogura and Phillips 1962; Dutton and Fichtl 1969; Wilhemson and Ogura 1972; Lipps and Hemler 1982; Bannon 1996), the deviations of thermodynamic variables from a horizontally uniform reference state are assumed to be small, and the local time derivative of density is neglected in the continuity equation to filter the acoustic waves. Thus, the original continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \qquad (1.1)$$

is replaced by

$$\nabla \cdot (\rho_0 \mathbf{V}) = 0, \qquad (1.2)$$

where ρ is the density, ∇ is the three-dimensional del operator, $\mathbf{V}$ is the three-dimensional velocity, and the subscript zero denotes a reference state that varies only vertically. To maintain the internal consistency of the system from the point of view of scale analysis and/or energetics, either the momentum or thermodynamic equation is usually modified. For example, Ogura and Phillips (1962) chose an isentropic atmosphere as the reference state, while Lipps and Hemler (1982) assumed that the reference-state potential temperature is a slowly varying function of the vertical coordinate. The pressure gradient force in the momentum equation is then approximated to maintain the consistency. Bannon (1996), on the other hand, maintains the consistency by

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modifying the thermodynamic equation introducing the concept of the "dynamic entropy." By "the anelastic system," we mean the Lipps–Hemler system in the rest of this paper unless otherwise noted.

We note, however, that the use of the anelastic continuity equation in (1.2) is more than needed to filter acoustic waves. Using the equation of state and the definition of potential temperature θ, we find

$$\frac{\partial \rho}{\partial t} = \frac{\rho}{\gamma p}\left(\frac{\partial p}{\partial t}\right)_\theta - \frac{\rho}{\theta}\left(\frac{\partial \theta}{\partial t}\right)_p, \tag{1.3}$$

where p is the pressure, $\gamma \equiv c_p/c_v$, and c_p and c_v are the specific heat at constant pressure and volume, respectively. The first and second terms on the right-hand side of (1.3) represent the local effects of isentropic compressibility and isobaric entropy change, respectively. The anelastic continuity equation neglects both of these effects. Durran (1989) showed that the inclusion of a linearized effect of the second term yields

$$\nabla \cdot (\bar{\rho}\bar{\theta}\mathbf{V}) = \frac{\bar{\rho}Q}{c_p \bar{\pi}}, \tag{1.4}$$

where the overbar denotes the horizontal mean state, which may have an arbitrary vertical structure, π is the Exner function given by $(p/p_{00})^\kappa$, p_{00} is a constant reference pressure, $\kappa \equiv R/c_p = 1 - 1/\gamma$, R is the gas constant, and Q is the heating rate per unit mass. Acoustic waves are still filtered because the first term on the right-hand side of (1.3) is neglected. Durran (1989) called (1.4) "the pseudo-incompressible equation." Nance and Durran (1994) showed that (1.4) becomes increasingly accurate as the flow becomes more nonhydrostatic. Durran (2008) further showed that the pseudo-incompressible equation is accurate if the Mach number is smaller than the Rossby number or 1, whichever is smaller. Durran and Arakawa (2007) showed that when (1.4) is used, energy is conserved with no modifications of the momentum and thermodynamic equations except for linearization. Durran (2008) presented further discussions of the pseudo-incompressible system and its generalizations.

The merit of using the anelastic or pseudo-incompressible approximation for small-scale motions is well recognized. Although our experience is rather limited, these approximations seem to hold well for most large-scale atmospheric motions as well. Nance and Durran (1994) pointed out that the errors incurred by using both the anelastic and pseudo-incompressible systems could be significantly less than the errors generated by the numerical methods. By analyzing the results of an anelastic model applied to the global domain, Smolarkiewicz et al. (2001) further pointed out, "the differences due to the higher-order truncation errors of legitimate modes of executing contemporary global models overwhelm the differences due to analytic formulation of the governing equations." Through this analysis, they conclude that nonhydrostatic anelastic models derived from small-scale codes adequately capture a broad range of planetary flows. Smolarkiewicz and Dörnbrack (2008) presented integrations of the anelastic and pseudo-incompressible systems applied to baroclinic development in the midlatitudes.

There are, however, conflicting views. Based on normal-mode analyses of fully compressible, pseudo-incompressible, anelastic and quasi-hydrostatic systems of equations applied to an f plane, Davies et al. (2003) concluded, "whilst of key importance for small-scale and process modeling, the anelastic equations are not recommended for either operational numerical prediction or climate simulation at any scale." They also pointed out that the pseudo-incompressible system appears to be viable for numerical weather prediction, but only at short horizontal scales.

A potentially more serious problem appears with the β effect when an anelastic model is applied to a hemispheric or global domain. With the anelastic continuity equation in (1.2), horizontal motions must inevitably be horizontally nondivergent. The situation is the same with the pseudo-incompressible equation in (1.4) without heating because the mean state is horizontally uniform. Then, as far as the barotropic mode with a fixed upper boundary is concerned, we are essentially back to the problem recognized during the early years of NWP. Wolff (1958) showed that forecast errors with a hemispheric nondivergent barotropic model are dominated by spuriously fast westward retrogression of ultra-long waves. This is anticipated from the retrogression speed of the nondivergent Rossby wave given by β/k^2 (Rossby et al. 1939), which unlimitedly increases as k decreases. Here β is the meridional gradient of the Coriolis parameter f and k is the zonal wavenumber. Rossby et al. (1939) pointed out, however, that conservation of the absolute *potential* vorticity, $(f + \zeta)/h$, instead of conservation of the absolute vorticity, $f + \zeta$, gives slower retrogression speeds. Here ζ is the vertical component of vorticity and h is the height of the interface between the lower dynamically active homogeneous layer and the upper dynamically inactive homogeneous layer. Based on this and the work by Bolin (1956), who used the height of tropopause for h, Cressman (1958) succeeded to reduce the errors in actual forecasts by introducing a correction term in the vorticity equation to represent "barotropic divergence." Wiin-Nielsen (1959) pointed out that the problem exists also for the barotropic mode in tropospheric baroclinic models. He noted that Cressman's choice of h is rather ambiguous and interpreted the required divergence term as a result of vertically varying static stability. It is difficult to see, however, how the vertical variation of static stability influences the barotropic (or external) Rossby wave. Wedi and Smolarkiewicz (2004, 2006), on the other hand, introduced divergence of the vertically integrated motion into their anelastic model by making the model top variable in space and time. In the present paper we point out that even purely horizontal motion can be divergent with compressibility so that its potential vorticity is given by $(f + \zeta)/\rho$, where the denominator represents the effect of compressibility on the change of $f + \zeta$.

Most large-scale models of the atmosphere use the quasi-hydrostatic system of equations (the primitive equations) instead of the anelastic system. In the quasi-hydrostatic system, the vertical component of the momentum equation is replaced by the hydrostatic equation. This filters vertically propagating sound waves, but it is done in a totally different way from the anelastic system. The quasi-hydrostatic system uses no approximation in the continuity equation and, therefore, compressibility is fully included as far as quasi-hydrostatic motions are concerned. To our knowledge, catastrophic errors for ultra-long waves such as those observed by Wolff (1958) with the nondivergent barotropic model have not been reported with the primitive equation models. We can think of various reasons for this. For example, the purely barotropic mode may not be a significant component in such models when they are applied to realistic situations. It is also possible that the improved treatment of planetary-scale topography commonly used in those models might have hidden the problem. In our point of view, however, the effect of compressibility on those waves included in the primitive equation models is at least one of the possible causes for the success of those models in predicting ultra-long waves.

It is well known that the quasi-hydrostatic approximation breaks down for motions with horizontal scales of the order of 10 km or less. There have been attempts to overcome this deficiency by including approximate nonhydrostatic effects without introducing vertically propagating sound waves. One of the earliest attempts along this line is the approach proposed by Miller (1974) (see also Miller and Pearce 1974; Miller and White 1984; White 1989), which uses the pressure as the vertical coordinate and the approximation $Dw/Dt \approx D(-\omega/\rho_s g)/Dt$ in the vertical component of the momentum equation. Here D/Dt is the material time derivative, w is the vertical velocity, $\omega \equiv Dp/Dt$, ρ_s is the density of the reference state, and g is the gravitational acceleration. They used the standard form of the quasi-hydrostatic continuity equation with the p coordinate without introducing the anelastic approximation. This is also an approximation since p in their system is not necessarily hydrostatic. Miller (1974) showed that, when viewed with the z coordinate, this approximation is equivalent to the use of the hydrostatic equation for the time derivative of density in the continuity equation in (1.1).

In the approach proposed by Laprise (1992), on the other hand, the hydrostatic pressure is used as the vertical coordinate. Despite the use of the hydrostatic pressure for the vertical coordinate, no approximation is used in the momentum and continuity equations and, therefore, the system is nonhydrostatic and fully compressible. In his "alternative approach," w is calculated using $w \approx Dz/Dt$. Thus, the vertical component of the momentum equation is not used as a prognostic equation for w. Instead, it is used as a diagnostic equation that determines the vertical gradient of the total pressure from known Dw/Dt. Bubnová et al. (1995) emphasized the merit of Laprise's approach saying "...all the big investments that have been put into developing complex environments for primitive equation models can be used with profit to do nonhydrostatic research experiments and, in some future, operational forecasts." Janjic et al. (2001) and Janjic (2003) extended Laprise's alternative approach to the case of a sigma coordinate based on the hydrostatic pressure. They also emphasized the advantage of Laprise's approach because the nonhydrostatic dynamics is introduced as an add-on module without interfering with the favorable features of the hydrostatic formulation.

