1. Introduction

Most atmospheric scientists are familiar with the type of inertia-gravity wave that is trapped along coastlines or the equator, known as a Kelvin wave (e.g., Thomson 1879; Gill 1982, chapter 10; Wang 2003). These waves play a prominent role in tropical dynamics including El Niño–Southern Oscillation (e.g., Battisti 1988). Perhaps less familiar within the atmospheric science community is the work by Lord Kelvin on vortex waves (Thomson 1880), in which he laid out the general strategy for obtaining a wide variety of solutions of waves traveling within infinitely extended, concentrated cylindrical vortices resembling the flow in tornadoes or dust devils some distance away from the lower boundary. These waves are usually referred to as “Kelvin waves” in the fluid physics literature, and they are also known as vortex waves, centrifugal waves, or inertial waves (Lugt 1989). Such waves play an important role in the vortex breakdown phenomenon as well as in the development of multiple-vortex tornadoes (e.g., Lewellen 1993; Rotunno 2013). Examples of these waves within tornadoes are shown in Fig. 1. Here so-called bending waves (to be introduced in section 3) are visible, which are often well pronounced during the rope stage of the tornado (Fig. 1b). The effects of vortex waves are also seen in Fig. 2, where Fig. 2a shows a multivortex structure that results from unstable growth of spiral modes (section 3), and Fig. 2b displays what appears to be a vortex breakdown. This phenomenon often occurs in vortices with nonzero axial velocity and it will be reviewed in section 8.

The need for this review arises from the observation that, despite the ubiquity of Kelvin vortex waves in tornado-like vortices, a detailed introduction to the structure and behavior of these waves appears to be absent from the atmospheric science literature. Although this topic is introduced in some fluid physics texts (e.g., Thomson 1880; Lamb 1932, section 158; Chandrasekhar 1961, chapter VII, especially p. 284; Drazin and Reid 1981, p. 75; Saffman 1992, chapters 11 and 12; Lim and Redekopp 1998; Fritts et al. 1998; Rossi 2000; Batchelor 2002, p. 559; Alekseenko et al. 2007, chapter 4), in a given presentation either only limited, or rather advanced, aspects of vortex waves are discussed. As a consequence, there is a large gap between the knowledge base offered in mainstream texts about atmospheric dynamics (e.g., Gill 1982; Holton and Hakim 2013; Markowski and Richardson 2010) and the advanced peer-reviewed literature on the remarkably rich Kelvin vortex wave dynamics. Thus, the present paper seeks to

- narrow the gap between the atmospheric-dynamics literature and the advanced fluid-physics literature by providing a detailed introduction to these waves. Except where the mathematical steps are readily available in mainstream texts, the complete mathematical development is included in this paper;
- retrace Kelvin’s original approach, and point to possible applications of his contributions to tornado-like flows, including multiple-vortex development and vortex breakdown.

To achieve this, an almost trivial generalization of Kelvin’s equations is included by allowing for a piecewise constant axial
flow, which admits a wide class of unstable solutions not considered in Kelvin’s original work. These solutions have been arrived at previously, but along with a review of these topics, it is demonstrated that the results follow from Kelvin’s approach. In the context of stability analyses, Kelvin’s (slightly modified) approach presented in this review has been superseded by much more advanced analysis techniques, but these are still based on analyses such as those presented here, and will be touched upon in section 8.

The remainder of the paper is structured as follows. In the next section a brief history of vortex waves will be offered and the relevance of these waves will be described, and section 3 offers an overview of the classification of vortex waves. In section 4 the governing equations describing infinitely extended columnar vortices will be presented following Kelvin’s approach. Subsequently, consecutively more refined scenarios are introduced that follow directly from Kelvin’s equations. Starting with the scenario of vanishing base-state axial flow, a cylindrical domain bounded by rigid walls will be considered (section 5) to gain an intuition for the structure and dynamics of these waves. Thereafter the Rankine vortex will be discussed in section 6, and two scenarios that allow for unstable wave growth will be considered in section 7, i.e., a Rankine vortex with upward motion in its core as well as a two-celled vortex with descending motion in its irrotational core, and rising motion outside of it. The unstable waves in the latter scenario provide a rudimentary model for multiple-vortex formation in tornado-like vortices. These ideas will then be applied to vortex breakdown in section 8. Limitations of Kelvin’s approach and its applicability to tornadoes will be addressed in section 9. Finally, section 10 offers concluding remarks and possible directions for future investigations.

