How is the Balance of a Forecast Ensemble Affected by Adaptive and Nonadaptive Localization Schemes?

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ABSTRACT
This paper investigates the effect on balance of a number of Schur product–type localization schemes that have been designed with the primary function of reducing spurious far-field correlations in forecast error statistics. The localization schemes studied comprise a nonadaptive scheme (where the moderation matrix is decomposed in a spectral basis), and two adaptive schemes: a simplified version of Smoothed Ensemble Correlations Raised to a Power (SENCORP) and Ensemble Correlations Raised to a Power (ECO-RAP). The paper shows, the author believes for the first time, how the degree of balance (geostrophic and hydrostatic) implied by the error covariance matrices localized by these schemes can be diagnosed. Here it is considered that an effective localization scheme is one that reduces spurious correlations adequately, but also minimizes disruption of balance (where the “correct” degree of balance or imbalance is assumed to be possessed by the unlocalized ensemble). By varying free parameters that describe each scheme (e.g., the degree of truncation in the schemes that use the spectral basis, the “order” of each scheme, and the degree of ensemble smoothing), it is found that a particular configuration of the ECO-RAP scheme is best suited to the convective-scale system studied. According to the diagnostics this ECO-RAP configuration still weakens geostrophic and hydrostatic balance, but overall this is less so than for other schemes.

1. Introduction

a. Sampling error and localization
Progress to improve the efficacy of ensemble data assimilation methods like the ensemble Kalman filter (EnKF) has been impeded by problems with sampling error. Sampling error arises when the number of ensemble members (N) is much less than the size of the state vector (n) (see e.g., Houtekamer and Mitchell 1998; Evensen 2003; Lorenc 2003). The true forecast error covariance matrix, $P$, can be estimated from an N-member ensemble as $P^{DE}(N) \in \mathbb{R}^{n \times n}$:

$$P^{DE}(N) = \frac{1}{N-1} \sum_{i=1}^{N} \delta x_i \delta x_i^T = \frac{1}{N-1} XX^T.$$  

Here the superscript DE stands for “dynamical ensemble,” $\delta x_i \in \mathbb{R}^n$ is the $i$th perturbation from the ensemble mean, and $X \in \mathbb{R}^{n \times N}$ is the matrix of ensemble perturbations $X = \{\delta x_1, \delta x_2, \ldots, \delta x_N\}$. Because of sampling error, this is only an estimate of $P$, and sampling errors are expected to be high when the true covariance (e.g., between elements $i$ and $j$, $P_{ij}$) is small (Houtekamer and Mitchell 1998). This is expected to be especially relevant in the far field, where the EnKF will yield anomalous analysis increments at points distant from each observation and will also lead to anomalous covariances in observation space through the term $HP^{DE}(N)H^T$.

It is difficult to distinguish genuinely large values of sample covariances from anomalously large ones that arise because of sampling error, but it is often possible to reduce sampling errors by damping covariances that are expected...
to be small. Points \( i \) and \( j \) that are separated by a large distance may be expected to be only weakly correlated, so these sample covariances should be damped or eliminated. This is the idea behind localization. For example, the elements \( [P^{DE}_{(N)}]_{ij} \) can be multiplied by a moderation function, \( \mu(r_i, r_j) \), which is unity when \( i = j \) and reduces to zero with separation between the locations \( r_i \) and \( r_j \) (Gaspari and Cohn 1999).

Localization can be applied in other ways, for example, by limiting observations that contribute to each grid point to those that fall within a localization radius (domain localization). This is used, for example, in the local ensemble transform Kalman filter (LEKF) (Ott et al. 2004) and is commonly used with the ensemble transform Kalman filter (ETKF) (Bishop et al. 2001) and the singular evolutionary interpolated Kalman filter (SEIK filter) (Pham 2001). Domain localization can give rise to loss of smoothness and so systems usually weight the observations visible to a particular analysis point according to the distance of each observation to that point in a similar way to the Gaspari and Cohn localization in model space (e.g., Houtekamer and Mitchell 2001; Hunt et al. 2007; Campbell et al. 2010; Jänic et al. 2011; Bowler et al. 2013; Holland and Wang 2013). This is problematic for observations whose observation operators are a function of a highly nonlocal part of the model space (such as measurements from radiometers) (Fertig et al. 2007; Bishop and Hodyss 2009b; Zhu et al. 2011). The focus of this paper is on the moderation function formulation (section 2) and how it affects geophysical balance properties of the prior ensemble. No actual data assimilation is done in this study.

b. The effect of localization on geophysical balances

The balance of the forecast ensemble can have an important influence on the analysis ensemble. For instance, consider an ensemble of background forecasts arranged to be in perfect linear balance and that have computed error covariance \( P^{DE}_{(N)} \). Since the ensemble is perfectly balanced, the unbalanced modes will lie in the null space of \( P^{DE}_{(N)} \) since those modes carry zero weight. In an unlocalized EnKF with a linear forward model \( \mathbf{H} \), that null space will then carry through to the computed error covariance matrix of the analysis ensemble, \( \mathbf{A} P^{DE}_{(N)} \), which can be seen with the covariance update equation

\[
\mathbf{A} P^{DE}_{(N)} = [I - P^{DE}_{(N)} \mathbf{H} (\mathbf{H} P^{DE}_{(N)} \mathbf{H}^T + \mathbf{R})^{-1} \mathbf{H}] P^{DE}_{(N)}
\]

The analysis ensemble will, therefore, also be perfectly balanced. In systems where the balanced and unbalanced modes are coupled [e.g., in nonlinear or slaved systems (e.g., Neef et al. 2006), or where the forecast ensemble is not in perfect balance], sampling error itself can affect the null space of \( P^{DE}_{(N)} \), so this balance link between forecast and analysis ensembles does not hold precisely in real systems. The argument though does serve to illustrate that balance in the background ensemble can be relevant to the entire analysis problem in unlocalized filters.

Introducing localization has an important bearing on this issue. Although localization helps to alleviate some of the sampling noise problems with the EnKF, it can severely modify the null space, which degrades the essential geophysical balance properties of the EnKF (Lorenc 2003; Kepert 2009; Greybush et al. 2011). This was demonstrated by Houtekamer and Mitchell (2005) who showed that anomalous rapid oscillations in surface pressure appeared in the (localized) EnKF analyses. In addition, representing balance incorrectly in an analysis system based on the Kalman update equations can also reduce the information synergy provided by different observations (Lorenc 1981). It is well known that inappropriate geostrophic imbalance in initial conditions is damaging to subsequent forecasts (Daley 1991) and so methods have been sought to reduce this effect. It is also natural to assume that inappropriate hydrostatic balance is also damaging. The simplest approach is to weaken the effect of the localization by choosing moderation functions with longer length scales (Houtekamer and Mitchell 2005), but this lessens the benefit of localization, and so requires more ensemble members. In addition, other factors, such as the information provided by observations, influence the localization length scales that should be used (Kirchgessner and Nerger 2014).

The distortion by the moderation functions affects mostly those variables that are strongly anisotropic (Kepert 2009) like \( u \) and \( v \) [the zonal and meridional winds; see e.g., Fig. 3 of Pailleux (1997)]. In Kepert (2009), the analysis is performed in terms of the more isotropic \( \psi \) and \( \chi \) (streamfunction and velocity potential) instead of \( u \) and \( v \), which was found to maintain a well-balanced ensemble. These are among the same variables that are localized in the Met Office’s hybrid data assimilation system (Clayton et al. 2013), namely, increments of \( \psi \), \( \chi \), \( p_u \) (unbalanced pressure), and \( \mu \) (relative humidity). In that system, balance is introduced explicitly by a balance operator [as used in the control variable transform of their system (Lorenc et al. 2000)], which gives increments of \( u \), \( v \), \( p \) (total pressure), \( \theta \) (potential temperature), and \( q \) (specific humidity).