The main thrust of this paper is to develop a system of dynamics equations that maintains close ties with *both* the primitive equation models for large scales and the anelastic (and Boussinesq) models for small scales, for each of which we have generations of valuable experience, while filtering vertically propagating sound waves of all scales. An obvious alternative to this approach is to use a fully compressible model with the split-explicit approach (Klemp and Wilhemson 1978; Skamarock and Klemp 1992, 1994; Klemp et al. 2007) or a semi-implicit scheme (e.g., Tanguay et al. 1990; Cullen et al. 1997; Côté et al. 1998). For a concise review of these methods, see Steppeler et al. (2003). In the approach presented in this paper, on the other hand, the vertically propagating sound waves are eliminated at their origin so that our effort in improving computational aspects can be more focused on motions of our interest.

The essence of the unified system presented in this paper is in the use of the continuity equation:

$$\frac{\partial \rho_{qs}}{\partial t} + \mathbf{\nabla} \cdot (\rho_{qs}\mathbf{V}) = 0, \qquad (1.5)$$

where ρ_{qs} is the quasi-hydrostatic density. This equation is a straightforward generalization of the anelastic continuity equation in (1.2) and the pseudo-incompressible equation in (1.4) and, when applied to quasi-hydrostatic motions, it includes both terms in the right-hand side of (1.3). The use of (1.5) obviously requires

$$\frac{\delta\rho}{\rho_{qs}} \ll 1, \tag{1.6}$$

where $\delta\rho \equiv \rho - \rho_{qs}$ is the nonhydrostatic density. This assumption should be better justifiable than the corresponding assumption commonly used in the anelastic systems because $\delta\rho$ is the deviation of ρ from the local quasi-hydrostatic value rather than the value of a prescribed reference state that varies only vertically. This point is especially important when the system is applied to a large horizontal domain such as the entire globe. The assumption in (1.6) alone, however, does not automatically justify the use of (1.5) because the time derivative of $\delta\rho$ cannot be neglected for vertically propagating sound waves because of their high frequencies. The unified system filters these waves through omitting the $\partial\delta\rho/\partial t$ term in (1.5). Since the $\partial\rho_{qs}/\partial t$ term is retained, this equation may still appear to be prognostic, but actually it is not, because ρ_{qs} is predicted not by this equation, but by the thermodynamic (and surface pressure tendency) equations.

The paper is organized as follows. Section 2 presents the definitions of the quasi-hydrostatic pressure and density and the equations for their predictions, including the condition on the time change of the quasi-hydrostatic pressure at the model top, and section 3 presents the dynamics of the unified system including the problem of determining nonhydrostatic pressure. Section 3 also presents an equation that governs the time change of total energy. Section 4 gives a computational procedure of the unified system that can be followed when the height coordinate is used. Section 5 presents the unified system when the quasi-hydrostatic pressure is used as the vertical coordinate. For the purpose of comparing the unified system with other systems, section 6 discusses small-amplitude perturbations on a resting horizontally uniform atmosphere in view of the dispersion relation and vertical structure of the normal modes on an f plane. The analysis is then extended to the midlatitude β plane with the quasi-geostrophic approximation. Section 7 presents a summary and further discussions. The form of energy conserved in this system is presented in appendix A. A version of the unified system based on the vector vorticity equation instead of the momentum equation is presented in appendix B.

2. Quasi-hydrostatic pressure

In this section we define quasi-hydrostatic values of pressure and density and then discuss how those values are predicted in the unified system. Throughout this paper, the virtual temperature effect is neglected for simplicity. Using the equation of state $p = \rho RT$ and the definition of potential temperature $\theta \equiv T/\pi$, we may write the momentum equation as

$$\frac{D\mathbf{V}}{Dt} = -2\mathbf{\Omega} \times \mathbf{V} - c_p\theta\nabla\pi - \mathbf{k}g + \mathbf{F}. \tag{2.1}$$

Here $\mathbf{\Omega}$ is the earth's angular velocity and $\mathbf{k}$ is the vertical unit vector. Replacing the vertical component of (2.1) by the hydrostatic equation, we *define* the vertical derivative of π_{qs} for a given vertical structure of θ by

$$\frac{\partial\pi_{qs}}{\partial z} \equiv -\frac{g}{c_p\theta}. \tag{2.2}$$

Integrating (2.2) with respect to z, we obtain

$$\pi_{qs} = (\pi_{qs})_S - \int_{z_S}^{z} \frac{g}{c_p\theta}\,dz, \tag{2.3}$$

where the subscript S denotes the earth's surface. Replacing π in $p = p_{00}\pi^{1/\kappa}$ by π_{qs}, we define p_{qs} by

$$p_{qs} \equiv p_{00}\pi_{qs}^{1/\kappa} \tag{2.4}$$

and ρ_{qs} by

$$\frac{\partial p_{qs}}{\partial z} \equiv -\rho_{qs}g. \tag{2.5}$$

These quasi-hydrostatic values do not necessarily represent a reference state because the vertical distribution of θ in (2.3) is arbitrary and, therefore, p_{qs} and ρ_{qs} do not necessarily have characteristic vertical structures. From the definitions of the quasi-hydrostatic state given by (2.2), (2.4), and (2.5), we find

$$\rho_{qs} = \frac{p_{qs}}{R\,\pi_{qs}\theta}\left(= \frac{p_{00}^{\kappa}}{R\,\theta}p_{qs}^{1-\kappa}\right). \tag{2.6}$$

This equation and the assumption in (1.6) implies that we are also assuming

$$\frac{\delta p}{p_{qs}} \ll 1 \tag{2.7}$$

though it is not formally used in the equations given in the text.

Equation (2.3) shows that $(\pi_{qs})_S$ and θ must be predicted to determine the time evolution of the quasi-hydrostatic state. To predict θ, we use the following thermodynamic equation:

$$\frac{D}{Dt}\ln\theta = \frac{Q}{c_p T}. \qquad (2.8)$$

To predict $(\pi_{qs})_S$, we apply the time derivative of (2.3) to $z = z_T$ to obtain

$$\frac{\partial}{\partial t}(\pi_{qs})_S = \frac{\partial}{\partial t}(\pi_{qs})_T - \int_{z_S}^{z_T} \frac{g}{c_p\theta^2}\frac{\partial\theta}{\partial t}\,dz, \qquad (2.9)$$

where the subscript T denotes the model top. Here z_T (as well as z_S) is assumed to be constant in time. The first term on the right-hand side, however, remains to be determined because $\partial(\pi_{qs})_T/\partial t = 0$ is not a correct condition at the model top even when $z_T \to \infty$. To see this, let us consider a small perturbation denoted by a prime on a horizontally uniform basic state denoted by an overbar. The perturbation part of (2.2) is given by

$$\frac{\partial\pi'_{qs}}{\partial z} \approx \frac{g}{c_p\bar{\theta}^2}\,\theta'. \qquad (2.10)$$

If $\theta' = 0$ (i.e., barotropic) at all heights, (2.10) shows that π'_{qs} is constant throughout the entire vertical column. The assumption of $(\pi'_{qs})_T = 0$ thus means $\pi'_{qs} = 0$ at all heights. Barotropic modes are then eliminated. Since $\pi'_{qs} \approx \kappa p'_{qs}(\bar{\pi}_{qs}/\bar{p}_{qs}) = (\kappa/p_{00}^{\kappa})(p'_{qs}/\bar{p}_{qs}^{1-\kappa})$, constant π'_{qs} means that p'_{qs} decreases in height as $\bar{p}_{qs}^{1-\kappa}\,(=\bar{p}_{qs}{}^{1/\gamma})$ does. This is consistent with the fact that free quasi-hydrostatic oscillations in an isothermal atmosphere, such as the Lamb wave (modified by rotation) and the barotropic Rossby wave (modified by compressibility), have the equivalent depth γH, where H is the scale height (e.g., Siebert 1961).

We see that (2.9) can be closed if we consider the mass budget of the entire vertical column. For this purpose, we rewrite the continuity equation in (1.5) as

$$\frac{\partial\rho_{qs}}{\partial t} = -\nabla\cdot(\rho_{qs}\mathbf{V}_H) - \frac{\partial}{\partial z}(\rho_{qs}w). \qquad (2.11)$$

Hereinafter the subscript H denotes the horizontal component and w is the vertical velocity. Integrating the time derivative of (2.5) with respect to z from z_S to z_T, using (2.11) and $w_S = \mathbf{V}_H\cdot\nabla z_S$, and assuming $z_T =$ const. in space as well as in time, we obtain the surface-pressure tendency equation given by

$$\frac{\partial}{\partial t}(p_{qs})_S = \frac{\partial}{\partial t}(p_{qs})_T - g\nabla_H\cdot\int_{z_S}^{z_T}\rho_{qs}\mathbf{V}_H\,dz. \qquad (2.12)$$

Though not fully justifiable, we have neglected $(\rho_{qs}w)_T$ in deriving (2.12) as is done in many models.

