Damping Characteristics of Horizontal Laplacian Diffusion Filters

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ABSTRACT
Horizontally diffusive computational damping terms are frequently employed in 3D atmospheric simulation models to enhance stability and to suppress small-scale noise. In configuring these filters, it is desirable that damping effects are concentrated on the smaller-scale disturbances close to the grid scale and that the dissipation is spatially isotropic. On Cartesian meshes, the isotropy of the damping can vary greatly depending on the numerical formulation of the horizontal filter. The most isotropic behavior appears to result from recursive application of a 2D Laplacian that combines both along-axis and diagonal contributions. Also, the recursive application of 1D Laplacians in each coordinate direction provides better isotropy than the recursive application of the 2D Laplacian represented with a five-point operator. Increased isotropy also permits a larger maximum diffusivity, which may be beneficial in certain filter applications. On hexagonal and triangular meshes, Laplacian operators exhibit excellent isotropy, owing to the more isotropic nature of the meshes. However, previous research has established that straightforward application of the Laplacian may yield a diffusion operator that damps both resolved physical modes and unresolved high-wavenumber (aliased) modes, but it does not converge to the proper analytic behavior. Special averaging is then required to recover an accurate representation for the Laplacian. A consequence of this averaging is that the resulting filters do not act on the aliased modes (the checkerboard mode in particular) and thus employing the unaveraged diffusion operators may be preferable. The damping characteristics and stability constraints are derived for both the unaveraged and averaged Laplacian filters for C-grid staggering on these meshes.

1. Introduction

Explicit numerical diffusion filters are frequently employed in atmospheric model simulations to control the amplitude of high-wavenumber modes in the evolving model fields. Since these modes are typically not accurately represented by the model numerics, the application of these filters may improve stability and remove small-scale noise without significant loss of accuracy in the resolved scales of the solutions. For this purpose, higher-order (hyperdiffusion) filters are often used to focus their influence on these smaller, more poorly resolved scales. However, the actual damping characteristics of these filters may be quite sensitive to the finite-difference representation that is used and to the coordinate mesh in which they are applied. An important consideration in constructing numerical schemes for these models is that the results are not significantly biased by the mesh structure on which they are implemented. In this regard, good isotropy in numerical diffusion filters is a desirable characteristic.

While there has been considerable documentation of filter characteristics in one dimension (i.e., Shuman 1957; Shapiro 1970, 1975; Raymond 1988), there has been relatively less detailed attention on the behavior of these filters in two dimensions, particularly on their isotropy and dependence on the coordinate framework in which they are implemented. On Cartesian meshes, the isotropy of the damping can vary greatly depending on the numerical formulation of the Laplacian filter. The isotropy also impacts the maximum stable diffusivity, which can be an important consideration for certain filter applications. On hexagonal and triangular meshes, Laplacian operators typically exhibit excellent isotropy, owing to the more isotropic natures of the meshes. For C-grid staggering, Wan (2009) and Gassmann (2011, 2013) have demonstrated that a straightforward application of the vector Laplacian for horizontal velocity leads to a diffusion operator that does not converge to the proper analytic behavior and that special averaging is required to recover an accurate representation for the Laplacian. A consequence of this averaging is that the resulting filters do not act on the unresolved high-wavenumber (aliased) modes (the checkerboard mode...
in particular). Since these filters are typically implemented to remove small-scale noise and enhance numerical stability, employing the unaveraged diffusion operators should be more effective even though they do not represent a proper Laplacian.

Many models that incorporate higher-order horizontal numerical filters provide little or no documentation on how they are actually implemented. This study documents the damping characteristics and stability constraints for two-dimensional horizontal diffusion for second- and higher-order explicit filters for several alternative finite-difference formulations on Cartesian meshes (section 2), and for both averaged and unaveraged filters on hexagonal (section 3) and triangular (section 4) mesh structures. These three coordinate meshes provide the basis for most global meshes.

The behavior of these formulations will be evaluated in the context of the simple diffusion equation \( \phi_t = K_n \nabla^2 \phi \) in its explicit finite-difference form

\[
\phi^{t+\Delta t} = \phi^t + K_n \Delta t D_n
\]

for various formulations of the \( n \)th-order diffusion operator \( D_n \), which is evaluated at time \( t \). To analyze the damping characteristics of individual Fourier modes for a particular diffusion operator, we represent the prognostic variable \( \phi \) in the form

\[
\phi(x, y, t) = A(k, l) \hat{\phi}(k, l) e^{i(kx+ly)},
\]

where \((k, l)\) are the wavenumbers in the \((x, y)\) directions, \( t = m \Delta t \), \( \hat{\phi}(k, l) \) is the Fourier coefficient, and \( A(k, l) \) is the amplification coefficient, or in other words, \( 1 - |A(k, l)| \) represents the fractional reduction in the amplitude of the Fourier mode in each time step. Technically, the numerical integration of (1) will remain stable provided \(|A| \leq 1\). However, any modes for which \( A(k, l) < 0 \) will produce spurious oscillations, which is a highly undesirable behavior. Thus, for practical purposes, we identify the maximum permissible (stable) value of the diffusion coefficient \( K_n \) in (1) as the value for which the minimum \( A(k, l) \) over all wavenumbers is equal to zero.

2. Cartesian meshes

a. Second-order filters

Shuman (1957), Shapiro (1970), and Haltiner and Williams (1980), among others, recognized that diffusive terms need to be included in numerical prediction models to suppress the growth of the potentially spurious high-wavenumber mode and to prevent the accumulation of energy near the grid scale. The most straightforward two-dimensional filter they proposed simultaneously applies 1D smoothers in the \( x \) and \( y \) directions. Although they presented the filter in terms of a smoothing function applied to a two-dimensional field, this smoother can also be written as applied to the diffusion equation (1) with \( D_2 = \nabla^2_a = \nabla_x^2 + \nabla_y^2 \) computed along the coordinate axes:

\[
\phi_{ij}^{t+\Delta t} = \phi_{ij}^t + \beta_a (\phi_{i+1,j}^t + \phi_{i-1,j}^t + \phi_{i,j+1}^t + \phi_{i,j-1}^t - 4\phi_{ij}^t).
\]

Here, \((i, j)\) refer to gridcell indices in the \((x, y)\) directions, respectively; \( \beta_a = K_2 \Delta t / \Delta^2 \); and for simplicity, \( \Delta = \Delta x = \Delta y \). The finite-difference stencil for \( \nabla_x^2 \) is displayed in Fig. 1a [this equation is equivalent to the five-point smoothing operator described by Shapiro (1970), setting his smoothing coefficient \( S = 4\beta_a \)].

![Cartesian Meshes and Diffusion Operators](image)

**Fig. 1.** Finite-difference stencils for Laplacian operators on a Cartesian mesh computed (a) along the coordinate axes and (b) by combining both along-axes and diagonal terms.

Representing (3) in terms of the individual Fourier modes, (2) yields the well-known amplification coefficient

\[
A_{a2}(k, l) = 1 - 4\beta_a \left[ \sin^2 \left( \frac{k\Delta x}{2} \right) + \sin^2 \left( \frac{l\Delta y}{2} \right) \right]
\]

(see Shuman 1957 or Shapiro 1970). From (4), it is clear that the maximum allowable dimensionless diffusion coefficient is \( \beta_{a2} = 1/8 \).