These are successful methods of maintaining balance, but they have limitations. Approaches like those of Clayton et al. (2013) require that the imposed balances—in this case strong hydrostatic balance and weak geostrophic balance—are appropriate for the system. These are questionable for convective-scale flows where imposing balance may be damaging [e.g., it is known that hydrostatic and nonhydrostatic models differ most at small horizontal and large vertical scales (Davies et al. 2003) so deep convection may be missed if the initial
Table 1. Summary of the four types of ensemble used in this paper. The “correlation function ensemble” and “localized ensemble” are specified with either the spectral, SENCORP, or ECO-RAP schemes.

<table>
<thead>
<tr>
<th>Ensemble</th>
<th>Description</th>
<th>No. of members</th>
<th>Ensemble member</th>
<th>Ensemble matrix</th>
<th>Covariance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dynamical ensemble (DE)</td>
<td>The ensemble of perturbations from the dynamical forecast model (prelocalization)</td>
<td>$N$</td>
<td>$\delta x_i$</td>
<td>$X$</td>
<td>$P_{DE}^{(N)}$</td>
</tr>
<tr>
<td>Correlation function ensemble (CE)</td>
<td>The ensemble whose covariance gives the correlation matrix that localizes</td>
<td>$K$</td>
<td>$\omega_i$</td>
<td>$K$</td>
<td>$\Omega_{CE}^{(K)}$</td>
</tr>
<tr>
<td>Localized ensemble (LE)</td>
<td>The ensemble whose covariance gives the localized covariances</td>
<td>$M$</td>
<td>$\tilde{x}_m$</td>
<td>$\tilde{X}$</td>
<td>$P_{LE}^{(N,K)}$</td>
</tr>
<tr>
<td>Smoothed ensemble (SE)</td>
<td>A spatially smoothed version of the dynamical ensemble</td>
<td>$N$</td>
<td>$\delta w_l$</td>
<td>$-$</td>
<td>$C$</td>
</tr>
</tbody>
</table>

Traditionally, moderation functions are prescribed and do not change with the flow, but over recent years schemes have been proposed that generate moderation functions that do change with the flow. For instance, in Anderson (2012) the sample correlation (in his case between an observation point and each analysis point) is moderated by a factor that depends on the sample correlation. This can reduce filter error, but still increases imbalance. Another two adaptive schemes are Smoothed Ensemble Correlations Raised to a Power (SENCORP) (Bishop and Hodyss 2007) and Ensemble Correlations Raised to a Power (ECO-RAP) (Bishop and Hodyss 2009a). Simplified versions of SENCORP and ECO-RAP each have a special factorization (square root) property that we exploit in this paper to study the way that they affect the balance of a localized ensemble (the factorization property is discussed in section 2a). These adaptive schemes, plus a static localization scheme that has a similar factorization property, are studied. The performance of each scheme is evaluated in terms of (i) the structure of the moderation functions that are implied and (ii) the effect on the degree of balance. It is beyond the scope of this paper to run assimilation experiments with these cases, but the balance diagnostics shown are applied to ensemble forecasts from a full weather forecasting model and so should be useful to guide the choice of localization scheme for operational use.

The structure of this paper is as follows. In section 2 we review the Schur product to define our notation and describe each localization scheme considered. In section 3 we introduce the ensemble and a vertical test profile used to test the localization schemes. In section 4 we describe the diagnostics and show how each scheme localizes and affects balance. In section 5 we use the best of the schemes found and apply them to all horizontal locations in the domain of the model. In section 6 we discuss the results and limitations of the present work, conclude the paper, and suggest further work.

2. The localization schemes

In this section the Schur product localization is introduced, and one static and two adaptive variants of localization schemes are described. The Schur product method replaces $P_{DE}^{(N)}$ in the EnKF in (1) with a localized version, $P_{DE}^{(N,K)} = P_{DE}^{(N)} \cdot \Omega_{DE}^{(K)}$, where $\Omega_{DE}^{(K)} \in \mathbb{R}^{n_N \times n_N}$ is a moderation matrix formed of moderation functions {for a single field, $[\Omega_{DE}^{(K)}]_{ij} = \mu(r_i, r_j)$, where $[\mu(r_i, r_j)] \leq 1$, $\mu(r_i, r_j) = 1$, and where $[\cdot]_i$ indicates matrix element} (Gaspari and Cohn 1999). The (K) subscript represents the rank of the moderation matrix. The abbreviations for localized ensemble (LE) and correlation function ensemble (CE) are explained below, and all abbreviations are summarized in Table 1. The main issues with this method are in the choice of $\Omega_{DE}^{(K)}$ for each localization scheme, in how it is implemented, and in any undesirable side effects of the localization, such as damage to the filter’s geophysical balance properties.

The term “correlation ensemble” is used as $\Omega_{CE}^{(K)}$ is like a correlation matrix. The terms “dynamical ensemble” and “correlation function ensemble” are adopted from (Zhu et al. 2011).
a. Ensemble representation of (Schur product) localized covariances

The aim here is to develop a new ensemble (the LE mentioned above) for each scheme considered, whose covariance gives the localized matrix $P_{(N,K)}^{LE}$. The balance diagnostics will be derived from the LE. This is done by first finding yet another ensemble (the CE mentioned above) whose covariance is $\Omega_{(K)}^{CE}$. Table 1 summarizes these ensembles. The LE and the CE are found from the columns of the square root matrices of $P_{(N,K)}^{LE}$ and $\Omega_{(K)}^{CE}$, respectively.

1) THE CORRELATION FUNCTION ENSEMBLE

Just as $P_{(N)}^{DE}$ can be written as the outer product of $N$ ensemble members in (1), the moderation matrix, $\Omega_{(K)}^{CE}$, can be written in terms of $K$ new ensemble members:

$$\Omega_{(K)}^{CE} = \frac{1}{K - 1} \sum_{k=1}^{K} \omega_k \omega_k^T = \frac{1}{K - 1} KK^T,$$

(2)

where $\omega_k \in \mathbb{R}^n$ form the $K$ new CE members. The matrix $K \in \mathbb{R}^{n \times K}$ is the (square root) matrix of the $K$ CE members $K = \{\omega_1, \omega_2, \ldots, \omega_K\}$. Each localization scheme that we consider will be decomposed (or factorized) into its own CE. An arbitrary rotation, $R \in \mathbb{R}^{K \times K}$ can also be appended to $K$, that is, $(K \rightarrow RK)$, where $RR^T = I$. It is set to the identity in our experiments, but can be set to other valid rotations to study how different square roots perform. In fact repeating the diagnostics shown in the results part of this paper with arbitrary rotations (found with sequences of random Jacobi rotations) makes no difference to our diagnostics.