Both (2.12) and (2.9) are obtained through vertically integrating the time derivative of the hydrostatic equation. Their physical meanings are different, however, because (2.12) relates the integral to mass budget while (2.9) relates it to thermodynamics. Naturally they must be consistent in view of (2.4). The time derivative of (2.4) applied to the earth's surface and the model top are given by

$$\frac{\partial}{\partial t}(p_{qs})_S = \frac{1}{\kappa}\left(\frac{p_{qs}}{\pi_{qs}}\right)_S\frac{\partial}{\partial t}(\pi_{qs})_S \qquad (2.13)$$

and

$$\frac{\partial}{\partial t}(p_{qs})_T = \frac{1}{\kappa}\left(\frac{p_{qs}}{\pi_{qs}}\right)_T\frac{\partial}{\partial t}(\pi_{qs})_T, \qquad (2.14)$$

respectively. Substituting (2.13) and (2.14) into (2.12) and eliminating $\partial(\pi_{qs})_S/\partial t$ using (2.9), we obtain

$$\frac{\partial}{\partial t}(\pi_{qs})_T = \frac{1}{(p_{qs}/\pi_{qs})_S - (p_{qs}/\pi_{qs})_T} \times\left[(p_{qs}/\pi_{qs})_S\int_{z_S}^{z_T}\frac{g}{c_p\theta^2}\frac{\partial\theta}{\partial t}dz - \kappa g\nabla_H\cdot\int_{z_S}^{z_T}\rho_{qs}\mathbf{V}_H\,dz\right]. \qquad (2.15)$$

Since the two terms in the brackets do not necessarily cancel, $\partial(\pi_{qs})_T/\partial t$ is generally finite even when $(p_{qs})_T = 0$. Since $p_{qs}/\pi_{qs} = p_{00}^{\kappa}p_{qs}^{1-\kappa} \to 0$ as $p_{qs} \to 0$, however, (2.14) shows that $\partial(p_{qs})_T/\partial t = 0$ holds when $(p_{qs})_T = 0$, as expected, but not when $(p_{qs})_T \neq 0$. With (2.15), (2.9) is closed and may be rewritten as

$$\frac{\partial}{\partial t}(\pi_{qs})_S = \frac{1}{(p_{qs}/\pi_{qs})_S - (p_{qs}/\pi_{qs})_T} \times\left[(p_{qs}/\pi_{qs})_T\int_{z_S}^{z_T}\frac{g}{c_p\theta^2}\frac{\partial\theta}{\partial t}dz - \kappa g\nabla_H\cdot\int_{z_S}^{z_T}\rho_{qs}\mathbf{V}_H\,dz\right]. \qquad (2.16)$$

3. Dynamics of the unified system and determination of the nonhydrostatic pressure

In this section we discuss the dynamics equations of the unified system, in which the continuity equation takes the form of (1.5). Since the unified system is a generalization of the anelastic system, the procedure is parallel to that of the anelastic system except that the continuity equation is exact for quasi-hydrostatic motion. When the momentum equation is used as the basic dynamical equation (instead of the vector vorticity equation as discussed in appendix B), the predicted three-dimensional velocity must satisfy the continuity equation. Thus, in parallel to the anelastic system, an elliptic equation must be solved for the nonhydrostatic pressure.

Using $D/Dt = \partial/\partial t + \mathbf{V} \cdot \nabla$ and (2.2), we rewrite the momentum equation in (2.1) as

$$\frac{\partial \mathbf{V}}{\partial t} = -(\mathbf{V} \cdot \nabla)\mathbf{V} - 2\boldsymbol{\Omega} \times \mathbf{V} - c_p\theta(\nabla_{\mathrm{H}}\,\pi_{qs} + \nabla\delta\pi) + \mathbf{F}, \tag{3.1}$$

where $\delta\pi \equiv \pi - \pi_{qs}$. Combining (3.1) with the continuity equation in (1.5), we obtain

$$\frac{\partial}{\partial t}(\rho_{qs}\mathbf{V}) = -\nabla \cdot (\rho_{qs}\mathbf{V}\mathbf{V}) - 2\boldsymbol{\Omega} \times \rho_{qs}\mathbf{V} - c_p\rho_{qs}\theta(\nabla_H\,\pi_{qs} + \nabla\delta\pi) + \rho_{qs}\mathbf{F}, \tag{3.2}$$

where $\nabla \cdot (\rho_{qs}\mathbf{V}\mathbf{V})$ is the divergence of the dyadic tensor $\rho_{qs}\mathbf{V}\mathbf{V}$. When υ_i is the ith component of $\mathbf{V}$ in the Cartesian coordinates $(x_1,\ x_2,\ x_3)$, the ith component of $\nabla \cdot (\rho_{qs}\mathbf{V}\mathbf{V})$ can be written as

$$[\nabla \cdot (\rho_{qs}\mathbf{V}\mathbf{V})]_i = \frac{\partial}{\partial x_j}(\rho_{qs}\upsilon_j\upsilon_i). \tag{3.3}$$

Taking the divergence of (3.2) and using the continuity equation (1.5) again, we obtain

$$\nabla \cdot (c_p\rho_{qs}\theta\nabla\delta\pi) = -\nabla \cdot [\nabla \cdot (\rho_{qs}\mathbf{V}\mathbf{V}) + 2\boldsymbol{\Omega}\ \rho_{qs}\mathbf{V} + c_p\rho_{qs}\theta\nabla_H\pi_{qs} - \rho_{qs}\mathbf{F}] + \frac{\partial^2\rho_{qs}}{\partial t^2}. \tag{3.4}$$

This is an elliptic equation to determine $\delta\pi$. In the anelastic models, a similar elliptic equation is solved, but usually for the deviation of pressure from a horizontally uniform hydrostatic state. In the unified system, on the other hand, (3.4) governs the deviation of pressure from the local quasi-hydrostatic pressure. Another important difference from the anelastic system is the existence of the last term in (3.4), which originates from the time derivative term in the continuity equation in (1.5). It thus represents a correction to the anelastic system. An expression for this term in the time-discrete case is presented in section 4.

Equation (3.4) requires boundary conditions. For vertical boundary conditions, it is a common practice in the anelastic models to use the vertical derivative of pressure obtained from the vertical component of the momentum equation applied to the upper and lower boundaries. We can do the same in the unified system, but in this way the spatially constant part of $\delta\pi$ cannot be determined. While this constant part does not matter for dynamics (Ogura and Charney 1962), it does matter for cloud microphysics as Schlesinger (1975) pointed out. Bannon et al. (2006) showed that this ambiguity could be removed by requiring total mass conservation. On the other hand, P. Smolarkiewicz (2008, personal communication) suggests using this freedom to conserve energy. Unless we enforce such kind of constraint on the constant part, the time sequence of $\delta\pi$ diagnostically determined at individual time steps may not be physical.

Recall that only the spatially varying part of $\delta\pi$ matters for dynamics and, therefore, only that part needs to be constrained for filtering vertically propagating sound waves. Then, if we are concerned with the most general filtered system, the spatially constant part of $\delta\pi$ should be predicted as is done (or effectively done) in a fully compressible nonhydrostatic model. Here we show that, by predicting the spatially constant part of $\delta\pi$, the system can conserve a properly defined energy. Appendix A derives the following equation from the equations of the unified system:

$$\frac{\partial}{\partial t}[\rho_{qs}(E_{qs} + c_p\delta T)] + \nabla \cdot [\mathbf{V}(\rho_{qs}E_{qs} + p)] = \frac{1}{\kappa}\frac{p_{qs}}{\pi_{qs}}\frac{\partial}{\partial t}\delta\pi. \tag{3.5}$$

In (3.5), E_{qs} is the quasi-hydrostatic energy per unit mass given by

$$E_{qs} \equiv \frac{1}{2}\mathbf{V}^2 + gz + c_\upsilon T_{qs}, \tag{3.6}$$

where

$$T_{qs} \equiv \pi_{qs}\theta = \frac{1}{R}\frac{p_{qs}}{\rho_{qs}}, \tag{3.7}$$

and δT is defined by

$$c_p\delta T \equiv \frac{\delta p}{\rho_{qs}}. \tag{3.8}$$

The $c_p\delta T$ term in (3.5) represents the enthalpy change per unit mass due to the change of δp through an adiabatic process. Equation (3.5) shows that

$$\frac{\partial}{\partial t}\overline{\overline{\rho_{qs}(E_{qs} + c_p\delta T)}} = 0 \tag{3.9}$$

if

$$\overline{\overline{\frac{p_{qs}}{\pi_{qs}}\frac{\partial}{\partial t}\delta\pi}} = 0, \tag{3.10}$$

where the double overbar denotes the volume mean over the entire domain. Then the mass-weighted mean of the energy $E_{qs} + c_p\delta T$ is conserved. Let $(\delta\pi)^*$ represent the solution of (3.4) with $\overline{\overline{(\delta\pi)^*}} = 0$. We see that $\delta\pi$ given by

$$\delta\pi = (\delta\pi)^* + \overline{\overline{\delta\pi}} \tag{3.11}$$

satisfies both (3.4) and (3.10) if $\overline{\overline{\delta\pi}}$ is determined by

$$\frac{\partial}{\partial t}\overline{\overline{\delta\pi}} = -\overline{\overline{\frac{p_{qs}}{\pi_{qs}}\frac{\partial}{\partial t}(\delta\pi)^*}}\Big/\overline{\overline{\frac{p_{qs}}{\pi_{qs}}}}. \tag{3.12}$$

Equation (3.12) means that $\overline{\overline{\delta\pi}}$ is prognostically determined. For an expression of (3.12) for a time-discrete case, see the next section.

Once $\delta\pi$ is determined, we can predict the horizontal velocity $\mathbf{V}_H$ using the horizontal component of (3.1) or (3.2). We could also predict w using the vertical component of (3.1) or (3.2). However, since the vertical component has already been used in deriving (3.4), w can be more simply determined from (2.11) with known $\partial\rho_{qs}/\partial t$. In this way, it is guaranteed that the continuity equation is exactly satisfied in a time-discrete case. We can show that this procedure is closely related to the determination of w using the Richardson equation (Richardson 1922, p. 118) for the quasi-hydrostatic system. To show this, we first rewrite (2.11) with (2.6) as

$$\begin{aligned}\frac{\partial w}{\partial z} &= -\left(\frac{\partial}{\partial t} + w\frac{\partial}{\partial z}\right)\ln\rho_{qs} - \frac{1}{\rho_{qs}}\mathbf{\nabla}_H\cdot(\rho_{qs}\mathbf{V}_H)\\ &= -\frac{1-\kappa}{p_{qs}}\left(\frac{\partial}{\partial t} + w\frac{\partial}{\partial z}\right)p_{qs} + \left(\frac{\partial}{\partial t} + w\frac{\partial}{\partial z}\right)\\ &\quad \ln\theta - \frac{1}{\rho_{qs}}\mathbf{\nabla}_H\cdot(\rho_{qs}\mathbf{V}_H).\end{aligned} \tag{3.13}$$

Further manipulating (3.13) using (2.5), (2.11), $(\rho_{qs}w)_T = 0$, and (2.5), we finally obtain

$$\begin{aligned}\frac{\partial w}{\partial z} = &-\frac{1}{\rho_{qs}}\mathbf{\nabla}_H\cdot(\rho_{qs}\mathbf{V}_H) - \frac{1-\kappa}{p_{qs}}\left[\frac{\partial}{\partial t}(p_{qs})_T\right.\\ &\left.-g\mathbf{\nabla}_H\cdot\int_z^{z_T}\rho_{qs}\mathbf{V}_H\,dz\right] - \mathbf{V}_H\cdot\mathbf{\nabla}_H\ln\theta + \frac{Q}{c_pT}.\end{aligned} \tag{3.14}$$

If the first term in the brackets is neglected, (3.14) essentially becomes the Richardson equation, which is still complicated. The complication is partly due to the use of the pressure tendency equation at all levels, as pointed out by Ooyama (1990), while only its application to the earth's surface given by (2.12) is needed. A more fundamental problem is that the Richardson equation is not physically illuminating. Recall that the anelastic approximation neglects the term on the left-hand side of (2.11). It is then appropriate to regard that term as a generally small correction to the anelastic approximation. Equation (3.14) splits this correction term into the sum of larger terms that exactly compensate each other when the motion is anelastic.