The dependence of the amplification coefficient on the wavenumbers \((k, l)\) as represented in (4) for second-order diffusion with \( \beta_{a2} = 1/8 \) is displayed in Fig. 2a for the allowable wavenumbers \((|k\Delta x| \leq \pi, |l\Delta y| \leq \pi)\) that can be represented on the Cartesian mesh. As expected, this coefficient is unity for the \((0,0)\) wavenumber (center of the panel) and then decreases as the absolute value of the wavenumber increases. However, the response is noticeably anisotropic; the amplitude drops off more rapidly along the diagonal in wavenumber space than along the wavenumber axes. The maximum damping occurs in the corners of the panel for the mode that is \(2\Delta\) in both the \(x\) and \(y\) directions (the checkerboard mode). With this maximum damping coefficient, only half of the amplitude per time step can be
FIG. 2. Amplification factor $A_{m}(k, l)$ with diffusion Laplacian computed along coordinate axes for (a) second-, (c) fourth-, and (e) sixth-order filters; and $A_{m}(k, l)$ with Laplacian computed including diagonal directions for (b) second-, (d) fourth-, and (f) sixth-order filters. The dimensionless diffusion coefficient is set to $\beta_{m} = 2^{-\nu_{m} / 2}$ and $\beta_{m} = 2^{-\nu_{m}}$, which provide the maximum overall $n$th-order damping.

removed from the $2\Delta$ plane waves that are oriented along the coordinate axes (either $k = 0$ or $l = 0$).

Shuman (1957) and Shapiro (1970) have also proposed a 2D filter that uses a nine-point stencil by applying the two 1D smoothers sequentially:

\[
\begin{align*}
\phi_{i,j}^{s} &= \phi_{i,j} + \beta_{2}(\phi_{i+1,j} + \phi_{i-1,j} - 2\phi_{i,j}) \\
\phi_{i,j}^{t*} &= \phi_{i,j} + \beta_{2}(\phi_{i,j+1} + \phi_{i,j-1} - 2\phi_{i,j}),
\end{align*}
\]

which combine to yield

\[
\begin{align*}
\phi_{i,j}^{t*} &= \phi_{i,j} + \beta_{2}(1 - 2\beta_{2})(\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j}) + \beta_{2}^{2}(\phi_{i+1,j+1} + \phi_{i+1,j-1} + \phi_{i-1,j+1} + \phi_{i-1,j-1} - 4\phi_{i,j}) \\
&\quad + \phi_{i-1,j-1} - 4\phi_{i,j}).
\end{align*}
\]

Again, (6) is the same as the nine-point sequential operator in Shapiro (1970) for his smoothing coefficient $S = 2\beta_{o}$. Applying (2) to (6) leads to the following amplification coefficient $A_{z2}$ for the sequential smoothing filter:

\[
A_{z2}(k, l) = \left[1 - 4\beta_{z2} \sin^{2}\left(\frac{k\Delta x}{2}\right)\right] \left[1 - 4\beta_{z2} \sin^{2}\left(\frac{l\Delta y}{2}\right)\right],
\]

and the maximum smoothing coefficient to maintain $A_{z2} \geq 0$ is $\beta_{z2} = 1/4$ (or $S = 1/2$). The amplification coefficient $A_{z2}$ for this maximum value of the smoothing coefficient is displayed in Fig. 2b. This response is highly isotropic and achieves maximum damping ($A_{z2} = 0$) around the entire periphery of the allowable wavenumber space. However, for smaller values of $\beta_{z2}$, the quadratic term in $\beta_{z2}$ in (7) becomes smaller relative to the linear terms and the response becomes less isotropic. For $\beta_{z2} \ll 1/4$, (7) becomes the same as (4) (using the Laplacian $\nabla^{2}_{d}$). Since model filters are usually designed to introduce the minimum dissipation required to control the high-wavenumber modes, using sequential smoothing (5) might not provide much improvement over the five-point smoothing operator (4).

A more consistent isotropic response for the 2D Laplacian operator can be achieved by including terms in the diagonal directions directly (not through sequential filtering as in (5)). With this approach, the second-order diffusion operator becomes

\[
D_{2} = \nabla^{2}_{c} \phi' = \frac{1}{2}(\nabla^{2}_{a} + \nabla^{2}_{d})\phi'
\]

\[
= \frac{1}{2\Delta^{2}}(\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j})
\]

\[
+ \frac{1}{4\Delta^{2}}(\phi_{i+1,j+1} + \phi_{i-1,j-1} + \phi_{i+1,j-1} + \phi_{i-1,j+1} - 4\phi_{i,j}),
\]

where $\nabla^{2}_{d}$ refers to the Laplacian computed along directions diagonal to the coordinate axes; and $\nabla^{2}_{c}$ represents the combined Laplacian, with a finite-difference stencil as depicted in Fig. 1b. Combining this alternative diffusion operator with (1) and (2) yields an amplification coefficient given by

\[
A_{z2}(k, l) = 1 - 4\beta_{z2}\left\{1 - \left[1 - \sin^{2}\left(\frac{k\Delta x}{2}\right)\right]\right[1 - \sin^{2}\left(\frac{l\Delta y}{2}\right)]\right\},
\]

which produces the maximum damping with the dimensionless diffusion coefficient, $\beta_{z2} = 1/4$. The
amplification coefficient (9) for this combined second-order diffusion operator (8) for \( \beta_{c2} = 1/4 \) is also as depicted in Fig. 2b and exhibits a highly isotropic behavior that is identical to the nine-point sequential smoother (7) for the maximum diffusion coefficients \( \beta_{c2} = 1/4 \). However, in contrast to (7), (9) retains this isotropic response for smaller values of the diffusion coefficient \( \beta_{c2} \), and thus is superior to the sequential filter represented by (5).

b. Higher-order filters (hyperdiffusion)

When explicit diffusive filters are employed in numerical integration models, the objective is typically to dissipate energy in the small-scale modes with minimal impact on the larger scales that are (hopefully) well resolved. Thus, higher-order damping operators (often called hyperdiffusion or hyperviscosity) are often used to more selectively confine the attenuation to the smaller scales. A straightforward way to construct a higher-order filter is through recursive application of the second-order Laplacian operator. In this manner, one can construct a fourth- or sixth-order filter using the diffusion operator

\[
D_4 = -\nabla^2 (\nabla^2 \phi) \tag{10}
\]

or

\[
D_6 = \nabla^2 [\nabla^2 (\nabla^2 \phi)] \tag{11}
\]

respectively. This approach has the simplicity of requiring the coding of only the second-order Laplacian, which is then applied sequentially. The amplification coefficient \( A(k, l) \), however, will depend on the form of the second-order Laplacian used to create the higher-order filters. Generalizing the expressions for the amplification coefficients (4) and (9) to higher order, obtained using the along-axis Laplacian (\( \nabla^2_a \)) or the combined Laplacian (\( \nabla^2_c \)), yields

\[
A_{4n}(k, l) = 1 - \beta_{4n} 2^n \left[ \sin^2 \left( \frac{k \Delta x}{2} \right) + \sin^2 \left( \frac{\Delta y}{2} \right) \right]^{n/2} \tag{12}
\]

and

\[
A_{6n}(k, l) = 1 - \beta_{6n} 2^n \left\{ 1 - \left[ 1 - \sin^2 \left( \frac{k \Delta x}{2} \right) \right] \left[ 1 - \sin^2 \left( \frac{\Delta y}{2} \right) \right] \right\}^{n/2} \tag{13}
\]

for \( n = 2, 4, 6 \) (or higher even integers). For these expressions, the maximum dimensionless diffusion coefficients [such that the minimum values of \( A_{4n}(k, l) \) and \( A_{6n}(k, l) \) are zero] are \( \beta_{4n} = K_{4n} \Delta \tau / \Delta^4 = 2^{-3n/2} \) and \( \beta_{6n} = K_{6n} \Delta \tau / \Delta^6 = 2^{-n} \).