2) THE LOCALIZED ENSEMBLE

From (1) and (2) the $i,j$ th matrix elements of $P_{(N)}^{DE}$ and $\Omega_{(K)}^{CE}$ are, respectively,

$$[P_{(N)}^{DE}]_{ij} = \frac{1}{N - 1} \sum_{l=1}^{N} [\hat{\delta}x_l]_i [\hat{\delta}x_l]_j \quad \text{and}$$

$$[\Omega_{(K)}^{CE}]_{ij} = \frac{1}{K - 1} \sum_{k=1}^{K} [\omega_k]_i [\omega_k]_j,$$

(3)

where $[\cdot]$ indicates the vector element. The CE must have the property that $[\Omega_{(K)}^{CE}]_{ii} = 1$ [i.e., that the sum of squares (over the $K$ members) of each element of the CE must equal to $K - 1$]. From (3) the $i,j$ th element of $P_{(N,K)}^{LE}$ is

$$[P_{(N,K)}^{LE}]_{ij} = [P_{(N)}^{DE}]_{ij} [\Omega_{(K)}^{CE}]_{ij},$$

$$= \frac{1}{(N - 1)(K - 1)} \sum_{l=1}^{N} \sum_{k=1}^{K} [\hat{\delta}x_l]_i [\omega_k]_j [\hat{\delta}x_l]_j [\omega_k]_j,$$

$$= \frac{1}{(M - 1)} \sum_{m=1}^{M} \hat{x}_m \hat{x}_m^T,$$

(4)

where $\hat{x}_m = \sqrt{(M - 1)/((N - 1)(K - 1))}[\hat{\delta}x_l]_i[\omega_k]_j \in \mathbb{R}^n$ form a set of $M$ combined ensemble members, which are made from every possible Schur product pair of DE and CE members. These new members compose the LE. Here $M = NK$ and $m$ is shorthand for every pair of $l$ and $k$ that appears in the line above (4). The increase in the number of members from $N$ in the DE to $M$ in the LE will lead to $\text{rank}(P_{(N,K)}^{LE}) > \text{rank}(P_{(N)}^{DE})$, and thus have a lower sampling error. The importance of (4) is that any diagnostic that is applied to the DE may be applied also to the LE, including the balance diagnostics in section 4.

b. The static (spectral) localization scheme

1) SPECTRAL REPRESENTATION OF A LOCALIZATION MATRIX

A convenient way of representing the static scheme is to build the CE of weighted spectral functions. It is useful to define how this is done first for a single variable $s$ [where $s$ in this paper represents errors in either zonal wind ($\overline{u}$), meridional wind ($\overline{v}$), pressure ($\overline{p}$), or temperature ($\overline{T}$)]. For the static spectral scheme considered in this paper, the eigen-representation,

$$\Omega_{(K)}^{CE,s} = \frac{1}{K - 1} F_s \Lambda_s F_s^T,$$

(5)

is a useful way of finding a “square root,” $K_s$, of $\Omega_{(K)}^{CE,s}$ (for variable $s$). The diagonal matrix $\Lambda_s \in \mathbb{R}^{K \times K}$ comprises $K$ nonzero eigenvalues, and $F_s \in \mathbb{R}^{K \times n}$ comprises the $K$ eigenvectors. This is a rank $K$ matrix (in practice $K \ll n$), which corresponds to $K_s = F_s \Lambda_s^{1/2}$, which is this variable’s CE (it also needs to be normalized so that the sum of squares of each row is $K - 1$; see the next section).

2) THE HORIZONTAL AND Vertical BASES USED IN THIS WORK

Columns of $F_s$ comprise 3D fields that are the products of horizontal plane waves and vertical modes (see below), and diagonal elements of $\Lambda_s$ comprise a variance spectrum. The variance spectrum is a known function of the horizontal total wavenumber of the plane waves and of the vertical mode index, and is found using the procedure described below. The elements of $K_s$ are then

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3 Note that the scheme described here is denoted ‘‘spectral’’ because the localization matrix is represented in a spectral basis; it does not mean that localization has been done in spectral space. ECO-RAP also makes use of the spectral operators discussed here.
where the horizontal domain has dimensions \(L_x \times L_y\). The subscripts on \([K]_{ik}\) have the following meanings: \(r\) represents a 3D position \((r_x, r_y, r_z)\) and \(k\) represents a horizontal wavenumber \(k_x, k_y,\) and vertical mode index \(k_z (k_x, k_y, k_z \in \mathbb{Z})\). The term \(\nu(r_z, k_z)\) is the value of \(k_z\)th vertical mode at level \(r_z\), \(\lambda^H_{ik}(k_x^2 + k_y^2)\) is the square root of the horizontal part of the variance spectrum (a function of total horizontal wavenumber), and \(\lambda^V_{ik}(k_z)\) is the square root of the vertical part of the variance spectrum. A variance spectrum is found by projecting the normalized \(\Theta(r, k)\) (see section 2b(3)). The phases \(\delta^H_i\) and \(\delta^V_j\) are chosen with the intention of allowing \(s\) to satisfy some imposed lateral boundary conditions consistent with the limited-area model that produced the ensemble. For wind components \(\delta^H_{1u} = -\pi/2, \delta^V_{1u} = 0, \delta^H_{2u} = 0, \delta^V_{2u} = -\pi/2\), which prohibit flow in or out of the domain. The remaining variables have

\[
\begin{align*}
\mu^H(r_{H_i}, r_{H_j}) & = \exp - \left( |r_{H_i} - r_{H_j}| \ell_H^2 \right)^2 \\
\mu^V(r_{z_i}, r_{z_j}) & = \exp - \left( |r_{z_i} - r_{z_j}| |\ell_V| \right)^2 
\end{align*}
\]

Here \(\ell_H\) and \(\ell_V\) are the horizontal and vertical length scales, respectively; \(r_{H_i} = (r_x, r_y)\); and \(r = (r_x, r_y, r_z)\).

Equation (7) describes the horizontal and vertical moderation functions approximated by the spectral localization scheme (they are also used in the ECO-RAP scheme, but not in SENCORP). The \(\lambda^H_{ik}\) spectrum in (6) is found by projecting \(\mu^H\) onto the plane waves (i.e., a Fourier transform) and then square rooting, and the \(\lambda^V_{ik}\) spectrum is found by projecting \(\mu^V\) onto the vertical modes (and square rooting). The value of \(K\) for the spectral scheme is the product of the number of horizontal plane waves with the number of vertical modes used. A localization scheme that is based on (6) will not perfectly recover forms in (7) because of two reasons. The first reason is the presence of the normalization discussed in section 2a(2), and the second reason is the high degree of truncation \((K \ll n)\). For this reason the realized length scales in the moderation functions will be different to those specified.

4) THE COMPLETE STATIC (SPECTRAL) SCHEME

Equation (5) forms the basis of a univariate static localization scheme where the moderation functions

\[
\delta^H_i = -\pi/2 \text{ and } \delta^V_j = -\pi/2, \text{ which represent Dirichlet boundary conditions. These conditions, however, are not seen in the correlations implied by (5) because of the normalization mentioned in section 2a(2). Equation (6) is valid for a continuous system and technical adjustments to (6) are needed to respect the staggering on the Arakawa C grid used for this study (not shown).}

The vertical modes are mutually orthogonal. Any reasonable basis set may be used, and we use eigenvectors of the vertical error covariance matrix for the unbalanced pressure control variable from a version of the Met Office variational data assimilation system (Lorenc et al. 2000; Bannister 2008). The first four vertical modes are plotted in Fig. 1. We have not studied the effect of using another basis, although this could be done in further work.

3) THE HORIZONTAL AND VERTICAL VARIANCE SPECTRA

To derive the horizontal and vertical variance spectra, \(\lambda^H_{ik}\) and \(\lambda^V_{ik}\), we first define the moderation function that we want the spectral scheme to represent:

\[
\mu^H(r_{H_i}, r_{H_j}) = \mu^H(r_{H_i}, r_{H_j}) \times \mu^V(r_{z_i}, r_{z_j}) 
\]
are parameterized as defined in (6) (we call the scheme “static” because the spectra do not adapt with the flow). The extension to multivariate static localization schemes is necessary to study balance. The scheme is summarized with the following form of the moderation ensemble:

\[
K_{\text{Spec}} = 
\begin{pmatrix}
F_{\delta s} A_{2s}^{1/2} \\
F_{\delta v} A_{s2}^{1/2} \\
F_{\delta p} A_{sp}^{1/2} \\
F_{\delta T} A_{T2}^{1/2}
\end{pmatrix}
\in \mathbb{R}^{n \times K},
\]

(8)

\[
[K_{\text{Spec}}]_{(r)} = [\omega]_{(r)} = c_{(r)} [F_s]_{(r)} [A_{s2}^{1/2}]_{kk}.
\]

(9)

The overbar in (8) indicates that the matrix is normalized such that the sum of squares of each row must be \( K - 1 \) [see section 2a(2)], which is accounted for by the normalizing factor \( c_{(r)} \) in (9). The localization matrix, \( \Omega_{(K)} \), implied by (8) has an autocorrelation matrix for variable \( s \) of the form \( F_s A_{s2}^{1/2} F_s^T \), and cross-localization submatrices between variables \( s_1 \) and \( s_2 \) of the form \( F_{s1} A_{s2}^{1/2} A_{s2}^{1/2} F_{s2} \) (both ignoring normalization).

c. The adaptive localization schemes

The nonadaptive localization scheme is based on the spectral method described in section 2b(4), which does not change with the flow. The SENCORP and ECO-RAP-based schemes on the other hand are flow dependent. SENCORP is itself based purely on the ensemble, and ECO-RAP is based on a combination of the ensemble and the spectral method.