4. Computational procedure with the height coordinate

This section discusses a procedure that can be followed in practical applications of the unified system with the z coordinate. The prognostic variables of the unified system are $(\pi_{qs})_S$, θ, and $(\rho_{qs}\mathbf{V}_H)$. The major diagnostic variables are π_{qs}, p_{qs}, ρ_{qs}, $\delta\pi$, and $(\rho_{qs}w)$ determined by (2.3), (2.4), (2.6), (3.4), with (3.12) and (2.11), respectively. Let the integer n denote a time level. Suppose that we know all variables except for $\delta\pi$ at time level n (and at past time levels if necessary) and we have a time-difference scheme for advancing $(\pi_{qs})_S^{(n)}$, $\theta^{(n)}$, and $(\rho_{qs}\mathbf{V}_H)^{(n)}$ to $(\pi_{qs})_S^{(n+1)}$, $\theta^{(n+1)}$, and $(\rho_{qs}\mathbf{V}_H)^{(n+1)}$ based on (2.16), (2.8), and the horizontal component of (3.2), respectively. We write these schemes symbolically as

$$(\pi_{qs})_S^{(n+1)} - (\pi_{qs})_S^{(n)} = G_1, \tag{4.1}$$

$$\theta^{(n+1)} - \theta^{(n)} = G_2, \tag{4.2}$$

and

$$(\rho_{qs}\mathbf{V}_H)^{(n+1)} - (\rho_{qs}\mathbf{V}_H)^{(n)} = G_3 - \Delta t(\rho_{qs}c_p\theta)^{(n)}\mathbf{\nabla}_H(\delta\pi)^n, \tag{4.3}$$

where Δt is the time step and the term that depends on $(\delta\pi)^n$ is explicitly written. For convenience, the value at the time level n is used for $(\rho_{qs}c_p\theta)$. To derive a time-discrete version of (3.4), we also need to specify a time difference scheme for the vertical component of (3.2), which may be symbolically written as

$$(\rho_{qs}w)^{(n+1)} - (\rho_{qs}w)^{(n)} = G_4 - \Delta t(\rho_{qs}c_p\theta)^{(n)}\frac{\partial}{\partial z}(\delta\pi)^n. \tag{4.4}$$

From $(\pi_{qs})_S^{(n+1)}$ and $\theta^{(n+1)}$ predicted by (4.1) and (4.2), respectively, $\rho_{qs}^{(n+1)}$ can be determined by (2.3), (2.4), and (2.6). To obtain a discrete version of (2.11), let us formally use a backward time-difference scheme to obtain

$$\begin{aligned}\frac{\partial}{\partial z}(\rho_{qs}w)^{(n+1)} = &-\mathbf{\nabla}_H\cdot(\rho_{qs}\mathbf{V}_H)^{(n+1)}\\ &-\frac{1}{\Delta t}[\rho_{qs}^{(n+1)} - \rho_{qs}^{(n)}].\end{aligned} \tag{4.5}$$

This time discretization has only the first-order accuracy. This is probably acceptable because the last term is supposed to represent a relatively small correction to the anelastic continuity equation. Moreover, ρ_{qs} is likely to change in time rather slowly compare to purely nonhydrostatic variables for which Δt is chosen. Applying $\mathbf{\nabla}_H\cdot$ and $\partial/\partial z$ to (4.3) and (4.4), respectively, taking the sum, and using (4.5), we obtain the following:

$$\nabla_H\cdot[(\rho_{qs}c_p\theta)^{(n)}\nabla_H(\delta\pi)^{(n)}]+\frac{\partial}{\partial z}\left[(\rho_{qs}c_p\theta)^{(n)}\frac{\partial}{\partial z}(\delta\pi)^{(n)}\right]=\frac{1}{\Delta t}\left(\nabla_H\cdot G_3+\frac{\partial}{\partial z}G_4\right)+\frac{1}{(\Delta t)^2}[\rho_{qs}^{(n+1)}+\rho_{qs}^{(n-1)}-2\rho_{qs}^{(n)}].\tag{4.6}$$

This is a time-discrete version of (3.4). Note that the right-hand side including ${\rho_{qs}}^{(n+1)}$ is known.

When we desire to fully determine $(\delta\pi)^{(n)}$ including $\overline{\overline{(\delta\pi)^{(n)}}}$, which is the volume mean of $(\delta\pi)^{(n)}$, we use the discrete version of (3.12) given by

$$\overline{\overline{(\delta\pi)^{(n)}}}-\overline{\overline{(\delta\pi)^{(n-1)}}}=\overline{\overline{[(p_{qs}^{1-\kappa})^{(n)}+(p_{qs}^{1-\kappa})^{(n-1)}][(\delta\pi)^{*(n)}-(\delta\pi)^{*(n-1)}]}}\Big/[(p_{qs}^{1-\kappa})^{(n)}+(p_{qs}^{1-\kappa})^{(n-1)}].\tag{4.7}$$

After solving (4.6) for $(\delta\pi)^{(n)}$, $(\rho_{qs}\mathbf{V}_H)^{(n+1)}$ can be determined by (4.3). Then $(\rho_{qs}w)^{(n+1)}$ can be found by a downward integration of (4.5) assuming $(\rho_{qs}w)_T^{(n+1)}=0$.

5. The unified system based on the quasi-hydrostatic pressure coordinate

One of the main points of the unified system is that it reduces to a quasi-hydrostatic model when the nonhydrostatic pressure is neglected. In this way, the system maintains a close tie with the existing primitive equation models. But practically all existing primitive equation models use the pressure coordinate or its variants, and thus there is an advantage of using such a coordinate in the unified system to have the same vertical structure as the conventional large-scale models. The merit of Laprise's approach of using the quasi-hydrostatic pressure as the vertical coordinate in nonhydrostatic models (see section 1) can be even greater for the unified system because the system explicitly deals with the quasi-hydrostatic values of thermodynamic state variables. In this section, we present the unified system based on the quasi-hydrostatic pressure coordinate.

Using the definition of p_{qs} given by (2.4), we may rewrite the hydrostatic equation [(2.2)] as

$$\frac{\partial z}{\partial p_{qs}}=-\frac{R}{g}\frac{\pi_{qs}}{p_{qs}}\theta.\tag{5.1}$$

Integrating (5.1) vertically with respect to p_{qs}, we obtain

$$z=z_S+\int_{p_{qs}}^{(p_{qs})_S}\frac{R}{g}\frac{\pi_{qs}}{p_{qs}}\theta\,dp_{qs}.\tag{5.2}$$

This corresponds to (2.3). The time derivative of (5.2) gives

$$\left(\frac{\partial z}{\partial t}\right)_{p_{qs}}=\frac{R}{g}\left(\frac{\pi_{qs}}{p_{qs}}\theta\right)_S\left(\frac{\partial p_{qs}}{\partial t}\right)_S+\int_{p_{qs}}^{(p_{qs})_S}\frac{R}{g}\frac{\pi_{qs}}{p_{qs}}\left(\frac{\partial\theta}{\partial t}\right)_{p_{qs}}dp_{qs}.\tag{5.3}$$

Thus, $(p_{qs})_S$ and θ must be predicted, as in the z-coordinate case, but this time to determine the time evolution of the height field. The thermodynamic equation in (2.8) to predict θ is now written as

$$\left(\frac{\partial\theta}{\partial t}\right)_{p_{qs}}=-\left(\mathbf{V}_H\cdot\nabla_{p_{qs}}+\omega\frac{\partial}{\partial p_{qs}}\right)\theta+\frac{Q}{c_p\pi},\tag{5.4}$$

where ω is defined by

$$\omega\equiv\frac{Dp_{qs}}{Dt}.\tag{5.5}$$

Using the hydrostatic equation in (2.5), the continuity equation in (2.11) can be rewritten as

$$\nabla_{p_{qs}}\cdot\mathbf{V}_H+\frac{\partial\omega}{\partial p_{qs}}=0.\tag{5.6}$$

It should be noted that, unlike in the usual quasi-hydrostatic p-coordinate system, (5.6) is a consequence of the *definitions* of p_{qs} and ω, and not of the quasi-hydrostatic approximation. Let us assume that the model top is a material surface with a constant p_{qs}. Then we have $\omega_T=0$. The vertical integral of (5.6) then gives

$$\omega_S=(\mathbf{V}_H)_S\cdot\nabla(p_{qs})_S-\nabla\cdot\int_{(p_{qs})_T}^{(p_{qs})_S}\mathbf{V}_H\,dp_{qs}.\tag{5.7}$$

Since the earth's surface is a material surface, ω_S can also be written as

$$\omega_S=\left(\frac{\partial p_{qs}}{\partial t}\right)_S+(\mathbf{V}_H)_S\cdot\nabla(p_{qs})_S.\tag{5.8}$$

Equating (5.7) and (5.8), we obtain

$$\left(\frac{\partial p_{qs}}{\partial t}\right)_S=-\nabla\cdot\int_{(p_{qs})_T}^{(p_{qs})_S}\mathbf{V}_H\,dp_{qs}.\tag{5.9}$$

Equation (5.3) is now closed. This procedure is simpler than that with the z coordinate mainly because we now have $(\partial p_{qs}/\partial t)_T = 0$.