The amplification factors as a function of wavenumber \((k, l)\) for the fourth- and sixth-order filters are displayed in Fig. 2 using the Laplacian \( \nabla^2_c \) computed along the coordinate axes for \( \beta_{4n} = 2^{-3n/2} \) (Figs. 2c,e), and using the Laplacian \( \nabla^2_c \) for the maximum diffusion coefficients \( \beta_{6n} \) (Figs. 2d,f). As expected, using either Laplacian operator, the damping effects are concentrated increasingly at higher wavenumbers as the order of the filter is increased. (Notice how the area enclosed by the innermost 0.9 contour line increases with increasing filter order.) However, using the \( \nabla^2_c \) Laplacian operator, the response actually becomes more anisotropic at the higher wavenumbers as the filter order is increased. For example, for the sixth-order filter using the maximum value of \( \beta_{6n} \) (Fig. 2e), \( A_{4n}(\pi \Delta, \pi \Delta) = 0 \), while for the 2D plane waves in the \( x \) and \( y \) directions, \( A_{4n}(\pi \Delta, 0) = A_{4n}(0, \pi \Delta) = 0.875 \). Using the Laplacian \( \nabla^2_c \) that includes terms along the diagonals, good isotropy is maintained in the higher-order filters (Figs. 2d.g).

c. Higher-order filters along coordinate axes

As an alternative to higher-order horizontal filtering by recursively applying a 2D Laplacian, filters can be constructed through a recursive application of a 1D Laplacian in each coordinate direction, such that

\[
D_4 = -\left[ \nabla^2_x (\nabla^2_x \phi) + \nabla^2_y (\nabla^2_y \phi) \right] \tag{14}
\]

and

\[
D_6 = \nabla^2_x [\nabla^2_x (\nabla^2_x \phi) + \nabla^2_y (\nabla^2_y \phi)] \tag{15}
\]

for fourth- and sixth-order filters, respectively. For these unidirectional diffusion operators (14) and (15), the amplification coefficients \( A_{un} \) obtained in combination with (1) and (2) are given by

\[
A_{un}(k, l) = 1 - \beta_{un} 2^n \left[ \sin^2 \left( \frac{k \Delta x}{2} \right) + \sin^2 \left( \frac{l \Delta y}{2} \right) \right] \tag{16}
\]

where the maximum value of the diffusion coefficient for which \( A_{un} \geq 0 \) is \( \beta_{un} = 2^{-(n+1)} \).

The dependence of \( A_{un} \) on the horizontal wavenumbers \((k, l)\) for the maximum damping coefficient \( \beta_{un} = 2^{-(n+1)} \) is displayed in Fig. 3 for the fourth- and sixth-order filters. The dependence of \( A_{un} \) on the horizontal wavenumber in Fig. 3 is noticeably anisotropic approaching the corners of the plotted wavenumber space. However, in comparing Fig. 3 with Figs. 2c and 2e, we see that \( A_{un} \) is more isotropic than the same order filter obtained by the recursive application of \( \nabla^2_a \) (note
the nearly circular shape of the 0.5 contours in Fig. 3 compared to their counterparts in Figs. 2c,e).

The Advanced Regional Prediction System (Xue et al. 1995) uses a fourth-order configuration of this filter, applied along coordinate lines, for computational smoothing. The WRF-ARW Model (Skamarock et al. 2008) utilizes a sixth-order monotonic along-axis horizontal filter for computational smoothing, implemented by Knievel et al. 2007), based on the numerical scheme proposed by Xue (2000).

In assessing the spectral characteristics of horizontal fields (kinetic energy, terrain height, etc.) in numerical simulation models, the spectra are often represented in terms of the absolute value of the horizontal wavenumber \( \kappa = \sqrt{k^2 + l^2} \). The amplification factors exhibited in Figs. 2 and 3 can be expressed as a composite function of \( \kappa \) by integrating around concentric shells for constant values of \( \kappa \):

\[
A_{\kappa} = \frac{1}{2\pi} \int_{2\pi} A(\kappa, \theta) d\theta,
\]

where \( \theta = \arctan(l/k) \). These curves are displayed in Fig. 4 and reflect the overall damping characteristics of the various filters using their maximum allowable diffusion coefficients \( \beta_{an} = 2^{-3n/2} \), \( \beta_{cn} = 2^{-n} \), and \( \beta_{un} = 2^{-n-1} \). They confirm that the filters using the \( \nabla_{2n}^{c} \) Laplacian operator permit much larger overall dissipation at the smallest scales than can be achieved using the \( \nabla_{a}^{2n} \) or \( \nabla_{u}^{2n} \) operators. Figure 4 also confirms that higher-order filtering along the individual coordinate axes (\( \nabla_{a}^{n} \)) allows for higher overall dissipation than can be achieved with recursive application of the 2D Laplacian (\( \nabla_{a}^{2n} \)).

Of course, the magnitude of dissipation is regulated by the value of the diffusion coefficient, and the curves for \( \nabla_{c}^{n} \) and \( \nabla_{u}^{n} \) can be made similar to those for \( \nabla_{a}^{n} \) by suitably reducing \( \beta_{cn} \) and \( \beta_{un} \). In fact, holding \( \beta_{an} \) at its maximum value and reducing \( \beta_{cn} \) from its maximum value by a factor of \( 2^{-n/2} \) and \( \beta_{un} \) by a factor of \( 2^{1-n/2} \), all three approaches produce exactly the same amplification profile \( A_{\kappa} \) along the \( l = 0 \) (or \( k = 0 \)) axis (\( \beta_{an} = \beta_{cn} = \beta_{un} = 2^{-3n/2} \)). For this value of the diffusion coefficients, Fig. 5 displays \( A_{\kappa} \) for second-, fourth-, and sixth-order filters for plane waves oriented perpendicular to the \( x \) axis (\( l = 0 \)) and also the corresponding \( A_{\kappa} \) for plane waves oriented at 45° to the coordinate axes (\( k = l \)). The differences between the corresponding profiles for \( l = 0 \) and \( k = l \) illustrate the anisotropy of the filters. Using the \( \nabla_{u}^{n} \) operator, damping at higher wave-numbers is significantly greater along the diagonal (\( l = k \)) than along the \( k \) axis (\( l = 0 \)). The \( \nabla_{u}^{n} \) operator has the most isotropic behavior, although the isotropy of the \( \nabla_{u}^{n} \) operator is also noticeably better than that for \( \nabla_{a}^{n} \).