1) THE SIMPLIFIED ADAPTIVE SENCORP LOCALIZATION SCHEME

In the simplified version of the “order \( Q \)” SENCORP scheme (Bishop and Hodyss 2007) that is studied in this paper, the moderation matrix is taken to have the following form\(^6\):

\[
\Omega_{(K)}^{\text{CE}} = C^Q.
\]

(10)

where \( C^Q \) means the “Schur power” (the Schur product of \( Q \) \( C \) matrices where \( Q \) is a positive even integer). The matrix \( C \in \mathbb{R}^{n \times n} \) is the correlation matrix formed from \( N \) spatially smoothed versions of the original ensemble members, \( \delta w_l \) \((1 \leq l \leq N)\), and are collectively called the smoothed ensemble (SE; in Table 1). The members are normalized after smoothing such that the variance of each element amongst the SE is 1 to ensure that the diagonal

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\(^6\) The full SENCORP localization matrix in Bishop and Hodyss (2007) is \([C^Q]^T\)’s, but this is too difficult to factorize for the large state space studied in this paper. Our simplified application is the case \( q = 1 \) and \( r = 1 \).
elements of $\mathbf{C}$—and hence $\mathbf{C}^{-Q}$—are unity. The matrix $\mathbf{C}$ is then found from $\mathbf{C} = \frac{1}{N-1} \sum_{i=1}^{N} \delta \mathbf{w}_i \delta \mathbf{w}_i^T$. A square root of $\Omega_{i(k)}^{\text{CE}}$ may be formed as a generalization of (4). The $i, j$th element of $\Omega_{i(k)}^{\text{CE}}$ is

$$
[\Omega_{i(k)}^{\text{CE}}]_{ij} = (\mathbf{C}_{ij})^Q = \left( \frac{1}{N-1} \right)^Q \sum_{l_1=1}^{N} \cdots \sum_{l_{Q}=1}^{N} [\delta \mathbf{w}_{l_1}]_i [\delta \mathbf{w}_{l_1}]_j^Q,
$$

where the $k$th SENCORP CE member is $\omega_k = \sqrt{(K-1)/(N-1)^Q} \omega_{lQ}$. $K = N^Q$ is the number of SENCORP moderation members, and $k$ is shorthand for every combination of $l_1, \ldots, l_{Q}$ that appears in the line above (11). The matrix containing the CE members is denoted $\mathbf{K}^{\text{SENCORP}} \in \mathbb{R}^{n \times N^Q}$. The smoothing step is performed to give correlation length scales in $\mathbf{C}$ that are longer than those in $\mathbf{P}^{\text{Spec}}$ and the degree of smoothing is chosen by experimentation.7 The Schur power of $Q$ acts to reduce correlations most among the SE that are small (these correlations arguably should be zero, but may not be zero because of sampling noise), but maintains correlations that are close to $\pm 1$. To preserve the sign of the sample covariances during the localization, $Q$ must be even. The definition of SENCORP’s CE in terms of the ensemble is the key adaptive feature of this method. Once the CE is found, the LE follows from (4).

2) THE ADAPTIVE ECO-RAP LOCALIZATION SCHEME

Although the definitions of the acronyms SENCORP and ECO-RAP (section 1c) may suggest that the difference between these two schemes is the inclusion of smoothing in SENCORP, this is not the defining difference. In fact, ECO-RAP is more like a mixture of the SENCORP and spectral approaches. The matrix $\mathbf{K}^{\text{ECORAP}}$ for the order-$Q$ ECO-RAP method Bishop and Hodyss (2009a) is

$$
\mathbf{K}^{\text{ECORAP}} = \mathbf{C}^{-Q} = \begin{pmatrix}
\mathbf{F}^{-1/2} A_{0}^{-1/2} A_{0}^{1/2} \\
\mathbf{F}^{-1/2} A_{0}^{-1/2} A_{0}^{1/2} \mathbf{F}^{-1/2} A_{0}^{-1/2} A_{0}^{1/2} \\
\mathbf{F}^{-1/2} A_{0}^{-1/2} A_{0}^{1/2} \mathbf{F}^{-1/2} A_{0}^{-1/2} A_{0}^{1/2}
\end{pmatrix} \in \mathbb{R}^{n \times K},
$$

where $\mathbf{C}^{-Q}$ is defined in section 2c(1). The number of CE members (the number of columns in $\mathbf{K}^{\text{ECORAP}}$) is the same as in $\mathbf{K}^{\text{Spec}}$, which depends on the number of horizontal plane waves and vertical modes. The matrix elements of (12) are

$$
[\mathbf{K}^{\text{ECORAP}}]_{i(r;k)} = [\omega_k]_{i(r)} = c_{(r)} \sum_{r,s} (\mathbf{C}^{-Q})_{i(r)s} \mathbf{F}_{(r,s)k} (A_{0}^{-1/2})_{kk}.
$$

3. THE CASE STUDY AND THE TEST PROFILE

The test ensemble comprises a 24-member ensemble of 3-h forecasts from a high-resolution weather forecasting model [the Met Office’s 1.5-km nowcasting model with a domain over the southern United Kingdom (Golding et al. 2014)]. The ensemble’s analysis perturbations are produced by the Met Office Global and Regional Ensemble Prediction System (MOGREPS) (Bowler et al. 2008) and adapted for this domain (Migliorini et al. 2011; Caron 2013; Baker et al. 2014). The MOGREPS perturbations are found by running an ETKF (Bishop et al. 2001) for the convective-scale domain. The system uses hourly cycling, and ETKF perturbations are added to the analysis from a variational system gradually using an incremental analysis update (IAU) over a time period from $T - 0.5$ to $T + 0.5$ h of the analysis. No localization is used in this particular configuration. Further details are found in the references above. The date chosen for the test is 20 September 2011, when a cold front passed over the southern United Kingdom. This case is interesting as the flow has deviations from geostrophic and hydrostatic balances and so should provide a good test of the localization schemes at the convective scale. Details of this case are documented in Baker et al. (2014). A particular location in the domain is chosen for a test profile (52.5°N, 2.3°W), which is precipitating and shows a detectable deviation.

7 In the limit of infinitely long length scales in $\mathbf{C}$, SENCORP will revert to the unlocalized system.
from hydrostatic balance. This point is indicated by a
crosshair in forthcoming figures.

4. Diagnosed correlation functions and balance
properties

The experiments performed for this work are listed in
Table 2. The free parameters that are explored include
the degree of truncation (via $N_{k_x}$ and $N_{k_z}$), the implied horizontal
and vertical length scales [$\ell_H$ and $\ell_V$ in (7)], the
degree of ensemble presmoothing, the order ($Q$), and
the influence parameters (to be described in section 4d). Not
all parameters are relevant to all schemes. There are two
different kinds of diagnostics shown in this paper.

1) Spatial correlation functions (univariate and multivar-
iate) are computed between a selection of variables
calculated from members of the DE giving $P_{(N)}^{DE}$ in (1),
the LE giving $P_{(N,K)}^{LE}$ in (4), and the CE giving $C_{(K)}^{CE}$ in
(3). These show how strong the degree of localization is
and how the schemes deal with multivariate aspects.
We have computed a large number of correlation functions
for each scheme, but we show only a small selection of these to demonstrate the important points.