2. A brief history of Kelvin vortex waves

Around the mid-1800s, Sir William Thomson, who was granted the title Lord Kelvin in 1892, pursued the idea of describing the previously discovered atoms in terms of microscopic knotted vortex rings, called “vortex atoms” (Thomson 1867; Fabre et al. 2006). The medium in which these vortex rings were thought to exist was the hypothesized all-pervading, homogeneous perfect fluid known as the aether. This idea was
Atkinson (1957) are generally credited for first documenting aerodynamicists, i.e., the vortex breakdown. Peckham and Atkinson (1957, p. 5), indicating turbulence and disappearing—as though the core was becoming more diffuse'' of the vortex structure, which appeared to ''bell out before vortices produced at the leading edge of ogival delta wings. At Fig. 31 for an example of such a breakdown in a tornado. On an undesirable drop of lift. The reader may skip ahead to ''cooperative instability'' and is sometimes visibly manifest as a vortex breakdown in tornadoes are shown by Lugt (1989).

Kelvin’s work entered the field of aerodynamic engineering following the discovery that lift-generating devices produce a pair of trailing vortices, which pose a hazard to aircraft that encounter these vortices, which consequently reduce airspace capacity (Hallock and Holzäpfel 2018). The stability of the wake vortices is directly related to the unstable growth of bending Kelvin modes (the different Kelvin modes will be introduced in the next section). This instability is known as “cooperative instability” and is sometimes visibly manifest as a contortion of aircraft contrails and the formation of contrail lobes (Lewellen and Lewellen 2001; Wu et al. 2006, p. 499; Schultz and Hancock 2016).

In the late 1950s, another phenomenon was discovered by aerodynamicists, i.e., the vortex breakdown. Peckham and Atkinson (1957) are generally credited for first documenting this phenomenon during the investigation of lift-generating vortices produced at the leading edge of ogival delta wings. At large angles of attack, the observation included a disintegration of the vortex structure, which appeared to “bell out before disappearing—as though the core was becoming more diffuse” (Peckham and Atkinson 1957, p. 5), indicating turbulence and an undesirable drop of lift. (The reader may skip ahead to Fig. 31 for an example of such a breakdown in a tornado.1 On the other hand, vortex breakdown was found to have beneficial effects in some applications such as stabilizing the flame in combustion chambers of gas turbines (e.g., Escudier 1988; Spencer and McGuirk 2008). Vortex breakdown has thus garnered considerable interest in the engineering and fluid physics community, and soon after its official discovery, Squire (1960) and Benjamin (1962) explained the vortex breakdown phenomenon as an axisymmetric analog of the hydraulic jump, which is related to the instability of wave perturbations, in this case axisymmetric Kelvin vortex waves, to propagate upstream (Squire 1960; Benjamin 1962). Among other vortex profiles, these studies also considered the Rankine vortex, and the vortex breakdown criterion discovered in these studies directly follows from Kelvin’s analysis. Vortex breakdown and its relation to Kelvin vortex waves will be reviewed in section 8.

Aside from the application to engineering problems, the roles of Kelvin vortex waves have been studied in the general context of transition to turbulence in rotating flows (e.g., Hopfinger et al. 1982). Importantly, these experiments revealed the existence of solitary-like vortex waves (Maxworthy et al. 1985). In the atmospheric science literature, Kelvin’s work does not appear to be mentioned frequently. One exception is Fultz (1959), who studied oscillations in convective motion. He observed vortex waves in his rotating tank experiments and pointed to a textbook by Bjerknes et al. (1933), where vortex waves are introduced. Apparently, however, Bjerknes et al. (1933) had been unaware of Kelvin’s treatment, prompting Fultz to note that “its significance in meteorological dynamics had to be rediscovered.” Kelvin vortex waves feature prominently in the studies by Andreassen et al. (1998), Fritts et al. (1998) and Fritts and Alexander (2003), which are concerned with turbulence generation as a result of shear instability as well as in breaking internal gravity waves (Fritts and Alexander 2003). These authors refer to vortex waves as “twist waves” (Arendt et al. 1997). Kieu (2016) extended Kelvin’s solutions with the goal of describing waves in the inner core region of tropical cyclones.