d. Horizontal velocity diffusion

Diffusion filters for horizontal velocity on meshes with C-grid staggering may be somewhat problematic, particularly for irregular meshes, since the direction of velocity components defined at each cell edge may vary from one cell to the next. To address this ambiguity, a vector Laplacian for the horizontal velocity is often written in terms of the horizontal divergence and vertical vorticity using the vector identity

\[
\frac{\partial}{\partial x} (u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}) + \frac{\partial}{\partial y} (u \frac{\partial u}{\partial y} + v \frac{\partial v}{\partial y}) = \nabla^{2} u + \nabla^{2} v,
\]

\[
\frac{\partial}{\partial x} (u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y}) + \frac{\partial}{\partial y} (u \frac{\partial v}{\partial y} + v \frac{\partial v}{\partial y}) = \nabla^{2} v - \nabla^{2} u,
\]
the second term is effectively a diffusion term for vertical vorticity. Evaluating (18) for a single Fourier component as in (2) produces a $2 \times 2$ determinant having two roots. Each of these roots individually has exactly the same damping characteristics (assuming C-grid staggering) as the scalar Laplacian as represented by $A_{2}$ in (4). One root reflects the damping of the divergent component of the horizontal wind [the first term on the rhs of (18)], while the other acts on the rotational component [the second term in (18)]. Combining the two terms in (18) recovers the amplification factor as in (4) for the full horizontal velocity. On the Cartesian mesh, both the divergence and vorticity damping terms individually and their combined form exhibit the same anisotropic response as is inherent in (4) and illustrated in Fig. 2a.

Writing the diffusion operator for the horizontal velocity as the combination of a horizontal divergence damping term and a vertical vorticity damping term has the odd property that the stability limits for each term are independent of whether the terms are applied individually or in combination. For example, applying the divergence damping term alone with a dimensional diffusion coefficient $K_{D}$, the resulting amplification factor as in (4) confirms the requirement that $\beta_{D} = K_{D} \Delta t / \Delta^2 \leq 1/8$. Similarly, applying the vorticity damping term alone with a coefficient $K_{Z}$ yields an amplification factor as in (4) and has the same stability limit, $\beta_{Z} = K_{Z} \Delta t / \Delta^2 \leq 1/8$. Applying both terms with $K_{D2} = K_{Z2}$, the stability constraints for both $\beta_{D2}$ and $\beta_{Z2}$ remain 1/8. The independent behavior arises because the damping of horizontal divergence has no effect on the vertical vorticity, and vice versa. This has been known since the early years of global modeling, supporting the recognition that damping of divergence and vorticity can be specified independently as appropriate for the particular application (see, e.g., Shuman and Stackpole 1969). In constructing higher-order filters for horizontal velocity, this same independence of the stability constraints is retained if the divergence and vorticity diffusion operators are recursively applied just to the divergence and vorticity damping terms, respectively.

While the basic vector Laplacian operator (18) dissipates kinetic energy, when implemented with a spatially variable diffusion coefficient, this coefficient must be placed between the differential operators to ensure that this dissipative property is retained:

\[
(\nabla \cdot K_{2} \nabla) \mathbf{V}_{H} \rightarrow \nabla(K_{D2} \text{Div}) - \nabla \times (K_{Z2} \mathbf{\zeta}).
\]

Here, the lhs of the arrow in (19) indicates the dissipative form for the scalar Laplacian, while the rhs represents the form of the vector Laplacian that guarantees...
energy dissipation. This is readily demonstrated by taking the dot product of $\mathbf{V}_H$ with the rhs of (19) to obtain
\begin{equation}
\mathbf{V}_H \cdot [\nabla (K_{D^2} \text{Div}) - \nabla \times (K_{\zeta^2} \xi)] = \nabla \cdot [K_{D^2} \text{Div} \mathbf{V}_H + K_{\zeta^2} \mathbf{V}_H \times \xi] - K_{D^2} \text{Div}^2 - K_{\zeta^2} \xi^2.
\end{equation}

The last two terms in (20) are negative definite and represent the kinetic energy dissipation associated with the divergence and vorticity pieces of the vector Laplacian, respectively. Thus, each term on the rhs of (19) will be energy dissipating even if different values are used for $K_{D^2}$ and $K_{\zeta^2}$. It should be noted, however, that with variable diffusion coefficients, the vector Laplacian form on the rhs of (19) is no longer formally equal to the scalar form on the lhs, even for $K_2 = K_{D^2} = K_{\zeta^2}$ [hence the arrow instead of an equal sign in (20)].

A fourth-order filter for horizontal velocity with variable diffusion coefficients can be constructed by applying (19) recursively, with coefficients defined as $K_{D^2} = K_{D^2}^1$ and $K_{\zeta^2} = K_{D^2}^2$ for each of the two applications. In this manner, the fourth-order equivalent of (20) can be derived that confirms kinetic energy dissipation represented by the negative definitive terms
\begin{equation}
-[(\nabla (K_{D^2} \text{Div})]^2 - [\nabla \times (K_{\zeta^2} \xi)]^2.
\end{equation}

3. Hexagonal meshes

a. Scalar diffusion

On a regular hexagonal mesh, each cell is surrounded by six adjacent cells, and thus for scalar variables $\phi$, a second-order horizontal diffusion operator is represented by the discrete Laplacian:
\begin{equation}
D_{h2} = \nabla_h^2 \phi = \nabla \cdot (\nabla \phi) = \frac{l_h}{A_h} \frac{1}{d_h} \sum_{\text{adj}=1}^6 (\phi_{\text{adj}} - \phi_0),
\end{equation}
where $A_h$ is the area of the hexagonal grid cell, $l_h$ is the hexagon edge length, $d_h$ is the distance between cell centers of adjacent hexagons, and the index adj refers to one of the six cells adjacent to the central cell as depicted in Fig. 6. This operator can also be applied to nonregular hexagons (or other polygons), such as those in a global icosahedral mesh, by accounting for the local $l_h$ and $d_h$ associated with the gradient term computed at each edge of the grid cell (Skamarock et al. 2012). For a regular hexagonal mesh, $l_h/(A_h d_h) = 2/(3d_h)$. Expressing $\phi$ in terms of its Fourier modes (2) and substituting it into the diffusion equation (1) yields the amplification coefficient for second-order diffusion on the hexagonal mesh:

\begin{equation}
A_{h2} = 1 - \frac{8}{3} \beta_{h2} [\sin^2(a + b) + \sin^2(a - b) + \sin^2(2a)],
\end{equation}

where $\beta_{h2} = K_{h2} \Delta t/d_h^2$, $a = kd_h/4$, and $b = \sqrt{3}d_h/4$. The terms within the brackets in (23) have a maximum amplitude of 9/4 for the smallest-scale modes that can be represented on the hexagonal mesh. Consequently, $\beta_{h2} = 1/6$ is the maximum dimensionless diffusion coefficient that ensures $A_{h2} \geq 0$ for all allowable wavenumbers.