2) Balance diagnostics that measure the degree of geo-

Figure 2 shows some correlation functions calculated
from the DE associated with the test profile at model
level 19 (~1.3 km over the sea). This corresponds to
experiment 0 in Table 2. The first row (Figs. 2a–c) are
$6T - 5T$ spatial correlations on three cross sections
calculated from the DE associated with the test profile at model
level 19 (~1.3 km over the sea). This corresponds to
experiment 0 in Table 2. The first row (Figs. 2a–c) are
$6T - 5T$ spatial correlations on three cross sections
calculated from the DE associated with the test profile at model
level 19 (~1.3 km over the sea). This corresponds to
experiment 0 in Table 2. The first row (Figs. 2a–c) are
$6T - 5T$ spatial correlations on three cross sections
calculated from the DE associated with the test profile at model
level 19 (~1.3 km over the sea). This corresponds to
experiment 0 in Table 2. The first row (Figs. 2a–c) are
$6T - 5T$ spatial correlations on three cross sections
through the domain (the correlations are between $6T$ at
each position in the field and $5T$ at the crosshair, blue is
negative correlation, and red is positive correlation). At
the validity time of the ensemble there is a cold front
passing over the United Kingdom from the west and
orientated along the southwest–northeast direction
(Baker et al. 2014). This is reflected in the anisotropy
evident in Fig. 2a. Apart from the region southeast of the

### Table 2. Summary of experiments. The variables $N_{k_x}$ and $N_{k_z}$ refer to the number of wavenumbers used in the x and z directions, respectively (with $N_{k_x} = N_{k_z}$). The total number of horizontal wavenumbers, $N_k$, is found by counting the number of horizontal wavenumbers satisfying $\sqrt{k_x^2 + k_z^2} \leq N_k$. For comparison with $K$ (the number of correlation function ensemble members), the number of grid points representing one field is $360 \times 288 \times 70 \times 7 \times 10^6$. Here $\ell_H$ and $\ell_V$ are the length scales specified in (7), which are used to compute the horizontal and vertical spectra in (6). “Horizontal” and “vertical” refer to the number of grid points (in the horizontal and vertical directions, respectively) in which the ensemble is presmoothed for the adaptive schemes. $Q$ is the order of the SENCORP/ECO-RAP schemes, and $\rho_H$ and $\rho_V$ refer to the ECO-RAP influence parameters.

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front, the sign of the correlations is positive in the horizontal, but there are some far-field features that may be due to sampling error. The vertical correlations (Figs. 2b,c) show mainly positive correlations up to level 45 (≈7 km) and have bands of correlations with an alternative sign above that. The local correlation structure shows patterns that slope with longitude and latitude. Some vertical correlations, especially those involving $\delta T$ and $\delta p$, may have legitimate far-field correlations (which arise due to “action at a distance” effects caused by hydrostatic balance for instance) so the effect of localization in the vertical requires special attention.

The second row (Figs. 2d,e) are $\delta u$--$\delta p$ spatial correlations (the correlations are between $\delta u$ at positions in the field with $\delta p$ at the crosshair). On level 19, there are mainly positive correlations to the north and negative correlations to the south of the crosshair, which is the large-scale pattern expected from geostrophic balance (Pailleux 1997; Bannister et al. 2011), although the pure geostrophic pattern is highly disturbed by the front. The vertical cross sections (Figs. 2e,f) show large-scale and small-scale structures, but have a clear band of negative correlation between levels ≈45 and ≈55 (from ≈7 to ≈11 km). These features are examples of the structures that an ensemble brings to data assimilation although the values in the far field may be noise.

The balance diagnostics corresponding to the vertical profile at the horizontal crosshair position are shown in Figs. 3a and 3b (black lines). Recall that perfect balance is indicated by a $\mathcal{M}'$--$\mathcal{W}'$ or $\mathcal{P}'$--$T'$ correlation of −1. The geostrophic covariance diagnostic (Fig. 3a) shows that geostrophic balance is not well obeyed in the DE, apart from around level 19 (this is the reason for choosing this level in Fig. 2). The low degree of geostrophic balance is perhaps not surprising given that we are measuring geostrophic balance at the grid scale and that a front is passing through. The hydrostatic diagnostic (Fig. 3b) shows that hydrostatic balance is almost perfectly satisfied (values are only just distinguishable from −1 and we note the different scales between the two panels). There are slight deviations around levels 20 (≈1.5 km), 28 (≈2.8 km), and 37 (≈4.8 km). These balance diagnostics for the unlocalized system are considered to be the target values, which we would like preserved after localization.

**Fig. 2.** Univariate and multivariate spatial correlation functions with no localization (experiment 0). Correlations of (a)–(c) temperature at field locations with temperature at the crosshair and (d)–(f) zonal wind at field locations with pressure at the crosshair. The cross sections correspond to the crosshairs. Positive (negative) correlations are red (blue) and have continuous (dotted) contours, but only every other contour is marked. Coastlines are drawn in yellow.

b. Diagnostics of the static (spectral) scheme

The implied spatial moderation functions for $\delta T$--$\delta T$ and $\delta u$--$\delta p$ found from the CE for the spectral scheme using a modest number of wavenumbers ($N_{k_x} = 9$ and $N_{k_z} = 5$) are shown in Figs. 4a–f. This corresponds to experiment 1 in Table 2. These fields are selected columns of the matrix $\Omega^{CE}_{(K)}$ implied by the spectral scheme [i.e., $\Omega^{CE}_{(K)} = \mathbf{K}^{Spec}K^{Spec} / (K - 1)$], where $K^{Spec}$ is defined by (8).
The moderation functions are smooth fields with length scales that vary on the scale of the domain. The value of this function at each position informs the assimilation how much to damp the sample correlations found from the DE. For example, the sample correlation between $d_T$ at the crosshairs with $d_T$ at any point in the field in Figs. 2a–c is modified by multiplying by the field values in Figs. 4a–c to give Figs. 4g–i. Furthermore, the sample correlation between $d_p$ at the crosshairs with $d_u$ at any point in the field in Figs. 2d–f is modified by multiplying by the field values in Figs. 4d–f to give Figs. 4j–l.

The moderation functions are intended to have the form of (7). If the spectral scheme reproduced these functions perfectly then the correlation functions for univariate correlations (Figs. 4a–c), would be isotropic in the horizontal (in terms of grid points rather than longitude and latitude) and would be decreasing monotonically away from the crosshair to zero (in the vertical and in the horizontal). This is clearly not the case, as is highlighted by the anisotropy in Fig. 4a and the presence of negative values in all panels. These shortcomings are consequences largely of the presence of the position-dependent normalization factor $c(r_s)$ in (9). Without $c(r_s)$ the moderation functions for $\delta T$–$\delta T$ would, for instance, be zero on the lateral boundaries [see section 2b(2)], and would be less anisotropic, but this factor is needed to make the covariance of the CE, $\Omega_{CE}$, a correlation function. Another effect is the low-rank CE (290 modes in this case, which is much less than the number of grid points representing one field $\sim 7 \times 10^6$). The negative lobes can be reduced in magnitude only marginally though by increasing the number of wavenumbers as in experiment 2 in Table 2 (not shown).

The implied moderation submatrix for $\delta T$–$\delta T$ has the form $F_{\delta T} \Lambda_{\delta T} F_{\delta T}^T$ [section 2b(4)], but for the different variables (e.g., $\delta u$–$\delta p$), it has the form $F_{\delta u} \Lambda_{\delta u}^{1/2} \Lambda_{\delta p}^{1/2} F_{\delta p}^T$. In this work the spectra represented by $\Lambda_{\delta u}^{1/2}$ and $\Lambda_{\delta p}^{1/2}$ are the same, but the horizontal basis functions are different [via the phases $\delta_{\delta u}$ and $\delta_{\delta p}$; see section 2b(2)]. To illustrate this, Figs. 4d–f show the slight difference that this phase difference can make to the implied spatial moderation functions.