In the field of tornado research, centrifugal waves attracted some attention in the 1970s and 1980s when research using tornado vortex chambers flourished (Ward 1972; Church et al. 1977; Rotunno 1979; Snow 1982; Church and Snow 1993). However, the focus of these analyses was the determination of how the flow parameters, most notably the swirl ratio (Davies-Jones 1973), led to different tornado structures including vortex breakdown and multiple vortices. The importance of centrifugal waves is mentioned in these studies in the context of vortex breakdown (Church et al. 1977; Rotunno 1979; Snow 1982; Fiedler and Rotunno 1986), but the structure and dynamics of these waves are not analyzed further. Nolan and Farrell (1999) analyzed these waves in detail, and they observed downward propagating axisymmetric waves in their numerically simulated tornado-like vortex. Their analysis suggests that these waves are less likely to be observed in flows with low swirl ratios, consistent with such flows being supercritical (section 8), but their linear analysis did not fully explain the behavior of these waves. More recently, Nolan (2012) revisited this phenomenon by studying the linear instability of fully nonlinear flows.

Because the vortex chamber experiments reproduced the observed transition of a single tornado vortex into multiple vortices, investigations into the stability of the tornado followed (e.g., Rotunno 1978; Walko and Gall 1984; Nolan 2012). This instability is a manifestation of unstable centrifugal waves, and will be discussed in section 7. With the exception of Nolan’s analyses (Nolan and Farrell 1999; Nolan 2012), these centrifugal waves apparently have not played important roles in interpreting tornadic flows since the 1980s, and Snow (1982).

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1 Peckham and Atkinson’s (1957) observation was not the first time the vortex breakdown phenomenon had been encountered. Antonescu (2017) compiled historic drawings of tornadoes, including a waterspout from 1648 near Sardinia, which exhibits what looks like a breakdown bubble (section 8 in the present paper) near the cloud base. Other historical observations of vortex breakdown in tornadoes are shown by Lugt (1989).
at the end of his review on tornado dynamics, added that “features that should be investigated...include the nature of the wavelike features often seen on the walls of condensation funnels” (Snow 1982, p. 963). It is fair to say that his recommendation has not materialized, and consideration of centrifugal waves in the context of tornadoes appears to be scarce. Aside from Nolan and Farrell (1999), some examples include Lugt (1989) and Trapp (2000), who discuss vortex breakdown in tornadoes and mesocyclones, respectively, and in this context briefly mention centrifugal waves; Bluestein et al. (2003) observed oscillations of a tornado’s intensity and as a possible explanation, centrifugal waves were invoked; in their textbook, Markowski and Richardson (2010) introduce the stability criterion for centrifugal oscillations, which correspond to an axisymmetric Kelvin mode.

This brief overview reveals that, while the relevance of vortex waves in tornadoes is generally acknowledged, the association with Kelvin’s work is not widely appreciated, and the structure and behavior of these waves has attracted somewhat limited attention in the atmospheric science community. To start the discussion of vortex wave dynamics, the basic types and naming conventions of these waves are introduced in the next section. The main distinction between the different wave types is the azimuthal wavenumber \( m \), which determines whether a wave is manifest as, e.g., axisymmetric or spiral perturbation. The azimuthal wavenumber is dimensionless and must be an integer to guarantee continuity of the wave perturbation. The vortex column serves as waveguide, and for infinitely extended vortex columns, each wave type (determined by \( m \)) may generally attain an arbitrary axial wavenumber \( k \).

3. Classification of Kelvin vortex waves

a. Axisymmetric modes, \( m = 0 \)

If \( m = 0 \), the perturbation is axisymmetric, resulting in a widening and narrowing of the vortex (e.g., in a Rankine vortex the radius of maximum winds extends and contracts). The axial wavenumber must be nonzero for axisymmetric waves to exist. A commonly used name of this mode goes back to Lord Rayleigh (Strutt 1902, p. 444). When discussing photographs of a liquid jet undergoing axisymmetric oscillations before breaking up into drops (as happens to a water stream emanating from a faucet), he noted: “…I have often been embarrassed for want of an appropriate word to describe the condition in question. But a few days ago, during a biological discussion, I found that there is a recognised, if not very pleasant, word. The cylindrical jet may be said to become varicose, and varicosity goes on increasing with time, until eventually it leads to absolute disruption.” Chandrasekhar (1961, p. 515), after citing this passage from Rayleigh, adds: “In recent times, ‘sausage instability’ has been used to describe the same condition; but this is also not a very ‘pleasant’ description, and varicosity instability would seem preferable.” The designations “sausage” or “sausaging” as well as “varicose” modes have been adopted widely to describe axisymmetric vortex waves, though in this paper these waves will be referred to mainly as axisymmetric waves. This wave can only propagate in the axial direction and it plays an important role in the onset of vortex breakdown (section 8).