The amplitude of $A_{h2}$ as a function of the horizontal wavenumbers ($k$, $l$) for the maximum diffusion $\beta_{h2} = 1/6$ is displayed in Fig. 7a over the range of wavenumbers that can be resolved on the hexagonal mesh and exhibits a highly isotropic response. This occurs because there are effectively three coordinate directions, each perpendicular to a pair of opposing edges of the hexagonal cell, thus reducing the directional variation in damping properties. The diffusion filter can be readily constructed at higher even orders through recursive application of the $\nabla_h^2$ operator as indicated in (10) and (11) for fourth- and sixth-order filters, respectively. For an $n$th-order filter, $A_{hn}$ becomes
\begin{equation}
A_{hn} = 1 - \left( \frac{8}{3} \right)^{n/2} \beta_{hn} [\sin^2(a + b) + \sin^2(a - b) + \sin^2(2a)]^{n/2}
\end{equation}
and produces the maximum overall $n$th-order damping for a dimensionless diffusion coefficient $\beta_{hn} = (1/6)^{n/2}$. The amplification coefficient $A_{h4}$ for a fourth-order filter
is shown in Fig. 7b. As expected, the response remains highly isotropic but with the attenuation concentrated at higher wavenumbers. This filter behavior is quite similar to that obtained in a Cartesian coordinate using the combined Laplacian \( \nabla^2 \) shown as in Figs. 2b and 2d.

**b. Horizontal velocity diffusion**

With C-grid staggering on regular hexagons, the horizontal velocity is represented by three velocity components located at the centers of the cell edges, as shown in Fig. 6. For a nonregular mesh, none of the horizontal velocity components may point in the same direction from cell to cell and thus the Laplacian is typically expressed in the vector divergence/vorticity form (18). To analyze this diffusion operator, we consider separately the contributions from divergence damping \( D_{hd}(u_i) = \frac{1}{d_h}(\text{Div}_1 - \text{Div}_0) \) (27) representing the horizontal velocity components in terms of their Fourier modes

\[
\hat{u}_j = e^{i(kx + ly)} \hat{u}_j, \quad j = 1, 2, 3, \tag{28}
\]

yields an equation for the amplification coefficient \( A_{hd} \) that is exactly the same as that for scalar diffusion \( A_{h2} \) in (23). This is to be expected, since the divergence is defined at cell centers (same as scalars), and taking the divergence of the horizontal momentum equations yields a scalar equation for divergence in which the diffusion of divergence is represented by the scalar Laplacian evaluated at cell centers. Thus, the maximum dimensionless diffusion coefficient multiplying \( D_{hd} \) to maintain \( A_{hd} \geq 0 \) is also \( \beta_{hd} = Kd_2 \Delta t d_i^2 \leq 1/6 \), and the amplification coefficient \( A_{hd} \) for this limiting value of \( \beta_{hd} \) is the same as that shown in Fig. 7a.

In considering the vorticity portion of the vector Laplacian, Gassmann (2011) demonstrated that the linear dependence of the horizontal velocity components at cell edges must be enforced to achieve a proper
finite-difference representation of the discrete form of the Laplacian. For regular hexagons, this is accomplished by computing the vorticity damping term using vorticities defined at the center of cell edges (Gassmann 2011, 2013) instead of at their more natural location at cell corners. With this representation on a regular hexagonal C grid, Gassmann (2013) confirmed that the Laplacian diffusion operator expressed in the vorticity/divergence form (18) has exactly the same finite-difference operator as that for the scalar Laplacian as in (22). Thus, the amplification coefficient $A_{h^2}$ for the vorticity portion of (18) is the same as that for scalar diffusion $A_{h^2}$ as defined in (23) and displayed in Fig. 7a.

For the hexagonal C grid, Thuburn (2008) pointed out that small-scale modes can exist in the vertical vorticity field that cannot be uniquely resolved. He illustrated this situation for a mode having $u_1 = u_2 = u_3 = $ constant, in which case the vorticity at cell corners (as depicted in Fig. 6) exhibits a checkerboard pattern, alternating in sign at adjacent cell corners. This mode, which appears to have zero wavenumber in the wind field, is actually an alias of higher-wavenumber modes, outside the resolvable range of wavenumbers.

A consequence of implementing a vector Laplacian filter in the manner proposed by Gassmann (2011, 2013) is that this filter formulation cannot act on the high-wavenumber aliased modes, such as the checkerboard mode discussed by Thuburn (2008). An alternative approach is to compute the vorticity damping term using vorticities defined at their natural locations at cell corners. This “unaveraged” form, utilized in the Model for Prediction Across Scales (MPAS; Skamarock et al. 2012; Ringler et al. 2013), damps the aliased modes along with the physical modes. Although it no longer formally represents the proper Laplacian for horizontal velocity [$\nabla^2 \rightarrow -(k^2 + p^2)$ in the limit as $d_h \rightarrow 0$], this requirement for formal convergence is not essential for filtering applications. However, if the Laplacian is used as part of the inviscid solver rather than as a filter (e.g., in semi-implicit time-integration schemes or in solving a vorticity/divergence form of the equations), then the proper (averaged) form of the Laplacian may be important to avoid exciting aliased modes (Wan 2009; Gassmann 2011).

For the unaveraged $D_{h^2}$, the $u_i$ equation can be written in terms of the horizontal velocity components as

$$D_{h^2}(u_i) = -\frac{1}{l_h}(\xi_2 - \xi_1) = -\frac{d_h}{l_h A_h} \left[ \sum_{j=1}^3 u_{j2} + \sum_{j=1}^3 u_{j1} \right]$$

(29)

Using (28) to represent (30) in terms of its Fourier components and forming the corresponding equations for the other two horizontal velocity components yields

$$\hat{D}_{h^2} \hat{u}_1 = -\frac{8}{d_h^2} \hat{u}_1 + \cos(a + b) \hat{u}_2 + \cos(a - b) \hat{u}_3$$

(31)

$$\hat{D}_{h^2} \hat{u}_2 = -\frac{8}{d_h^2} \hat{u}_2 + \cos(a + b) \hat{u}_3 + \cos(2a) \hat{u}_1$$

(32)

$$\hat{D}_{h^2} \hat{u}_3 = -\frac{8}{d_h^2} \hat{u}_3 + \cos(a - b) \hat{u}_1 + \cos(2a) \hat{u}_2$$

(33)

Setting the determinant of (31)–(33) equal to zero leads to a cubic equation for $\hat{D}_{h^2}$ that has one zero root and two real roots given by

$$\hat{D}_{h^2} = -\frac{8}{d_h^2} \left\{ \frac{3}{2} \pm \left[ \cos^2(a + b) + \cos^2(a - b) + \cos^2(2a) - \frac{3}{4} \right]^{1/2} \right\}.$$  

(34)

Combining (34) with the Fourier representation of the diffusion equations (1) and (2), we obtain the amplification coefficients $A_{h^2}$:

$$A_{h^2} = 1 - 8\beta_{h^2} \left\{ \frac{3}{2} \pm \left[ \cos^2(a + b) + \cos^2(a - b) + \cos^2(2a) - \frac{3}{4} \right]^{1/2} \right\},$$

(35)

where $\beta_{h^2} = K_{l2} \Delta t / d_h^2$.