The localized spatial correlations (i.e., from the LE) of experiment 1 are shown in Figs. 4g–l, which are equivalent to multiplying Figs. 2a–f by Figs. 4a–f. The correlations maintain the broad structures from the DE in Fig. 2, but are, by design, more compact. The selection of localized multivariate correlations keeps their local structures, which is essential for the localized system to have any chance of maintaining balance.

For reference, Figs. 3c–e (red lines) are the localized correlation functions for $\delta T$–$\delta T$ in the longitude, latitude, and vertical directions along the crosshairs, which show more clearly how they differ from the unlocalized correlation functions (black lines). The two are similar in the horizontal (Figs. 3c,d) within a degree or so from the crosshair (where the correlation is unity), but the
Localization tends to be stronger (and becomes negative) in the southerly direction than in the northerly direction (Fig. 3d). The two lines are similar in the vertical (Figs. 3e) within about 15 levels of the crosshair, but there is virtually no localization from level 19 downward, reflecting the structure of the first vertical basis function shown in Fig. 1.

The balance diagnostics for experiment 1 are overlaid in Figs. 3a and 3b (red lines). For the geostrophic balance diagnostic, the localized ensemble follows closely the unlocalized ensemble, which is an encouraging result, but perhaps not surprising given that \( \ell_H \) is reasonably large (~200 grid points, or ~300 km, Table 2). When the horizontal length scale, \( \ell_H \) is reduced to 100 grid points (experiment 3, in Table 2), the close agreement lessens as expected (not shown). The spectral scheme though dramatically loses the hydrostatic balance present in the unlocalized system. This is a surprising result given that the unlocalized and localized
correlation functions are virtually indistinguishable in the vicinity of the crosshair (Fig. 3e), but evidently the vertical derivatives used in the hydrostatic balance diagnostic are different. Using this scheme to localize in a convective-scale ensemble-based data assimilation scheme may therefore anomalously induce convective storms.

c. Diagnostics of the adaptive SENCORP scheme

The spectral scheme uses the same moderation functions, irrespective of the flow regime. SENCORP on the other hand constructs moderation functions that are determined purely from the DE. The simplified SENCORP scheme has a number of parameters that can be adjusted, namely, the order, $Q$, and the degree of smoothing of the DE members to make the SE, which in turn make the CE as in (11). As for the spectral scheme in section 4b, we examine the appearance of the moderation functions and the effect that the scheme has on balance.

1) EFFECT OF THE ORDER, $Q$

Figures 5 and 6 show the same selection of spatial moderation functions as in section 4b, but for SENCORP with $Q = 2$ (experiment 4) and $Q = 4$ (experiment 5), respectively (no presmoothing of the DE is performed at this stage). Because of computer memory limitations the $Q = 4$ results use only 16 instead of 24 members.\footnote{For $Q = 4$ and $N = 16$ (experiment 5), $K = 65536$, but for $Q = 4$ and $N = 24$, $K = 331776$, which hits our computer’s memory limits.}

The 14th column of Table 2 shows the number of members, $K$ in the LE. Figures 5a–f can be compared with the same panels in Fig. 4 for the spectral case (for brevity we do not show the SENCORP localized spatial correlations). The moderation functions do not have the smooth structure of the spectral scheme because they are computed directly from the ensemble, but they have no anomalous negative correlations.

The localization effect for SENCORP has mixed severity compared to the spectral scheme. For $Q = 2$ (Fig. 5) the degree of localization for temperature (Figs. 5a–c) is more severe in most regions. An exception is the positive correlation feature in the southwest of the domain of Fig. 5a. SENCORP decides to damp correlations where they are weak (according to the “flow of the day”), rather than how far they are, and this feature has strong correlations. Another example is the multivariate $\delta u - \delta p$ correlation in the far south of the domain in Figs. 5d–f, even though the multivariate correlations are virtually zero close to the crosshair. Increasing the order to $Q = 4$ (but $N = 16$, Fig. 6) reproduces similar patterns as $Q = 2$, but with much tighter localization. This is how SENCORP is intended to work with increasing $Q$.

The localized spatial correlation functions for $\delta T - \delta T$ along the crosshairs are shown in Figs. 7c–e, which shows the localizing effect of SENCORP experiments 4 and 5 (solid red and blue lines, respectively) against the unlocalized correlations (black lines). All lines differ in the vicinity of the crosshair (where correlations are close to
unity) more so than for the spectral scheme in the corresponding panels of Fig. 3. As shown in Figs. 7a and 7b, any geostrophic and hydrostatic balance properties of the LE are virtually destroyed in these SENCORP experiments (more so for \( Q = 4 \) than for \( Q = 2 \)). This may be due to the tight localization implied by this scheme. In the case of hydrostatic balance, although SENCORP severely reduces the degree of balance (correlation values are much higher than the target unlocalized values of \( -1 \)), the SENCORP values do vary in phase with the unlocalized values (i.e., when the black curve deviates most from \(-1\), so do the SENCORP curves). Overall, these SENCORP balance properties are unsatisfactory for data assimilation.

2) EFFECT OF PRESMOOTHING

The full SENCORP scheme in Bishop and Hodys (2007) has more freedom to influence the length scales than the simplified SENCORP (e.g., by adjusting the parameter \( q \) mentioned in footnote 6). It may be argued that the unsatisfactory balance properties of experiments 4 and 5 may be due to SENCORP localizing too much in the vicinity of the crosshairs. Two further SENCORP experiments are performed (both have \( Q = 2 \)) by first presmoothing the DE over 10 grid points in the horizontal and vertical directions (experiment 6) and then over 50 grid points (experiment 7). The presmoothing does reduce the degree of localization, as shown in Figs. 7c–e (dotted red line for experiment 6), which becomes indistinguishable from the unlocalized correlation functions for experiment 7 (not shown). Compared with the unsmoothed case (experiment 4), the balance diagnostics are generally improved for experiment 6 (Figs. 7a,b, dotted red line), but still inadequate for data assimilation. The balance diagnostics are worse for experiment 7 (not shown).

d. Diagnostics of the adaptive ECO-RAP scheme

The evaluation of the matrix of CE members for ECO-RAP in (12) is more expensive in computer time than comparable spectral and SENCORP schemes. Equation (12) requires the computation of \( C^Q \in \mathbb{R}^{n \times n} \) acting on a matrix that is \( \in \mathbb{R}^{n \times k} \), which is a prohibitive task for typical \( n \). To make ECO-RAP practical for the test cases used in this paper, some approximations to (12) are made. The full evaluation of \( K_{\text{ECORAP}} \) [adapted from (13)] is

\[
[K_{\text{ECORAP}}]_{(s)} = [\omega_k]_{(s)} = \sum_{s_1=1}^{n_1} \sum_{s_2=1}^{n_2} \sum_{s_3=1}^{n_3} (C^Q)_{(s_1)(s_2)(s_3)} F_{(s_1)s_2}^{(s_3)} (A_s^{1/2})_{kk},
\]  

(14)
where \( r' = (r'_x, r'_y, r'_z) \); \( s' \) and \( s'' \) represent variables; and the domain has \( n_x, n_y, \) and \( n_z \) points in the \( x, y, \) and \( z \) directions, respectively. Assuming that short-range correlations in \( C^{Q} \) are the most important, (14) can be approximated by

\[
[K^{\text{ECORAP}}]_{(r)(s)} = [\omega_{(r)(s)}] \approx \tilde{c}_{(r)} \sum_{s'} r'_x \rho_H r'_y \rho_H r'_z \rho_V (C^{Q})_{(r)(s')} F_{(r')(s'')} (A^{1/2}_{(r)} A^{1/2}_{(s)})_{kk},
\]

where \( \rho_H \) is the maximum distance in both horizontal directions of correlations considered in \( C^{Q} \), and \( \rho_V \) is the maximum distance of the vertical correlations considered. In this paper we will call these the ECO-RAP influence parameters. Equation (15) is used in this paper, so a number of experiments are performed with different values of \( \rho_H \) and \( \rho_V \) (12th and 13th columns in Table 2).