As in the z-coordinate case, the spatially varying part of the nonhydrostatic pressure δp can be determined by requiring that the velocity field predicted by the momentum equation satisfy the continuity equation, which now takes the form of (5.7). We begin with the momentum equation written in the following form:

$$\frac{D\mathbf{V}}{Dt} = -2\boldsymbol{\Omega} \times \mathbf{V} - \frac{1}{\rho_{qs}}\nabla_{\mathrm{H}} p_{qs} - \frac{1}{\rho_{qs}}\left(\nabla_{\mathrm{H}} + \mathbf{k}\frac{\partial}{\partial z}\right)\delta p - \mathbf{k}g\frac{\delta\rho}{\rho_{qs}} + \mathbf{F}. \tag{5.10}$$

Here the assumption in (1.6) and the definition of ρ_{qs} given by (2.5) have been used. Transforming the vertical coordinate in (5.10) from z to p_{qs} and taking the horizontal and vertical components, we obtain

$$\left(\frac{\partial \mathbf{V}_H}{\partial t}\right)_{p_{qs}} = -\mathbf{J}_H - \left(\frac{1}{\rho_{qs}}\nabla_{p_{qs}} + g\nabla_{p_{qs}} z\frac{\partial}{\partial p_{qs}}\right)\delta p \tag{5.11}$$

and

$$\left(\frac{\partial w}{\partial t}\right)_{p_{qs}} = -J_z + g\frac{\partial}{\partial p_{qs}}\delta p, \tag{5.12}$$

where the three-dimensional vector $\mathbf{J}$ is defined by

$$\mathbf{J} \equiv -(\mathbf{V}\cdot\nabla_{p_{qs}})\mathbf{V} - \omega\frac{\partial \mathbf{V}}{\partial p_{qs}} - 2\boldsymbol{\Omega}\times\mathbf{V} - g\nabla_{p_{qs}} z - \mathbf{k}\frac{\delta\rho}{\rho_{qs}}g + \mathbf{F}. \tag{5.13}$$

From (5.11), we obtain

$$\left(\frac{\partial}{\partial t}\right)_{p_{qs}} \nabla_{P_{qs}}\cdot\mathbf{V}_H = -\nabla_{P_{qs}}\cdot\mathbf{J}_H - \nabla_{P_{qs}}\cdot\left(\frac{1}{\rho_{qs}}\nabla_{p_{qs}} + g\nabla_{p_{qs}} z\frac{\partial}{\partial p_{qs}}\right)\delta p. \tag{5.14}$$

From $w = (\partial/\partial t + \mathbf{V}_H\cdot\nabla)_{p_{qs}} z + \omega\,\partial z/\partial p_{qs} = (\partial/\partial t + \mathbf{V}_H\cdot\nabla)_{p_{qs}} z - \omega/\rho_{qs} g$, on the other hand,

$$\omega = -\rho_{qs} g(w - w_C), \tag{5.15}$$

where

$$w_C \equiv \left(\frac{\partial}{\partial t} + \mathbf{V}_H\cdot\nabla\right)_{p_{qs}} z. \tag{5.16}$$

Substituting (5.15) into the continuity equation in (5.6) results in

$$\nabla_{p_{qs}}\cdot\mathbf{V}_H - \frac{\partial}{\partial p_{qs}}[\rho_{qs} g(w - w_C)] = 0. \tag{5.17}$$

Since w_C includes the term $\partial z/\partial t$, (5.17) could be viewed as a prognostic equation for z. The point of the unified system is, however, z is predicted through (5.3) and, therefore, the continuity equation in (5.17) is used as a diagnostic equation. In this way, vertically propagating sound waves are filtered.

Taking the time derivative of (5.17) and then using (5.14) and (5.12), we finally obtain

$$\nabla_{p_{qs}}\cdot\left(\frac{1}{\rho_{qs}}\nabla_{p_{qs}} + g\nabla_{p_{qs}} z\frac{\partial}{\partial p_{qs}}\right)\delta p + g^2\frac{\partial}{\partial p_{qs}}\left(\rho_{qs}\frac{\partial}{\partial p_{qs}}\delta p\right) = -\nabla_{p_{qs}}\cdot\mathbf{V}_H + g\frac{\partial}{\partial p_{qs}}(\rho_{qs}J_z) - g\frac{\partial}{\partial p_{qs}}\left[w\left(\frac{\partial\rho_{qs}}{\partial t}\right)_{p_{qs}} - \left(\frac{\partial}{\partial t}\right)_{p_{qs}}(\rho_{qs}w_C)\right]. \tag{5.18}$$

Time discretization of (5.18) can follow (4.6) using the already predicted values of $\rho_{qs}^{(n+1)}$ and $z^{(n+1)}$. Determination of the spatially constant part of δp can follow the procedure for determining the spatially constant part of $\delta\pi$ described in sections 3 and 4.

If desired, the vertical coordinate used in the equations presented here can further be transformed to a sigma coordinate as Janjic et al. (2001) and Janjic (2003) did in their nonhydrostatic model.

6. Small-amplitude perturbations on a resting atmosphere

To compare the unified system with other commonly used systems, this section discusses small-amplitude perturbations on a resting, horizontally uniform atmosphere. For simplicity, the motion is assumed to be adiabatic and frictionless. The standard z coordinate is used for this analysis.

a. Linearized equations

Let an overbar and a prime denote the basic state and perturbation, respectively. Linearizing the equation of state and the definition of θ applied to the perturbation, we obtain

$$\frac{\rho'}{\bar\rho} = \frac{1}{c_s^2}\frac{p'}{\bar\rho} - \frac{\theta'}{\bar\theta}, \tag{6.1}$$

where c_s is the speed of sound given by $c_s^2 \equiv \gamma RT$. Also, we obtain $\pi' \approx \kappa(\bar{\pi}/\bar{p})p'$ from the definition of π. Then, using the equation of state applied to the basic state written in the form $\bar{p} = \bar{\rho} R \bar{\pi} \bar{\theta}$, we can show

$$c_p \bar{\theta} \pi' \approx \frac{p'}{\bar{\rho}}. \tag{6.2}$$

Linearizing the horizontal and vertical components of the momentum equation [(2.1)] with the "traditional approximation" (Eckart 1960; Phillips 1966, 1968) and using (6.2), we obtain

$$\frac{\partial \mathbf{V}'_H}{\partial t} = -f\mathbf{k} \times \mathbf{V}'_H - \nabla_H \left(\frac{p'}{\bar{\rho}}\right) \tag{6.3}$$

and

$$\delta \frac{\partial w'}{\partial t} = -\frac{\partial}{\partial z}\left(\frac{p'}{\bar{\rho}}\right) + \underline{\underline{\frac{p'}{\bar{\rho}} \frac{d \ln \bar{\theta}}{dz}}} + \frac{\theta'}{\bar{\theta}} g. \tag{6.4}$$

Here $\delta = 1$ and $\delta = 0$ represent the nonhydrostatic and quasi-hydrostatic systems, respectively. The anelastic systems proposed by Ogura and Phillips (1962) and Lipps and Hemler (1982) drop the double-underlined term. Linearization of (2.8) without heating gives

$$\frac{\partial}{\partial t}\left(\frac{\theta'}{\bar{\theta}}\right) = -w' \frac{d \ln \bar{\theta}}{dz}. \tag{6.5}$$

On the other hand, linearization of (1.1) and the use of (6.1) and (6.5) give

$$\underline{\frac{1}{c_s^2} \frac{\partial}{\partial t}\left(\frac{p'}{\bar{\rho}}\right)} = -\frac{1}{\bar{\rho}}\left[\nabla_H \cdot (\bar{\rho} \mathbf{V}'_H) + \frac{\partial}{\partial z}(\bar{\rho} w')\right] - \underline{\underline{w' \frac{d \ln \bar{\theta}}{dz}}}. \tag{6.6}$$

The anelastic continuity equation neglects both the single- and double-underlined terms while the pseudo-incompressible equation (Durran 1989) neglects only the single-underlined term.

b. Normal-mode analysis on an *f* plane

In the rest of this section, we analyze the dispersion relation and vertical structure of the normal modes for various systems of equations using a Cartesian horizontal coordinate (x, y), first on an f plane without the quasigeostrophic approximation and then on a midlatitude β plane with the quasigeostrophic approximation. For simplicity, we assume that the motion is uniform in y as in Rossby et al. (1939). An isothermal resting atmosphere is used as the basic state.