The amplification coefficient $A_{h^2}$ associated with the minus sign on the rhs of (35) damps the resolved modes of the vorticity field. For this mode the maximum damping occurs at the minimum value of the term in square brackets in (35), which is zero for $a = \pi/3$ and $b = 0$. Thus, the maximum dimensionless diffusion coefficient for which $A_{h^2} > 0$ is $\beta_{h^2} = 1/12$. The additional amplification coefficient $A_{h^2}^+$ corresponding to the plus sign on the rhs of (35) arises from high-wavenumber modes outside the wavenumber range shown in Fig. 7 that are aliased back into the range of resolved wavenumbers. For these modes, the maximum damping occurs when the term in square brackets in (35) reaches its maximum value of $3/2$. Although this maximum damping appears to occur for $k = l = 0$, it actually corresponds to the aliased checkerboard mode. At this value, the minimum $A_{h^2}$ goes to zero for the dimensionless diffusion coefficient $\beta_{h^2} = 1/24$. The wavenumber dependence for $A_{h^2}^-$ and $A_{h^2}^+$ is shown in Fig. 8 for this maximum value of the more restrictive dimensionless diffusion coefficient $\beta_{h^2} = 1/24$. Damping of the resolved mode exhibits a highly isotropic behavior, with the
amplification coefficient $A_{\Delta}^{z}$ reaching a minimum of about 0.6 at the highest resolvable wavenumbers for the maximum diffusion coefficient that allows damping of the aliased modes to remain stable. For $\beta_{hD} = 1/24$, the maximum $A_{\Delta}^{z}$ reaches about 0.4 in Fig. 8b, confirming that the unaveraged vorticity damping can be effective in damping the full range of these aliased modes.

The stability constraint associated with the aliased mode of the unaveraged vorticity damping is highly restrictive, with a maximum diffusion coefficient 2 times smaller than the maximum coefficient $\beta_{hD}$ for the other vorticity root and 4 times smaller than the maximum coefficient $\beta_{hD}$ for the divergence damping portion of the vector Laplacian. For this reason, MPAS has implemented separate diffusion coefficients for the divergence and vorticity components of the operator [similar to the form expressed in (19)], limiting the damping coefficient for vorticity to satisfy its stability constraint while using a significantly larger coefficient for the divergence damping term (Skamarock et al. 2014). Recursive application of the second-order divergence and vorticity damping operators then achieves a higher-order biharmonic filter (Skamarock et al. 2012; Ringler et al. 2013).

4. Triangular meshes

a. Scalar diffusion

For the triangular mesh, each cell has one of two orientations, either upward pointing or downward pointing, as indicated in Fig. 9. Each cell shares edges with three other cells and thus the straight-forward second-order diffusion operator for cell-centered scalars $\phi_{\Delta}$ and $\phi_{\nabla}$ can be written as

$$D_{\Delta} = \nabla^{2} \phi_{\Delta} = \nabla \cdot (\nabla \phi_{\Delta}) = \frac{l_i}{A_i} \frac{1}{d_i} \sum_{\text{adj}=1}^{3} (\phi_{\text{adj}} - \phi_{\Delta}),$$

(36)

$$D_{\nabla} = \nabla^{2} \phi_{\nabla} = \nabla \cdot (\nabla \phi_{\nabla}) = \frac{l_i}{A_i} \frac{1}{d_i} \sum_{\text{adj}=1}^{3} (\phi_{\text{adj}} - \phi_{\nabla}),$$

(37)

respectively, where $l_i = d_i$ is the triangle edge length, $A_i$ is area of the triangular cell, and $d_i$ is the distance between adjacent cell centers, such that $l_i/(A_i d_i) = 4/(3d_i^2) = 4/d_i^2$. The adj index refers to the three cells that share edges with the central cell. Since the cell-centered scalar variables on the triangular mesh correspond to the corners of the dual hexagonal mesh, the damping characteristics $A_{\Delta}^{z} = A_{hD}^{z}$ as represented in (35) and displayed in Fig. 8, and the stability limits for the dimensionless diffusion coefficients $\beta_{hD} = K_{hD}/d_i^2$ are the same as for $\beta_{hD}^{z}$. Here again the minus sign in (35) corresponds to the resolved modes, while the plus sign represents aliased modes. The damping exhibits the same isotropy as on the hexagonal mesh and provides strong damping of the aliased modes (Fig. 8b), although the presence of the aliased modes again significantly constrains the maximum damping for the physical modes (Fig. 8a).
Wan (2009) pointed out that this straightforward formulation for the Laplacian on a triangular mesh has only first-order accuracy and does not converge to the proper analytic behavior. To correct these deficiencies, he proposed that the Laplacian operators be evaluated by replacing the cell-centered scalar quantities in (36) and (37) with the average of the scalar values in the three rhombi (each formed by two adjacent triangles) that share a particular triangle. For regular triangles, this is equivalent to forming a Laplacian from scalar values that have been interpolated to the center of cell edges [which are subsequently interpolated to cell centers to replace the cell-centered values on the rhs of (36) and (37)]. With this representation, (36) and (37) become decoupled for regular triangles, such that the Laplacian for an upward-pointing triangle (36) depends only on scalar values in the surrounding upward-pointing triangles and similarly for downward-pointing triangles (37), as recognized by Gassmann (2011). With this averaging on the rhs of (36) and (37), the scalar Laplacian is the same as that for the hexagonal mesh with cell spacing $d_h$. Thus, the amplification coefficient $A^\phi_h$ is identical to $A_{h2}$ as defined in (23) and displayed in Fig. 7a for the maximum diffusion coefficient $\beta^\phi_h = K^\phi_h \Delta t / d_h^2 = \beta_{h2} = 1/2$. This maximum diffusion coefficient is 4 times larger than the maximum coefficient permitted for the unaveraged scalar damping and allows much larger damping of the resolved modes at the higher wavenumbers. However, because of the decoupling of the upward- and downward-pointing triangles, this averaged Laplacian operator is not an effective diffusion filter for damping the aliased modes.

**b. Horizontal velocity diffusion**

The discrete horizontal Laplacian on a triangular mesh with C-grid staggering is typically evaluated, as for a hexagonal mesh, in terms of horizontal divergence and vertical vorticity as in (18). On the triangular mesh, the natural location to define divergence is at cell centers:

$$\text{Div} = \frac{l}{A_t} \text{sgn} \sum_{j=1}^{3} u_j,$$  \hspace{1cm} (38)

where $u_j$ are the horizontal velocities at the three edges of the triangular cell and sgn = 1 for upward-pointing triangles and sgn = −1 for downward-pointing triangles. This cell-centered divergence is located at the corners of the dual-hexagonal mesh, and therefore the vector Laplacian damping for divergence has an amplification coefficient $A^\text{div}_h$ that is the same as that for the unaveraged scalar Laplacian on triangles and for the unaveraged vorticity on hexagons. Thus, $A^\text{div}_h = A^\phi_h = A^\zeta_h$ as defined in (35) and displayed in Fig. 8. The divergence damping portion of the horizontal velocity diffusion acts on both the physical and aliased modes with the same range of stability as for the scalar fields.