b. Spiral modes, \( |m| > 0 \)

Waves with azimuthal wavenumbers \( |m| \gtrsim 1 \) are referred to as helical or spiral modes. If \( |m| = 1 \), the waves displace the vortex axis and are called “bending” modes. For all other choices of \( m \) the vortex axis remains centered. An example of how an azimuthal wave perturbation is related to the vortex structure is shown in Fig. 3, demonstrating that a wavenumber of two corresponds to an elliptic deformation of the vortex core. The sign of the azimuthal wavenumber determines the handedness of the spiral. As shown in Fig. 4, \( m > 0 \) implies left-handedness and \( m < 0 \) implies right-handedness.2

If the slope of the phase lines in the \((\theta, z)\) plane wrapping around the cylindrical vortex is taken to be \( dh/d\theta \), where \( h \) is the local height of the phase line and \( \theta \) is the azimuthal angle,
this slope is identical to the pitch of the wave, given by $-m/k$ (Fabre et al. 2006). Defined this way, pitch is proportional to the distance by which the helix advances during one revolution (i.e., to the axial wavelength of the helix). Saffman (1992) and Alekseenko et al. (2007) use slightly different definitions of pitch, which only apply to $|m| = 1$ but are proportional to each other as well as to the definition used here. It follows that, e.g., left-handed spirals have a negative pitch and axisymmetric waves have zero pitch. For a given axial (vertical) wavenumber, the pitch becomes larger in magnitude as the number of spirals winding around the vortex (i.e., $m$) is increased.

An azimuthal wavenumber of $m = 2$ implies a double helix or double spiral structure. Waves with $k = 0$ and $m \neq 0$ are sometimes called “fluted” modes (Maxworthy 1988), which are 2D waves that only propagate in azimuthal direction. Hopfinger et al. (1982), Hopfinger and Browand (1982), and Maxworthy et al. (1985) coined the term “kink wave” for helical solitary waves ($m = \pm 1$) they observed in their experiments. These waves resembled the soliton solution by Hasimoto (1972).

The azimuthal angular phase speed of spiral modes is given by $\omega/m$, where $\omega$ is the wave frequency. Assuming that the angular velocity of the vortex $\Omega$ is positive, implying cyclonic rotation, then if the azimuthal phase speed is larger than the angular velocity of the vortex, the waves are said to be “cograde,” meaning these waves propagate downstream relative to the motion within the vortex. If the angular speed of the wave is less than $\Omega$, but the waves still move in the same sense as the azimuthal vortex flow, the wave is said to be “retrogade.” These waves propagate upstream relative to the vortex flow, but are advected downstream. Finally, the waves are “countergrade” if they move in the opposite sense than the vortex, now also for a stationary observer (Fabre et al. 2006). From these basic definitions and conventions the rich structure of vortex waves is already becoming apparent. The next section introduces the governing equations describing the dynamics of these waves.

4. Governing equations

One goal of this paper is to reintroduce Kelvin’s original mathematical treatment, which is summarized in Fig. 5. Starting with the linearized, inviscid, incompressible equations of motion in cylindrical coordinates, the existence of a base state flow is assumed, and the equations are linearized about this base state. Subsequently, a normal-mode solution is inserted, and the resulting simplified equation set is solved for the radial and azimuthal flow components. One then specifies the base state flow for a given vortex configuration, uses the mass continuity equation, and arrives at a linear ordinary differential equation (ODE) whose solution governs the radial wave structure. Application of the boundary conditions finally leads

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3 In plasma physics, the term kink wave is used to describe bending modes ($m = \pm 1$), which play an important role, e.g., in coronal loops in the solar atmosphere (e.g., Nakariakov and Ofman 2002). There are additional uses of the term kink wave in physics.
to the complete flow field as well as the dispersion relation, from which the wave frequencies and speeds, as well as the existence and growth rates of unstable modes, may be inferred.