Our analysis on an f plane is almost parallel to that performed by Davies et al. (2003) except that we use a different transformation of the dependent variables. From (6.3) with $p' = p'_{qs} + \delta p'$, we can derive the divergence and vorticity equations as

$$\frac{\partial}{\partial t} \frac{\partial u'}{\partial x} = f \frac{\partial v'}{\partial x} - \frac{\partial^2}{\partial x^2}\left(\frac{p'_{qs}}{\bar{\rho}} + \frac{\delta p'}{\bar{\rho}}\right) \tag{6.7}$$

and

$$\frac{\partial}{\partial t} \frac{\partial v'}{\partial x} = -f \frac{\partial u'}{\partial x}, \tag{6.8}$$

where u and v are the x and y components of velocity, respectively, and f is a constant Coriolis parameter. Using $d \ln \bar{\theta}/dz = \kappa/H$ in (6.4), where H is the scale height, the vertical component of the momentum equation may be written as

$$\delta \frac{\partial w'}{\partial t} = -\left(\frac{\partial}{\partial z} - \underline{\underline{\frac{\kappa}{H}}}\right)\left(\frac{p'}{\bar{\rho}}\right) + b', \tag{6.9}$$

where $b' \equiv g\theta'/\bar{\theta}$. From the definition of p_{qs}, we have

$$\left(\frac{\partial}{\partial z} - \underline{\underline{\frac{\kappa}{H}}}\right)\left(\frac{p'_{qs}}{\bar{\rho}}\right) = b'. \tag{6.10}$$

Then (6.9) gives

$$\delta \frac{\partial w'}{\partial t} = -\left(\frac{\partial}{\partial z} - \underline{\underline{\frac{\kappa}{H}}}\right)\left(\frac{\delta p'}{\bar{\rho}}\right). \tag{6.11}$$

Equations (6.5) and (6.6), on the other hand, give

$$\frac{\partial}{\partial t} b' = -g \frac{\kappa}{H} w' \tag{6.12}$$

and

$$\underline{\frac{1}{c_s^2} \frac{\partial}{\partial t}\left(\frac{p'_{qs}}{\bar{\rho}} + \varepsilon \frac{\delta p'}{\bar{\rho}}\right)} = \frac{w'}{H} - \left[\frac{\partial u'}{\partial x} + \left(\frac{\partial}{\partial z} + \underline{\underline{\frac{\kappa}{H}}}\right) w'\right]. \tag{6.13}$$

Equations (6.7)–(6.8) and (6.10)–(6.13) form a closed system for the dependent variables u', v', w', p'_{qs}/ρ, b', and $\delta p'/\bar{\rho}$. Recall the following definitions:

- Fully compressible: all underlined terms are retained with $\varepsilon = 1$, $\delta = 1$;
- Unified: all underlined terms are retained with $\varepsilon = 0$, $\delta = 1$;
- Pseudo-incompressible: terms with single underline are omitted with $\delta = 1$;

- Anelastic (Lipps–Hemler): terms with single and double underlines are omitted with $\delta = 1$;
- Quasi-hydrostatic: all underlined terms are retained with $\delta = 0$.

From (6.11), (6.10), and (6.13), it is obvious that the Lipps–Hemler anelastic model applied to an isothermal atmosphere requires κ (~0.286) << 1 at least for deep motions. As Bannon (1995) pointed out, this condition is also required for the approximation $\theta'/\theta_0 \approx T'/T_0$ used in their model. This requirement suggests that applications of the anelastic system to the stratosphere need some caution. The pseudo-incompressible model of Durran (1989) is free of this requirement.

We now consider normal modes governed by (6.7)–(6.8) and (6.10)–(6.13) that have the following form:

$$\bar{\rho}^{1/2}(u', v', w', b') = \mathrm{Re}[(\hat{u}, \hat{v}, \hat{w}, \hat{b})e^{i(kx+mz-\nu t)}] \quad \text{and}$$
$$\bar{\rho}^{-1/2}(p'_{qs}, \delta p') = \mathrm{Re}[(\hat{p}_{qs}, \delta\hat{p})e^{i(kx+mz-\nu t)}], \tag{6.14}$$

where k and m are the horizontal and vertical wavenumbers, respectively, and ν is the frequency, We consider barotropic and baroclinic modes separately.

1) Barotropic mode

By ''barotropic mode,'' we mean nonbuoyant motions ($b' = 0$). Equation (6.12) shows that such motions are horizontal ($w' = 0$). Then (6.11) and (6.10) show that both $\delta p'/\bar{\rho}$ and $p'_{qs}/\bar{\rho}$ vary exponentially in height. Thus, if we assume that these variables are zero at the model top, they are zero at all height. This is not acceptable at least for $p'_{qs}/\bar{\rho}$ as we discussed in section 2. Using (6.14) in (6.7), (6.8) and (6.13) with $w' = 0$, we find the following dispersion relation:

$$\underline{k^2\nu^3} - k^2(\underline{f_0^2} + k^2c_s^2)\nu = 0. \tag{6.15}$$

A solution of (6.15) is $\nu = 0$, representing the stationary barotropic geostrophic motion.

(i) The pseudo-incompressible and anelastic systems

These systems neglect the underlined term in (6.15). Consequently, $\nu = 0$ is the only solution of (6.15).

(ii) The fully compressible, unified, and quasi-hydrostatic systems

The assumption of $\varepsilon = 0$ for the unified system and that of $\delta = 0$ for the quasi-hydrostatic system do not influence the dispersion relation (6.15). Thus, the fully compressible, unified and quasi-hydrostatic systems have identical solutions given by

$$\nu^2 = f^2 + k^2c_s^2, \tag{6.16}$$

which gives the frequency of the Lamb wave modified by the Coriolis force.

2) Baroclinic modes

Using the transformation of the dependent variables given by (6.14), we obtain the dispersion relation for baroclinic modes. As in the case of barotropic modes, $\nu = 0$ is a solution, representing the stationary baroclinic geostrophic motion. For other modes, the dispersion relation is given by

$$\underline{\varepsilon\delta k\nu^4} - [\underline{N^2k} + \delta k^3c_s^2 + \underline{\varepsilon\delta kf^2} + c_s^2k(m^2 + \mu^2)]\nu^2$$
$$+ [\underline{N^2kf^2} + k^3c_s^2N^2 + c_s^2kf^2(m^2 + \mu^2)] = 0, \tag{6.17}$$

where

$$\mu \equiv \frac{1}{H}\left(\frac{1}{2} - \underline{\underline{\kappa}}\right). \tag{6.18}$$

For solutions satisfying $w_S = w_T = 0$ at the upper and lower boundaries, the vertical wavenumber of the solutions is constrained to the form given by

$$m = \frac{\pi n}{z_T}, \quad n = 1, 2, 3, \ldots, \tag{6.19}$$

where n is the integer vertical wavenumber and z_T is the height of the upper boundary as previously defined. In (6.19), we assume that $z_S = 0$ at the lower boundary.

Figure 1 shows frequencies of these modes (solid lines) as well as that of the Lamb wave (dashed line) as functions of the horizontal wavenumber for selected values of n for (Fig. 1a) the fully compressible, (Fig. 1b) anelastic, (Fig. 1c) pseudo-incompressible, (Fig. 1d) unified, and (Fig. 1e) quasi-hydrostatic systems. Only positive frequencies are shown. The fully compressible system (Fig. 1a) yields three distinct modes, one representing vertically propagating sound waves, one representing inertia-gravity waves and one representing the Lamb wave. Vertically propagating sound waves are filtered by all the systems (Figs. 1b–e). The unified, pseudo-incompressible, and anelastic systems do the filtering without significant distortions in the dispersion of the inertia–gravity mode while the quasi-hydrostatic system seriously distorts the dispersion of that mode with large horizontal wavenumbers.

It is evident in Fig. 1 that the fully compressible, unified, pseudo-incompressible, and anelastic systems produce virtually identical dispersion relation for the

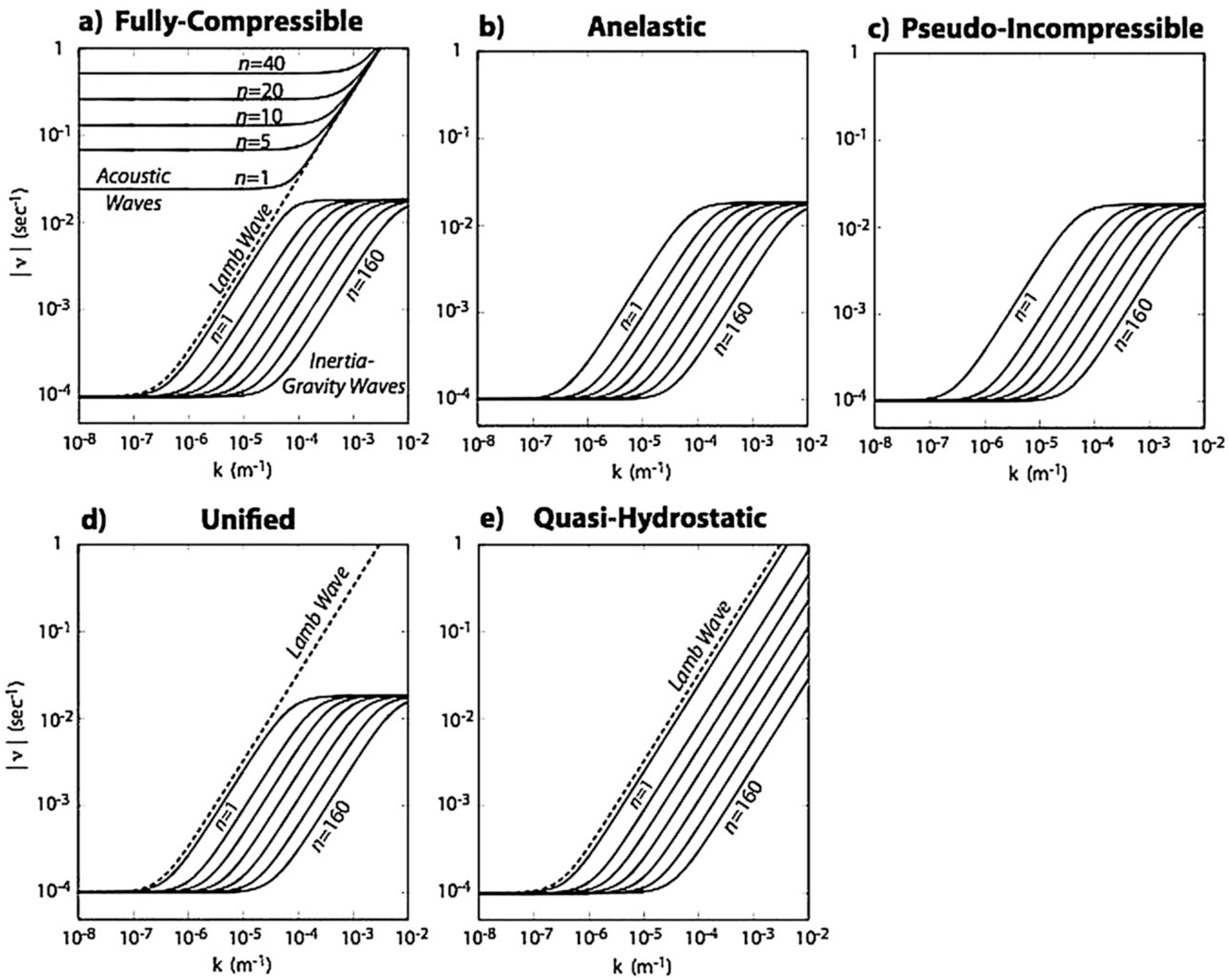

FIG. 1. Frequencies of normal modes on an *f* plane as functions of horizontal wavenumber for (a) the fully compressible, (b) anelastic, (c) pseudo-incompressible, (d) unified, and (e) quasi-hydrostatic systems. See the text for more details.