Wan (2009) and Gassmann (2011) proposed that the individual cell divergences in (25) should be replaced by averaged values in order to recover second-order accuracy and the proper convergence for the Laplacian. This averaging is the same as that described above in evaluating the scalar Laplacian, namely, to average the divergences in the three rhombuses that share a particular triangle. With this averaging, the amplification factors $A^\text{div}_h = A^\phi_h = A^\zeta_h$ for the individual Fourier modes, as represented in (37) and displayed in Fig. 7a.

For the vorticity portion of the horizontal velocity diffusion, $\zeta$ can be written as

$$\zeta = \frac{d}{A_h} \sum_{j=1}^{3} (u_j^+ - u_j^-),$$  \hspace{1cm} (39)

where $u_j^\pm$ denotes the velocities normal to the six edges adjacent to the corner, with the $u_j^+$ oriented in a cyclonic direction relative to the corner and $u_j^-$ pointing anticyclonically. Here, $A_h = \sqrt{3} d_h/2$ is the area of the dual-grid hexagons formed by connecting the cell centers of the six triangles adjacent to the corner. Since the corners of the triangular cells coincide with the centers of the dual-hexagonal mesh, the amplification coefficient for the vorticity portion of the vector Laplacian is identical to the amplification factor $A_{h2}$ for the scalar Laplacian on hexagons as expressed in (35) and shown in Fig. 7a.
While the unaveraged operators for scalars and divergence in Laplacian filters have the advantage of damping both the physical and aliased modes (including the checkerboard mode), the stable range for the second-order diffusion coefficient is reduced from that for the averaged operators by a factor of 4. If increased damping is desired in filtering horizontal velocity, then there may be a benefit to computing the divergence damping and vorticity damping terms separately as in (19) using a larger diffusion coefficient for the vorticity damping (in contrast to a larger coefficient for the divergence damping, which might be beneficial on a hexagonal mesh).

The Icosahedral Nonhydrostatic (ICON) model, developed jointly by the German Weather Service and the Max Planck Institute for Meteorology, is perhaps the most prominent modeling system employing a global triangular mesh. In an early hydrostatic version of the ICON model, Wan et al. (2013) employed a fourth-order filter for horizontal velocity based on a biharmonic Laplacian using an unaveraged cell-centered divergence, as characterized by (38). They recognized the stability constraint $K_d \Delta t / d_t^2 < 1/64$, which is the same as the stability constraint on $(\beta_{2D})^2$ and $(\beta_{3D})^2$, as discussed above. They noted that in using the same coefficient for divergence and vorticity portions of the vector Laplacian, this constraint on the divergence damping may not permit the desired levels of diffusion acting on the physical modes.

In designing horizontal filters for the nonhydrostatic version of the ICON model, Zängl et al. (2015) elected not to construct a fourth-order filter for horizontal velocity using a divergence–vorticity formulation as in (18), noting that it exhibited grid imprinting and did not converge to the correct Laplacian behavior. They instead developed an alternative approach to directly compute a Laplacian diffusion for the horizontal velocity that is used for a second-order Smagorinsky diffusion and for a fourth-order background hyperdiffusion. To form a Laplacian for the normal velocity located at the center of a triangle edge, they first do a radial basis function (RBF) reconstruction of the same 2D horizontal velocity component at the four vertices of the rhombus formed by combining the two triangles that share that edge. They then construct the Laplacian using essentially the same stencil as that shown in Fig. 1a for a Cartesian mesh but accounting for the differing stencil spacing in the two directions [see Fig. 1 in Zängl et al. (2015)].

To analyze this diffusion operator for a regular triangular mesh, we note that the construction of the 2D horizontal velocity at a triangle vertex is just the appropriate average of the normal velocities on the six edges that share that same corner:

$$ u_i = \frac{1}{4} \sum_{adj=1}^{2} [u_i^{(adj)} - u_j^{(adj)} - u_k^{(adj)}]. $$

Here, $u_{i\text{w}}$ refers to the $u_i$ velocity component interpolated to a vertex, and $j, k$ refer to the other two velocity components. The sum over $adj = 1, 2$ represents the two values for each of the three velocity components that are defined on the six edges adjacent to the vertex where $u_{i\text{w}}$ is located. The diffusion operators for the three velocity components then become

$$ D_d(u_1) = \frac{4}{9d_t^2} [u_{1v_1} - 2u_1 + u_{1w_2} + 3(u_{1v_3} - 2u_1 + u_{1w_4})], $$

$$ D_d(u_2) = \frac{4}{9d_t^2} [u_{2v_2} - 2u_2 + u_{2w_3} + 3(u_{2v_4} - 2u_2 + u_{2w_5})], $$

$$ D_d(u_3) = \frac{4}{9d_t^2} [u_{3v_3} - 2u_3 + u_{3w_4} + 3(u_{3v_5} - 2u_3 + u_{3w_6})], $$

respectively, where the subscripts $v_1$ and $v_2$ refer to interpolated velocities at the two vertices opposite the edge on which $u_i$ is located, and $v_3$ and $v_4$ refer to interpolated velocities at the two vertices that bound the edge on which $u_i$ is located. Representing the horizontal velocity components in terms of the Fourier modes as in (28) and using (40) to evaluate velocities interpolated to vertices, (41)–(43) become three equations for $D_d$ in terms of the Fourier coefficients of the three horizontal velocity components. Setting the determinant of these three equations to zero produces a cubic equation for $D_d$, which must be solved numerically for the three roots. Combining these roots with the Fourier representation of the diffusion equation (1) and (2) yields the three amplification factors $A_d$ displayed in Fig. 10. Here, the amplitudes shown in Figs. 10a and 10b correspond to resolved modes, while those in Fig. 10c represent aliased modes (maximum damping at zero wavenumber, which actually corresponds to the checkerboard mode). These amplification factors are displayed for a dimensionless diffusion coefficient $\beta_d = K_d \Delta t / d_t^2 = 1/16$, which is the maximum stable damping coefficient for the aliased checkerboard mode. Figure 10 actually plots $|A_d|$ because the two resolved roots become complex at high wavenumbers, which may introduce a phase shift as well as damping in those modes. Thus, constructing a diffusion term for horizontal velocity using an interpolated Laplacian as in (41)–(43) also damps the aliased modes and allows a somewhat wider range of damping for the physical mode than can be achieved using the unaveraged vector Laplacian, although at greater computational expense.
5. Summary

Horizontally diffusive computational damping terms are frequently employed in 3D atmospheric simulation models to enhance stability and to suppress small-scale noise. In configuring these filters, two desirable features are 1) that damping effects are concentrated on the smaller-scale disturbances close to the grid scale and 2) that the dissipation is spatially isotropic. The above-mentioned analyses confirm that, as expected, diffusive effects shift progressively to higher wavenumbers as the order of the filter is increased. However, particularly on Cartesian meshes, the isotropy of the damping can vary greatly depending on the discrete formulation of the Laplacian filter. Substantial anisotropy is apparent using recursive application of the basic five-point Laplacian operator (3) computed only along the coordinate axes ($\nabla_a^2$). With this filter operator, there is significantly more damping along the diagonal in wavenumber space than along the axes (Figs. 2a,c,e, 5). For example, as seen in Fig. 5, the effective damping in the higher wavenumber range varies by about a factor 2 or more as the orientation of disturbances shifts by 45° on the grid. This anisotropic behavior in the discrete Laplacian operator becomes more pronounced when the operator is applied recursively to produce higher-order hyperdiffusion.