1) EFFECT OF THE ECO-RAP INFLUENCE PARAMETERS

It was found by experimentation that the degree of geostrophic balance exhibited by the LE produced by ECO-RAP degrades with increasing horizontal ECO-RAP influence parameter (not shown). In our experiments we set \( \rho_H = 0 \) so we are left with a scheme that is like the spectral scheme in the horizontal and like the ECO-RAP scheme in the vertical and is a significant saving of computer effort.\(^9\) It is not clear from our results why \( \rho_H \) should have such an effect on geostrophic balance, but it may be due to ensemble noise in \( C^{Q} \) degrading the (already reasonable) spectral scheme. This subsection is concerned with the effect of \( \rho_V \) only [spectral parameters are as in experiment 1, the order is \( Q = 2 \), and only a small amount of pre-smoothing (two units) is done in the horizontal and vertical directions].

Figure 8 shows the localized balance and spatial correlation diagnostics for \( Q = 2 \) for three values of \( \rho_V \)

\(^9\)Note though that even with \( \rho_H = 0 \) and \( \rho_V = 0 \) the scheme would not be identical to the spectral scheme because of the multivariate effect that \( C^{Q} \) has at each location.
(experiments 8–10, red, blue, and solid green lines, respectively), compared with the unlocalized (experiment 0, black line) results. In terms of geostrophic balance (Fig. 8a), ECO-RAP appears to perform reasonably. Although it is not as close to the target correlations as the spectral scheme (Fig. 3a), it shows similar patterns of behavior in the vertical. The case with \( \rho_V = 2 \) (red line) performs better than cases for other values of \( \rho_V \). The significant gain from ECO-RAP though is found in the hydrostatic balance diagnostics (Fig. 8b) for the higher values of \( \rho_V \) shown: \( \rho_V = 24 \) (experiment 9, blue line) and \( \rho_V = 32 \) (solid green line). Broadly speaking we find that, up to a point, the larger the value of \( \rho_V \) the closer the hydrostatic balance diagnostics are to the target values (although results are poor around levels 45 and 50). The test with \( \rho_V = 2 \) gives similar (poor) hydrostatic results to the spectral scheme (Fig. 3b, red line). This is a result that might be expected since the larger \( \rho_V \) is, the more flow dependent the scheme is and the farther away the scheme is from being purely spectral. The horizontal localized correlation functions (Figs. 8c,d) show similar behavior to the spectral case (see Figs. 3c,d), which might be expected given \( \rho_H = 0 \) (it is not identical to the spectral scheme because of mixing of information between vertical levels, and between variables in ECO-RAP).

The effect of \( \rho_V \) on the degree of vertical localization is mixed (Fig. 8c). For instance, around level 40 the experiment with \( \rho_V = 2 \) localizes to a greater degree than the other ECO-RAP experiments, but around level 61 this is reversed. The vertical influence parameter value tested that gives good results with respect to both balances and with respect to its effectiveness to localize is judged to be \( \rho_V = 32 \) (experiment 10). Parameter \( \rho_V \) can be increased further, and the largest possible values lead to the unapproximated ECO-RAP scheme in the vertical. Large values do have a negative impact on geostrophic and hydrostatic balances though, and can be less effective at localizing. For \( \rho_V = 64 \) (\( \rho = 2 \)) for instance, the localized \( \delta T \sim \delta T \) correlation functions are indistinguishable from the unlocalized functions, but for \( \rho = 4 \) there is a clear localization effect (not shown). Unfortunately we do find that for \( \rho_V = 64 \) this \( \rho = 4 \) experiment performs worse than the \( \rho = 2 \) experiment at maintaining both balances.

Although the original reason for introducing the two ECO-RAP influence parameters was for computational efficiency, their presence has shown how the ECO-RAP
scheme can be tuned. We believe that these findings are potentially important for possible use in convective-scale ensemble data assimilation.

2) Effect of the order, Q

Maintaining the value, \(\rho_V = 32\), from section 4d(1), we now investigate the effect of changing the order of the scheme. Figure 8 shows the results for \(Q = 4\) (experiment 11, dotted green line). There is some benefit to increasing \(Q\): it does improve slightly the degree of localization over the equivalent experiment with \(Q = 2\) (experiment 10, solid green line), especially in Fig. 8d, and it gives similar (sometimes better, sometimes worse) performance for the balance diagnostics in Figs. 8a and 8b. The main improvement is around level 19 for geostrophic balance, but this is not found to be a consistent improvement at other locations. For this reason we take experiment 10 (\(Q = 2\)) as the best result to examine in section 5.

An analog of Fig. 2, but for experiment 10 is Fig. 9. The \(\delta T - \delta T\) moderation functions (Figs. 9a–c) have a similar structure to those of the spectral scheme in Figs. 4a–c, but are clearly modified by the ensemble. One improvement from the spectral scheme is the reduced number of oscillations in the vertical. Additionally the localization is not as tight as produced by SENCORP (Figs. 5 and 6). There are still negative values, however, and there are jumps in the magnitude of the moderation functions toward the north and northeastern boundaries, where values \(\rightarrow -1\). These jumps could be damaging to far-field localized correlations, especially as they could induce large gradients in analysis increments. The \(\delta u - \delta p\) functions (Figs. 9d–f) also have a similar structure to those of the spectral scheme and, unlike SENCORP, are not close to zero at the crosshair. These \(\delta u - \delta p\) functions do not have the problem at the boundaries like the \(\delta T - \delta T\) functions.

5. Beyond single profiles

The balance diagnostics shown in section 4 are for a single profile. In this section we show the same balance diagnostics, but for the horizontal domain at one level (level 19). Figure 10 comprises plots of the geostrophic (left panels) and hydrostatic (right panels) balance diagnostics for the same case discussed in section 3. Figures 10a and 10b are found from the DE (experiment 0, no localization) and the next three rows are found from the LE of configurations of the spectral (experiment 1), SENCORP (experiment 4), and ECO-RAP (experiment 10) schemes that are judged to be the best of each. The darker the shading, the more balanced the ensemble (correlation \(\rightarrow 1\)). The balance diagnostics are computed at the grid scale where the DE has regions that are geostrophically balanced and others that are unbalanced (Fig. 10a). The most balanced region at this level is to the west of the Bristol Channel. The DE is though strongly hydrostatically balanced everywhere (Fig. 10b), but slightly less for the band stretching from the southwest to the northeast of the domain (along the
Fig. 10. Maps of (a),(c),(e),(g) geostrophic and (b),(d),(f),(h) hydrostatic correlation diagnostics calculated for level 19 for (a),(b) the unlocalized system; (c),(d) the spectral scheme; (e),(f) SENCORP; and (g),(h) ECO-RAP. Regions with darker shading are more balanced than those with lighter shading and the square encloses the location of the profiles studied in earlier figures. In (b), the white outline has been added to show the region of slightly relaxed hydrostatic balance (the difference in shading is otherwise not visible on printed versions of this paper). The RMS values shown in (c)–(h) are the RMS differences between each diagnostic and the target values averaged over the domain for this level. The horizontal localization values shown in (c),(e), and (g) are the percentage of the localized to unlocalized $\partial T/\partial \zeta$ horizontal length scales (measured as the full-width–half-maximum and averaged in the longitudinal and latitudinal directions).
cold front).\textsuperscript{10} As before, these DE diagnostics are regarded as the target values that we would like the LE of each scheme to reproduce. As a measure of the closeness of each scheme to the target values, the root-mean-square (RMS) differences between the balance diagnostics of each scheme’s LE and the target are given in each panel. In addition, as a measure of the ability of each scheme to localize spurious correlations, estimates of the percentage reductions in horizontal length scales of $\delta T - \delta T$ correlations are also given (on the left-hand panels).