inertia–gravity mode. The $n = 1$ case of the pseudo-incompressible system is, however, worse than the anelastic system. This does not mean, however, that the solutions for u', v', w', b', p'_{qs}, and $\delta p'$ of the anelastic system are better than those of the pseudo-incompressible system. To show this, we define the vertical phase angle φ by

$$\varphi \equiv \arctan\left(\frac{\pi n}{\mu z_T}\right). \tag{6.20}$$

In the definition of μ given by (6.18), all terms are kept in the fully compressible, unified, pseudo-incompressible, and quasi-hydrostatic systems, while the term with double underline is omitted in the anelastic system. Consequently, the vertical phase angle is different for the anelastic system from the others. The difference is maximum (approximately 30°) for $n = 1$ and decreases with increasing n. This is due to the failure of the anelastic system in correctly recognizing the effect of static stability.

c. Normal-mode analysis on a midlatitude β plane with the quasigeostrophic approximation

The normal-mode analysis presented in section 6b is extended to a midlatitude β plane. Since our focus here is on the Rossby wave, we use the quasigeostrophic (and quasi-hydrostatic) approximations for clarity of the results. In this analysis, (6.7), (6.8), and (6.13) are replaced by

$$f_0 \frac{\partial v'}{\partial x} = \frac{\partial^2}{\partial x^2}\frac{p'_{qs}}{\bar{\rho}}, \tag{6.21}$$

$$\frac{\partial}{\partial t}\frac{\partial v'}{\partial x} = -\beta v' - f_0 \frac{\partial u'}{\partial x}, \tag{6.22}$$

and

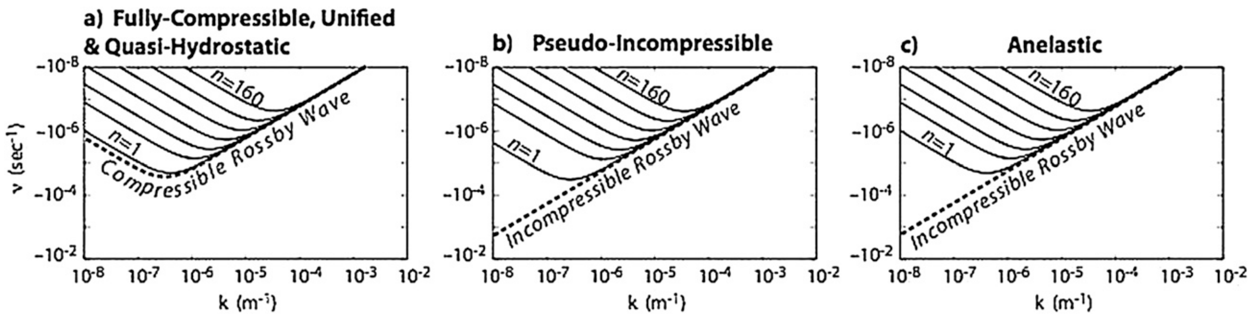

FIG. 2. Frequencies of normal modes on a midlatitude β plane with the quasigeostrophic approximation as functions of horizontal wavenumber for (a) the fully compressible, unified, and quasi-hydrostatic, (b) pseudo-incompressible, and (c) anelastic systems. See the text for more details.

$$\underline{\frac{1}{c_s^2}\frac{\partial}{\partial t}\frac{p'_{qs}}{\bar{\rho}}} = \frac{w'}{H} - \left[\frac{\partial u'}{\partial x} + \left(\frac{\partial}{\partial z} + \frac{\kappa}{\underline{\underline{H}}}\right)w'\right]. \qquad (6.23)$$

1) BAROTROPIC MODE

(i) The pseudo-incompressible and anelastic systems

These systems neglect the underlined term in (6.13), which gives $u' = 0$ for horizontal motion. The dispersion relation then becomes

$$\nu = -\frac{\beta}{k}. \qquad (6.24)$$

This frequency gives the westward retrogression speed of the prototype Rossby wave, which becomes infinite as $k \to 0$.

(ii) The fully compressible, unified, and quasi-hydrostatic systems

All of these systems have the dispersion relation given by

$$\nu = -\frac{k\beta}{k^2 + \underline{(f_0/c_s)^2}}. \qquad (6.25)$$

In a sharp contrast to (6.24), (6.25) gives $\nu \to 0$ as $k \to 0$.

2) BAROCLINIC MODES

For these modes, the dispersion relation is given by

$$\nu = -\frac{k\beta}{k^2 + f_0^2\left[\underline{\frac{1}{c_s^2}} + \frac{1}{\kappa g H}H^2(m^2 + \mu^2)\right]}. \qquad (6.26)$$

With the upper boundary at a height comparable to the scale height H, the second term in the brackets dominates over the first term so that, unlike the barotropic mode, the differences of the anelastic/pseudo-incompressible systems from the others are relatively minor.

Figure 2 shows frequencies of the barotropic (dashed lines) and baroclinic (solid lines) Rossby modes for the fully compressible, unified, and quasi-hydrostatic system (Fig. 2a), the pseudo-incompressible system (Fig. 2b), and the anelastic system (Fig. 2c). The overall performance of the unified, pseudo-incompressible, anelastic, and quasi-hydrostatic systems relative to the fully compressible system is summarized in Table 1. In the table, ''not modified'' and ''modified'' are relative to the fully compressible system. In summary, as far as the normal modes are concerned, the unified system maintains the characteristics of the fully compressible system almost exactly except that it filters vertically propagating sound waves.

7. Summary and conclusions

This paper presents a system of equations that can cover a wide range of horizontal scales from turbulence to planetary waves while filtering vertically propagating sound waves of all scales. The continuity equation of the system includes the time derivative of quasi-hydrostatic density, which can be predicted using the thermodynamic equation and the tendency equation for the quasi-hydrostatic surface pressure. The system can therefore be viewed as a generalization of the anelastic system while it is fully compressible for quasi-hydrostatic motions. The system can also be viewed as a generalization of the quasi-hydrostatic (usually simply called ''hydrostatic'') system since no approximation is introduced into the momentum equation. In this way, the system maintains close ties with *both* the primitive equation models for large scales and the anelastic (and Boussinesq) models for small scales. As in the anelastic system, the spatially varying part of the nonhydrostatic Exner function is determined through solving an elliptic equation. A computational procedure that can be followed in a time-discrete model is presented. The paper also presents the unified system with the quasi-hydrostatic pressure as the vertical coordinate. Appendix B shows

TABLE 1. A summary of the normal-mode analysis.

	Dispersion relation				Solutions				
						Amplitudes			
	Acoustic waves	Lamb waves	Gravity waves	Rossby waves	Vertical phase	$\delta p'$	p'_{qs}	u'	v'
Unified	Filtered	Not modified	Not modified	Not modified	Not modified	Not modified	Not modified	Not modified	Not modified
Pseudo-incompressible	Filtered	Filtered	Slightly modified for deep waves	Modified for ultra-long waves	Not modified	Slightly modified	Not modified	Slightly modified	Slightly modified
Anelastic	Filtered	Filtered	Not modified	Modified for ultra-long waves	Modified unable to recognize stability	Slightly modified	Slightly modified	Slightly modified	Slightly modified
Quasi-hydrostatic	Filtered	Not modified	Greatly modified for short waves	Not modified	Not modified	Zero no nonhydrostatic pressure	Not modified	Not modified	Not modified

that the unified system can also use the vector vorticity equation instead of the momentum equation.

Through normal-mode analysis, it is shown that the unified system reduces the westward retrogression speed of the barotropic Rossby wave through the inclusion of horizontal divergence due to compressibility. It also removes the large systematic error of the anelastic system in the vertical structure. While vertically propagating sound waves are filtered, the Lamb wave is included in the unified system as in the usual primitive equation models. Because of the close analogy between the Lamb wave and shallow-water gravity waves, we hope that the multipoint explicit differencing (MED) technique originally developed for shallow-water gravity waves by Konor and Arakawa (2007) will be effective in stabilizing the Lamb wave with high Courant numbers.

It is shown that a properly defined energy can be conserved in this system with no heating and friction. Whether the energy is conserved or not, however, depends on how we determine the spatially constant part of the Exner function, which does not influence the dynamics of the system. Conservation also depends on the definition of nonhydrostatic temperature, which appears only in the right-hand side of (2.8) representing the diabatic effect. Thus energy conservation in this system is a matter of interpretation as far as adiabatic cases are concerned.

In conclusion, the unified system seems to be a promising system as the dynamics core of global cloud-resolving models although its computational efficiency relative to that of fully compressible models is yet to be assessed.