Of the several filter operators analyzed above on a Cartesian mesh, the best isotropy is achieved when the 2D Laplacian combines the along-axis and diagonal contributions [i.e., $\nabla_a^2$ as in (8) and Fig. 1b]. With this approach, the inherent anisotropy in both $\nabla_a^2$ and $\nabla_d^2$ is largely compensating, such that there is little directional variation in the damping characteristics (Figs. 2b,d,f, and 5). Higher-order filters applying operators in each horizontal direction separately ($\nabla_u^n$) also exhibit good isotropy over most of the range of resolvable wavenumbers.

Because of the anisotropy of a discrete filter operator, the maximum damping (with explicit numerics) that can be achieved for disturbances will depend on their orientation with respect to the coordinate axes. The combined $\nabla_u^n$ operator provides the most flexibility and can be set large enough to remove all of the 2Δ modes along the wavenumber axes in a single time step (Figs. 2b,d,f). The maximum stable dimensionless diffusion coefficient $\beta_{u2} = 1/4$ for the combined operator is twice as large as the maximum $\beta_{a2}$ for the basic Laplacian in (3). The higher-order along-axis filters ($\nabla_u^n$) also allow a reasonable damping range and can remove up to half of the 2Δ modes (along the wavenumber axes) in each time step. The stable range for the basic five-point Laplacian operator ($\nabla_a^n$) becomes more restrictive with increasing order; at sixth order it permits a maximum removal of only 1/8 of the 2Δ mode along the

![Fig. 10. Amplification factor $|A_{12}(k, l)|$ for horizontal velocity diffusion for the three roots of the amplification equation based on an interpolated Laplacian as represented in (41) and (43). (a),(b) Two resolved modes and (c) an aliased mode. The amplification factors are displayed for $\beta_{u} = 1/16$, which provides the maximum damping for the aliased mode.](image-url)
wavenumber axes in each time step to maintain stability (Figs. 2e, 5c).

Perhaps the most significant practical benefit of increased isotropy in Laplacian filters on Cartesian meshes is the increased maximum stable diffusivity that can be achieved. Although numerical diffusion filters are typically employed using filter coefficients that are as small as possible to control small-scale noise, they may also be used with larger amplitudes to remove (or control) unwanted modes or to represent unresolved physical processes in the model. Examples include 3D divergence damping (Skamarock and Klemp 1992), 2D divergence damping, particularly at upper levels (Jablonowski and Williamson 2006), and stabilization of unresolved flow discontinuities (such as frontal boundaries). The use of a horizontal Laplacian diffusion for upper-level sponge layers for gravity wave adsorption (Klemp and Lilly 1978) provides another good illustration of the benefit of an increased maximum stable diffusivity. As discussed by Klemp et al. (2008), using a traditional horizontal Laplacian filter to absorb gravity wave energy near the top of the model domain, stability constraints may preclude specification of a damping coefficient large enough to effectively absorb the gravity waves. For this sponge-layer application, the intent is to absorb resolved gravity wave modes (not near the grid scale) that may require larger magnitudes of the diffusion and thus more isotropic filters will be beneficial.

On the hexagonal mesh, the straightforward application of the scalar Laplacian operator as in (22) provides excellent isotropy in the damping characteristics (see Fig. 7), owing to the more isotropic nature of the grid itself. Filters for the horizontal velocity are typically represented by a vector Laplacian in terms of divergence and vorticity, as in (18). For C-grid staggering, Gassmann (2011, 2013) demonstrated that a proper Laplacian could be constructed by suitably averaging the vorticity on the hexagonal mesh (to the center of cell edges for regular hexagons). With this averaging, however, the filter cannot act on unresolved aliased modes (such as the checkerboard mode). Alternatively, the Laplacian in (18) can be evaluated with vorticity defined at cell corners as is done in the MPAS model (Skamarock et al. 2012; Ringler et al. 2013). Although this representation does not provide a proper Laplacian for the horizontal velocity, it does provide damping for divergence and vorticity, which also acts on the aliased modes. These aliased modes in the discrete representation of the vorticity damping, however, constrain the magnitude of the second-order diffusion coefficient to be a factor of 4 smaller than can be used for the divergence damping to maintain stability. This limitation can be partially overcome by specifying a larger damping coefficient for the divergence damping than for the vorticity damping.

The unaveraged discrete diffusion operator for scalars and divergence and vorticity on a triangular mesh can provide effective high-wavenumber filtering of both the resolved and aliased modes as indicated in Fig. 8, although it does not accurately represent a proper Laplacian (Wan 2009; Gassmann 2011). However, as for the hexagonal mesh, the presence of the aliased modes does restrict the range of damping for the resolved modes (minimum amplification factor of about 0.6 in Fig. 8a). For horizontal velocity filtering using the vector Laplacian, the tighter stability constraint for the divergence damping term suggests that employing different diffusion coefficients for the divergence and vorticity damping terms may be beneficial. Wan (2009) and Gassmann (2011) demonstrated that a Laplacian for cell-centered variables (i.e., scalars and horizontal divergence) with the proper convergence properties could be constructed by suitably averaging these variables (to the center of cell edges for regular triangles). With this formulation, a wider range of damping can be achieved for the resolved modes, as illustrated in Fig. 7. However, a detracting consequence of this averaging is that the upward- and downward-pointing triangular grid cells become decoupled and that the filter cannot act on the aliased modes (the checkerboard mode in particular).

It is interesting to note that the diffusive behavior of the scalar Laplacians and the divergence and vorticity portions of the vector Laplacian on both hexagonal and triangular meshes have one of two forms depending on the location of the variables on the mesh. Laplacian diffusion for cell-centered variables on hexagonal cells (corresponding to corner variables on the dual triangles) all damp the resolved modes with an amplification coefficient $A_{h2} = A_{hD} = A_{hC}$ as given by (23) and illustrated in Fig. 7, whereas cell-centered variables on triangular cells (corresponding to corner variables on the dual hexagons) all damp both the physical and aliased modes, such that $A_{h2} = A_{hD} = A_{hC}$ as represented in (35) and displayed in Fig. 8. For this latter set of diffusion operators, appropriate averaging of the variables, as discussed above, will produce a proper Laplacian with amplification coefficient $A_{h2} = A_{hD} = A_{hC} = A_{h2}$.

The ICON model has implemented yet another approach for filtering the horizontal velocity on a triangular mesh by forming a scalar Laplacian directly for each velocity component from reconstructed velocity components at nearby vertices (Zängl et al. 2015). This formulation damps both the resolved and aliased modes, and it appears to allow a wider range of amplification coefficients than the unaveraged operator discussed above. However, the approach is computationally more
expensive and may cause some shifting of the phase of disturbances as a result of an imaginary component of the amplification coefficient $A_H$.

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