For geostrophic balance the spectral scheme (Fig. 10c) shows patterns that agree well with the target values and this is the best scheme according to the RMS difference. For hydrostatic balance though (Fig. 10d) the spectral scheme is the worst according to the RMS difference. These results are consistent with the profiles in Fig. 3. The SENCORP scheme gives both geostrophic (Fig. 10e) and hydrostatic (Fig. 10f) balances that do not follow well the target values, as is consistent with the profiles in Fig. 7. ECO-RAP shows geostrophic balance (Fig. 10g) that has the correct patterns, but values that are not balanced enough, and a similar conclusion may be made for hydrostatic balance (Fig. 10h). The ability of ECO-RAP to follow the correct level of geostrophic balance has some success (for instance by eye, ECO-RAP looks significantly better than SENCORP, but the gain of ECO-RAP over SENCORP in terms of the RMS diagnostic is not huge). ECO-RAP is the best scheme for hydrostatic balance. Again, these results are consistent with the profiles in Fig. 8.

6. Discussion and conclusions

We have demonstrated how well the three Schur product–type localization schemes described in section 2 are able to simultaneously remove assumed spurious correlations from sample covariances and maintain the geostrophic and hydrostatic balance properties of the unlocalized ensemble. The three schemes have not been designed to specifically conserve balance, but have been considered before in the literature. They are based on (i) a decomposition of the moderation matrix in a spectral basis, (ii) a simplified version of SENCORP (Bishop and Hodyss 2007), and (iii) the ECO-RAP method (Bishop and Hodyss 2009a). Scheme (i) is a static scheme (in that the degree of localization does not depend on the flow), and schemes (ii) and (iii) are adaptive (in that the degree of localization does depend on the flow). SENCORP modifies the sample correlation between two points by multiplying by a factor that tries to maintain large correlations and damp small correlations (irrespective of the separation). ECO-RAP is like the spectral scheme, but instead of a purely spectral basis, the cosine basis members are modified by the ensemble. Results are derived from a test ensemble from the Met Office’s MOGREPS system, adapted for the convective scale. This work shows how the balance properties of a localized covariance matrix can be formed via the construction of a “localized ensemble” (Table 1), which has that same covariance matrix. The localized ensemble is then studied by computing its geostrophic and hydrostatic balance properties and comparing them to those derived from the “dynamical ensemble” (the unlocalized ensemble).

Localization affects balances because multiplying by a moderation function that reduces with distance affects fields and gradients of fields in different ways (Lorenc 2003) and balance is often defined as the equalization of a field (e.g., wind for geostrophic balance) and the derivative of another field (e.g., pressure). Longer localization length scales ease this problem, but reduce the efficacy of the localization scheme. Although we could have reduced these problems by using different variables, and then localizing (e.g., Kepert 2009), we choose to localize directly in the model variables ($u$, $v$, $p$, and $T$) as we want to test the adaptive schemes rather than the effect of the change of variables itself. Adaptive localization schemes offer a possibility to overcome the balance problem as the degree of localization is dependent on the flow itself and not just on a prescribed moderation function.

Out of the candidates there is no one scheme that maintains the correct level of both geostrophic and hydrostatic balance closely, and hydrostatic balance is particularly difficult to maintain. Although the non-adaptive spectral scheme is best at maintaining the correct level of geostrophic balance, we do find that a particular configuration of ECO-RAP is the best scheme overall. This minimizes inappropriate imbalance, but still is able tofilter far-field correlations. Horizontal and vertical “influence parameters” were invented for the C\textsuperscript{$Q$} part of ECO-RAP. These were introduced for efficiency, but their use proved to be essential for ECO-RAP to give best results. SENCORP localizes very effectively, but it also destroys the balance properties the most.

We do acknowledge certain weaknesses of the work performed in this paper. The spectral and ECO-RAP schemes rely on a truncation in the horizontal and vertical wavenumber spectra that may introduce artifacts in

\textsuperscript{10} The reason for choosing the specific profile location earlier in this paper (at the center of the boxes in Fig. 10) is that it coincides with a region of relatively high geostrophic balance, but is (just) within the region where hydrostatic balance is slightly relaxed.
the implied moderation functions, such as nonreducing values in the vertical direction. We also find negative moderation function values although these affect mainly the far field where the moderation functions are smaller anyway (except for a problem identified with ECO-RAP close to part of the domain boundary). In parts of the schemes where length scales have a prescribed element (spectral and ECO-RAP) we have used the same length scales for all variables for simplicity. Even though, as stated above, the simplified SENCORP scheme is poor at maintaining balance, this result may say little about the performance of the full SENCORP scheme (see footnote 6). There are also potential weaknesses in the balance diagnostics, which are local measures of balance only and so are susceptible to grid-scale noise [nonlocal balance diagnostics are considered (e.g., by Vetra-Carvalho et al. 2012), but they do not consider localization].

The schemes themselves add expense to the data assimilation system, whether applying them as a direct Schur product or via the square root formulation used here and in the LETKF (Bishop and Hodyss 2009b). The cost to calculate a $K$ matrix scales in different ways for different schemes. In terms of cost per grid box and per variable these are as follows: spectral $N_k$, SENCORP $N^0Q$, ECO-RAP (without using influence parameters) $nN_kN$, and ECO-RAP (with influence parameters) $(2\rho_H + 1)^2(2\rho_V + 1)nN_kN$. Here $N_k$ is the number of wavenumbers (combined horizontal and vertical) used in the spectral decomposition, $N$ is the number of DE members, $Q$ is the “order” of the SENCORP or ECO-RAP scheme, $n$ is the size of the state vector, $\rho_H$ and $\rho_V$ are the ECO-RAP influence parameters, and $n_5$ is the number of variables (e.g., $u, v$, etc.). Clearly from an efficiency point of view, the aim is to use schemes that use the smallest acceptable values of these parameters.

Without implementing the schemes in a realistic ensemble data assimilation/forecasting system there is no quantitative indication of first, how close the localized geostrophic and hydrostatic balance diagnostics should be to the unlocalized diagnostics, and second, how much localization is sufficient. It is hoped though that this work can help guide operational developers to the best localization scheme. It is also worth pointing out here that forecast ensembles can also be used to calibrate the $B$ matrix in variational or hybrid systems (Fisher 2003; Buehner et al. 2010; Fairbairn et al. 2014), and this process is likely to be more accurate if the forecasts are appropriately balanced due to improved localization. Implementing a scheme like ECO-RAP in an operational scheme may be beneficial, but as with all modifications to an assimilation system, there are likely to be complications beyond the simple implementation (e.g., the need to recalibrate any inflation factor as the forecasts are cycled). This might be especially true of modifications that change the degree of balance since the growth rates of perturbations in forecasts are likely to be affected considerably.

Some of the findings of this work may require further investigation; for example, to understand why using a nonzero value of $\rho_H$ in ECO-RAP is found to decrease the level of geostrophic balance where it is obeyed by the ensemble. Nonzero $\rho_H$ values do increase the cost of ECO-RAP dramatically (see above), so here we are able to consider only small values of $\rho_H$. The cause of the boundary problems with ECO-RAP (Fig. 9) may also need to be investigated.

It would be interesting to extend this work to a wider range of balance diagnostics, to find ways of testing the full SENCORP method and other methods such as in Anderson (2012), to study alternative basis functions to those tested in the spectral and ECO-RAP schemes, and to test the schemes in a realistic EnKF system. One idea provided by J. Flowerdew (2013, personal communication) is, instead of relaxing long-range correlations to zero, relax them to climatology.

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REFERENCES