a. Linearization of the momentum and mass-continuity equations

The radial, azimuthal (or tangential), and axial (here, vertical) velocity components, respectively, are written as \((u, v, w)\). The azimuthal velocity component is also called swirl velocity. The equation for the radial velocity \(u\) is given by (e.g., Drazin and Reid 1981, p. 71):

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + v \frac{\partial u}{\partial \theta} + w \frac{\partial u}{\partial z} - \frac{V^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r},
\]

where \(r\) is the radial distance from the center, \(\theta\) is the azimuth, and \(p\) is the pressure perturbation relative to a hydrostatic reference state. The density \(\rho\) is assumed to be constant in this treatment. The last term on the lhs is the centripetal acceleration, which appears when writing the equations of motion in cylindrical coordinates. For the azimuthal component \(v\) we have

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + v \frac{\partial v}{\partial \theta} + w \frac{\partial v}{\partial z} + \frac{V}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial \theta},
\]

where the last term on the lhs describes the effect of angular-momentum conservation, which like the centripetal acceleration, enters the equation when using cylindrical coordinates. The vertical momentum equation reads as

\[
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + v \frac{\partial w}{\partial \theta} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z},
\]

Since gravity does not appear in the equation, the implication is that the air is neutrally stratified (i.e., buoyancy is zero). Finally, the mass continuity equation for incompressible flows is given by

\[
\nabla \cdot \mathbf{v} = \frac{1}{r} \frac{\partial}{\partial r} (r \rho \frac{\partial v}{\partial r}) + \frac{\partial}{\partial \theta} (\rho v) = 0.
\]

The \(u/r\) term arises from the axisymmetry of the coordinate system (i.e., whenever the radial velocity \(u\) is nonzero there is nonzero radial divergence).

For the base-state vortex flow \(V(r)\) to be a solution of the above equations, a state of cyclostrophic balance is implied. The azimuthal base-state velocity \(V(r)\) may have an arbitrary dependence on \(r\). The axial base-state flow \(W\) is constant, and there is no radial base-state flow. Then,

\[
\begin{align*}
\mathbf{u} &= u(r, \theta, z, t), \\
v &= V(r) + v(r, \theta, z, t), \\
w &= W + w(r, \theta, z, t), \\
p &= P(r) + p'(r, \theta, z, t),
\end{align*}
\]

where \(P(r)\) is the base-state pressure and the primed variables denote perturbations from the base state. Throughout the analysis, \(V\) is assumed to be positive, so the base-state vortex spins cyclonically. To linearize the governing equations, this decomposition is inserted into Eqs. (1)–(4). For the radial momentum equation, one finds

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{r} [V + v'] \frac{\partial u}{\partial \theta} + \frac{1}{r} \frac{\partial}{\partial \theta} \left[ W + w' \frac{\partial u}{\partial z} - \frac{(V + v')^2}{r} \right] \\
&= -\frac{1}{\rho} \frac{\partial}{\partial r} (P + p').
\end{align*}
\]
Neglecting the products of perturbation variables and thereby linearizing the equation, gives
\[
\frac{\partial \psi}{\partial t} + \frac{V \psi}{r} \frac{\partial \psi}{\partial \theta} + W \frac{\partial \psi}{\partial z} = \frac{2 \psi}{r} + \frac{1}{\rho} \frac{\partial \rho}{\partial r} = 0.
\] (10)

The last two terms on the rhs cancel because the base state is cyclostrophically balanced, resulting in
\[
\frac{\partial \psi}{\partial t} + \frac{V \psi}{r} \frac{\partial \psi}{\partial \theta} + W \frac{\partial \psi}{\partial z} = 0.
\] (11)

The analogous procedure for the azimuthal velocity component gives
\[
\frac{\partial \psi}{\partial t} + \frac{V \psi}{r} \frac{\partial \psi}{\partial \theta} + W \frac{\partial \psi}{\partial z} = \frac{1}{\rho} \frac{\partial \rho}{\partial r}.
\] (12)

The last term on the lhs represents radial angular momentum advection. The vertical momentum equation becomes, after linearization,
\[
\frac{\partial \psi}{\partial t} + \frac{V \psi}{r} \frac{\partial \psi}{\partial \theta} + W \frac{\partial \psi}{\partial z} = \frac{1}{\rho} \frac{\partial \rho}{\partial z}.
\] (13)