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APPENDIX A

Derivation of the Energy Equation

Rewriting the definition of θ using the equation of state, we obtain

$$\ln \theta = (1 - \kappa) \ln p - \ln \rho + \text{const}, \qquad \text{(A.1)}$$

while from (2.6), which is a consequence of the definition of the quasi-hydrostatic state, we obtain

$$\ln\theta = (1-\kappa)\ln p_{qs} - \ln\rho_{qs} + \text{const.} \qquad (A.2)$$

From the definition of T_{qs} given by (3.7) and (A.2), we can also express $\ln\theta$ as

$$\ln\theta \equiv (1-\kappa)\ln T_{qs} - \kappa\ln\rho_{qs} + \text{const.} \qquad (A.3)$$

From (A.2) and (A.3) with the adiabatic thermodynamic equation $D\ln\theta/Dt = 0$, we may write

$$\frac{D}{Dt}\ln\rho_{qs} = (1-\kappa)\frac{D}{Dt}\ln p_{qs} \qquad (A.4a)$$

$$= \frac{1-\kappa}{\kappa}\frac{D}{Dt}\ln T_{qs}. \qquad (A.4b)$$

Subtracting (A.2) from (A.1) and using (1.6) and (2.7), on the other hand, we obtain

$$\frac{\delta\rho}{\rho_{qs}} \approx (1-\kappa)\frac{\delta p}{p_{qs}}. \qquad (A.5)$$

We also have

$$\frac{\delta\pi}{\pi_{qs}} \approx \kappa\frac{\delta p}{p_{qs}}. \qquad (A.6)$$

Linearizing the pressure gradient force $-\nabla p/\rho$ using (1.6), the momentum equation without the friction force may be written as

$$\frac{D\mathbf{V}}{Dt} = -2\boldsymbol{\Omega}\times\mathbf{V} - \frac{1}{\rho_{qs}}\left(\nabla p - \frac{\delta\rho}{\rho_{qs}}\nabla p_{qs}\right) - \mathbf{k}g. \qquad (A.7)$$

Multiplying (A.7) by $\rho_{qs}\mathbf{V}$, using $w = Dz/Dt$, and substituting $\nabla\cdot\mathbf{V}$obtained from the continuity equation in (1.5) rewritten in the following form:

$$\frac{D}{Dt}\ln\rho_{qs} + \nabla\cdot\mathbf{V} = 0, \qquad (A.8)$$

we obtain

$$\rho_{qs}\frac{D}{Dt}\left(\frac{1}{2}\mathbf{V}^2 + gz\right) = -\nabla\cdot(p\mathbf{V}) - p\frac{D\ln\rho_{qs}}{Dt} + \frac{\delta\rho}{\rho_{qs}}\mathbf{V}\cdot\nabla p_{qs}. \qquad (A.9)$$

Using $p = p_{qs} + \delta p$, (A.4b) for $-p_{qs}D\ln\rho_{qs}/Dt$, (A.4a) for $-\delta pD\ln\rho_{qs}/Dt$, and then (A.5), (A.9) may be rewritten as

$$\rho_{qs}\frac{DE_{qs}}{Dt} = -\nabla\cdot(p\mathbf{V}) - (1-\kappa)\frac{\delta p}{p_{qs}}\frac{\partial p_{qs}}{\partial t}, \qquad (A.10)$$

where E_{qs} is the quasi-hydrostatic energy defined by

$$E_{qs} \equiv \frac{1}{2}\mathbf{V}^2 + gz + c_v T_{qs}. \qquad (A.11)$$

Using (3.8) and (A.6), the last term in (A.10) may be rewritten as

$$\begin{aligned}
-(1-\kappa)\frac{\delta p}{p_{qs}}\frac{\partial p_{qs}}{\partial t} &\equiv -\frac{\partial}{\partial t}\delta p + p_{qs}\frac{\partial}{\partial t}\left(\frac{\delta p}{p_{qs}}\right) + \kappa\frac{\delta p}{p_{qs}}\frac{\partial p_{qs}}{\partial t} \\
&= -\frac{\partial}{\partial t}(c_p\rho_{qs}\delta T) + \frac{1}{\kappa}p_{qs}\frac{\partial}{\partial t}\left(\frac{\delta\pi}{\pi_{qs}}\right) \\
&\quad + \frac{\delta\pi}{\pi_{qs}}\frac{\partial p_{qs}}{\partial t} \\
&\equiv -\frac{\partial}{\partial t}(c_p\rho_{qs}\delta T) + \frac{1}{\kappa}\frac{p_{qs}}{\pi_{qs}}\frac{\partial}{\partial t}\delta\pi \\
&\quad - \frac{\delta\pi}{\pi_{qs}}\left(\frac{1}{\kappa}\frac{p_{qs}}{\pi_{qs}}\frac{\partial}{\partial t}\pi_{qs} - \frac{\partial p_{qs}}{\partial t}\right).
\end{aligned} \qquad (A.12)$$

From (2.4), the sum of the terms in the last pair of parentheses vanishes. Using this result and the Eulerian form of (A.10), we finally obtain

$$\frac{\partial}{\partial t}\rho_{qs}(E_{qs} + c_p\delta T_{qs}) + \nabla\cdot[\mathbf{V}(\rho_{qs}E_{qs} + p)] = \frac{1}{\kappa}\frac{p_{qs}}{\pi_{qs}}\frac{\partial}{\partial t}\delta\pi. \qquad (A.13)$$

This is (3.5) in the text.

APPENDIX B

Computational Procedure with the Vector Vorticity Equation

In the cloud-resolving model developed by Jung and Arakawa (2008), the horizontal component of the three-dimensional vorticity equation is used instead of the momentum equation. From the curl of (2.1), we can derive the vector vorticity equation as

$$\frac{D\boldsymbol{\omega}}{Dt} = -\boldsymbol{\omega}\nabla\cdot\mathbf{V} + (\boldsymbol{\omega}\cdot\nabla)\mathbf{V} + c_p\nabla(\pi_{qs} + \delta\pi) \times \nabla\theta + \nabla\times\mathbf{F}, \qquad (B.1)$$

where ω is the three-dimensional vorticity, $\nabla\times\mathbf{V}$. The horizontal component of (B.1) is

$$\frac{D\boldsymbol{\omega}_{\mathrm{H}}}{Dt} = -\boldsymbol{\omega}_H\nabla\cdot\mathbf{V} + (\boldsymbol{\omega}\cdot\nabla)\mathbf{V}_H + c_p\mathbf{k} \times \left(\nabla_H\theta\frac{\partial}{\partial z} - \frac{\partial\theta}{\partial z}\nabla_H\right)(\pi_{qs} + \delta\pi) + (\nabla\times\mathbf{F})_H. \qquad (B.2)$$

In the Jung–Arakawa model, the vertical component of $\boldsymbol{\omega}$ is diagnosed from its horizontal component using the identity

$$\nabla \cdot \boldsymbol{\omega} = \nabla_H \cdot \boldsymbol{\omega}_{\mathrm{H}} + \frac{\partial \omega_z}{\partial z} \equiv 0. \tag{B.3}$$

To derive equations that relate w to the horizontal components of velocity or vorticity, we first rewrite the continuity equation in (2.13) as

$$\frac{1}{\rho_{qs}} \frac{\partial}{\partial z}(\rho_{qs} w) = -\nabla \cdot \mathbf{V}_H - \left(\frac{D}{Dt}\right)_H \ln \rho_{qs}, \tag{B.4}$$

where $(D/Dt)_H \equiv \partial/\partial t + \mathbf{V}_H \cdot \nabla_H$. Differentiating (B.4) with respect to z, adding $\nabla_H^2 w$ to both sides, and using $\nabla_H \times \boldsymbol{\omega}_{\mathrm{H}} \equiv \mathbf{k}\nabla_H(\partial \mathbf{V}_H/\partial z - \nabla_H{}^w)$, we obtain

$$\nabla_H^2 w + \frac{\partial}{\partial z}\left[\frac{1}{\rho_{qs}} \frac{\partial}{\partial z}(\rho_{qs} w)\right] = -\mathbf{k} \cdot \nabla_H \times \boldsymbol{\omega}_{\mathrm{H}} - \frac{\partial}{\partial z}\left(\frac{D}{Dt}\right)_H \ln \rho_{qs}. \tag{B.5}$$

When the last term is dropped, (B.5) becomes a diagnostic equation that relates w to $\boldsymbol{\omega}_H$. This diagnostic equation is used in the anelastic model presented by Jung and Arakawa (2008), which replaces the elliptic equation for the Exner function. In the unified system based on the vector vorticity equation, (B.5) is used to update w from the predicted $\boldsymbol{\omega}_H$ and ρ_{qs}. Using the backward scheme to express the last term as in section 4, (B.5) may be discretized as

$$\nabla_H^2 w^{(n+1)} + \frac{\partial}{\partial z}\left\{\frac{1}{\rho} \frac{\partial}{\partial z}[\rho w^{(n+1)}]\right\} = -\mathbf{k} \cdot \nabla_H \times \omega_H^{(n+1)} - \frac{\partial}{\partial z}\left[\frac{\ln \rho_{qs}^{(n+1)} - \ln \rho_{qs}^{(n)}}{\Delta t} + (\mathbf{V}_H \cdot \nabla_H \ln \rho_{qs})^{(n)}\right]. \tag{B.6}$$

Using already known $w^{(n+1)}$ and $\rho^{(n+1)}$ in the continuity equation in (4.5), we can diagnose $\nabla \cdot \mathbf{V}_H^{(n+1)}$ at an arbitrary level (e.g., at the model top). Then, the horizontal divergence equation applied to the model top determines $(\delta\pi)_T^{(n+1)}$ except for a horizontally constant part through a Poisson equation. A downward integration of the vertical component of the momentum equation in (3.3) from this temporary value of $(\delta\pi)_T^{(n+1)}$ determines $(\delta\pi)^{(n+1)}$ at all height except for a spatially constant part. We can then follow (3.11) and a time-discrete version of (3.12) to obtain the final value of $(\delta\pi)^{(n+1)}$.

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