Finally, for the continuity equation one obtains
\[
\frac{\partial \psi}{\partial t} + \frac{V \psi}{r} \frac{\partial \psi}{\partial \theta} + \frac{1}{\rho} \frac{\partial \rho}{\partial \theta} = 0.
\] (14)

Now a normal mode solution is assumed,
\[
\psi = r u e^{i(kz + m\theta - \omega t)},
\] (15)
\[
v = v e^{i(kz + m\theta - \omega t)} ,
\] (16)
\[
w = w e^{i(kz + m\theta - \omega t)} ,
\] (17)
\[
p = p e^{i(kz + m\theta - \omega t)} ,
\] (18)
where the “hatted” variables are complex amplitudes that only depend on \(r\), and \(\omega = \omega_r + i\omega_i\) is the complex phase speed. The axial wavenumber \(k\) is assumed to be real in this analysis. It is implied that only the real part of the solution has physical relevance (e.g., Markowski and Richardson 2010, chapter 6; Holton and Hakim 2013, chapter 5), which is sometimes written explicitly by adding the complex conjugate to the above normal modes (e.g., Fabre et al. 2006). Upon inserting these normal modes into the linearized governing equations, one finds that
\[
\begin{align*}
\frac{i \omega g + 2 \frac{V}{r} \frac{\partial \psi}{\partial \theta}}{\psi} &= \frac{1}{\rho} \frac{\partial \rho}{\partial r} , \quad (19) \\
\frac{i \omega g - \left[ \frac{V}{r} + \frac{1}{\rho} \frac{\partial \rho}{\partial r} \right] \frac{\partial \psi}{\partial r}}{\psi} &= \frac{\partial p}{\partial r} . \quad (20)
\end{align*}
\]

4 Kelvin (Thomson 1880) used a standing wave solution of the form \(\cos(kz) \cos(-m\theta + \omega t)\), presumably because he sought to apply his solutions to oscillations of vortex rings.
The matching condition, each side: contributions from the base state and its perturbation on vent infinite pressure gradient accelerations. Including the sure is enforced across the perturbed free boundary to pre-

\[ D\frac{R}{dt} = u' = \frac{\partial r'}{\partial t} + V \frac{\partial r'}{\partial r} + W \frac{\partial r'}{\partial z} \]  

(28)

and linearized using Eqs. (6) and (7):

\[ u' = \frac{\partial r'}{\partial t} + \frac{V}{r} \frac{\partial r'}{\partial r} + W \frac{\partial r'}{\partial z} \]  

(29)

Inserting the expression for \( r' \) [Eq. (27)] in Eq. (29) results in

\[ u' = -i\omega' + i\frac{V}{r} m r' + i W k r' = -i g r' \]  

(30)

and solving for the boundary displacement gives

\[ r' = \frac{i u'}{g} \]  

(31)

The matching condition, \( r'_1 = r'_2 \), is thus

\[ \frac{u'_1}{g_1} = \frac{u'_2}{g_2} \]  

(32)

or equivalently,

\[ \frac{\dot{u}_1}{g_1} = \frac{\dot{u}_2}{g_2} \]  

(33)

Next, the dynamic boundary condition is specified.

2) Dynamic boundary condition

For the dynamic boundary condition, continuity of pressure is enforced across the perturbed free boundary to prevent infinite pressure gradient accelerations. Including the contributions from the base state and its perturbation on each side:

\[ P_1(R_b) + \rho_1'(R_b; z, \theta, t) = P_2(R_b) + \rho_2'(R_b; z, \theta, t). \]  

(34)

Expanding pressure in a Taylor series around \( R \) approximately gives

\[ \left[ P_1(R) + \left. \frac{dP_1}{dr} \right|_{r=R} \right] r' + \left[ \rho_1'(R, z, \theta, t) + \left. \frac{\partial \rho'_1}{\partial r} \right|_{r=R} \right] r' \]  

\[ = \left[ P_2(R) + \left. \frac{dP_2}{dr} \right|_{r=R} \right] r' + \left[ \rho_2'(R, z, \theta, t) + \left. \frac{\partial \rho'_2}{\partial r} \right|_{r=R} \right] r'. \]  

(35)

Retaining only the first-order terms yields the dynamic boundary condition:

\[ \left[ P_1(R) + \left. \frac{dP_1}{dr} \right|_{r=R} \right] r' + \rho_1'(R, z, \theta, t) = \left[ P_2(R) + \left. \frac{dP_2}{dr} \right|_{r=R} \right] r' + \rho_2'(R, z, \theta, t). \]  

(36)

The first term in the brackets on either side of this equation represents the base-state pressure at the unperturbed boundary, and the second term in the brackets describes the variation of the base-state pressure along the perturbed boundary. The third term is the perturbation pressure at the unperturbed boundary.

Since the dynamic boundary condition is also fulfilled when there are no perturbations, \( P_1 = P_2 \), and the condition reduces to (see also Gallaire and Chomaz 2003)

\[ \left. \frac{dP_1}{dr} \right|_{r=R} r' + \rho_1'(R, z, \theta, t) \]  

(38)

The displacement \( r' \) may be eliminated using the kinematic boundary condition, Eq. (31), giving

\[ \left. \frac{dP_1}{dr} \right|_{r=R} u'_1 + \rho_1'(R, z, \theta, t) = \left. \frac{dP_2}{dr} \right|_{r=R} u'_1 + \rho_2'(R, z, \theta, t), \]  

(39)

or

\[ \rho_1'(R, z, \theta, t) = \rho_2'(R, z, \theta, t) + \frac{u'_1}{g_1} \left[ \left. \frac{dP_2}{dr} - dP_1 \right|_{r=R} \right]. \]  

(40)

Equivalently,

\[ \hat{\rho}_1(R) = \hat{\rho}_2(R) + \frac{u'_1}{g_1} \left[ \left. \frac{dP_2}{dr} - dP_1 \right|_{r=R} \right]. \]  

(41)

Instead of \( u_1/g_1 \), one may also use \( u_2/g_2 \), per Eq. (33). With this, all the tools needed to find the wave solutions are available. Kelvin remarked that “crowds of exceedingly interesting cases present themselves” (Thomson 1880, p. 157). Following Kelvin, the first scenario considered is the simplest one, with the goal of gaining an intuition for the structure and general behavior of the waves.

5. Solid-body rotation in a bounded domain

a. General solution and boundary condition

The case of solid body rotation in a bounded domain may be realized using a rotating, fluid-filled cylindrical vessel with rigid boundaries, which is a special case of the Couette flow (e.g., Kundu and Cohen 2008, p. 303). This scenario has been studied extensively in laboratory experiments (e.g., Chandrasekhar 1961 chapter VII; Fultz 1959). Here,

\[ W_1 = W = 0, \]  

(42)

\[ V_1(r) = V(r) = \Omega r, \]  

(43)

where \( \Omega \) is the uniform angular velocity of the fluid, and the rigid cylindrical boundary is located at \( r = R \). With this,

\[ dV/dr = V/r = \Omega. \]  

These conditions are inserted in Eqs. (24) and (26). First, it is noted that \( d \) [Eq. (25)] becomes

\[ d = 4\Omega^2 - (\omega - m\Omega)^2 \]  

(44)

and

\[ g = \omega - m\Omega. \]  

(45)

Inserting this in Eq. (24) gives

\[ \ddot{u} = i \frac{g}{k[4\Omega^2 - g^2]} \left[ \frac{dV}{dr} - \frac{2\Omega m}{r} \right]. \]  

(46)
For the azimuthal velocity, Eq. (26), one finds
\[
\dot{v} = \frac{g}{k(4\Omega^2 - \xi^2)} \left[ 2\Omega \frac{d\xi}{dr} - mg \frac{dr}{r} \right].
\]

(47)

These expressions for the velocity components are substituted into the continuity equation, Eq. (22), giving
\[
\frac{ig}{k(4\Omega^2 - \xi^2)} \left[ \frac{d}{dr} \frac{d\xi}{dr} - 2\Omega m \frac{1}{r} \frac{d\xi}{dr} + \frac{2\Omega m}{r} \dot{\xi} \right] + \frac{i}{k(4\Omega^2 - \xi^2)} \left[ \frac{1}{r} \frac{d}{dr} \frac{\xi}{r} \right] + \frac{g}{k(4\Omega^2 - \xi^2)} \left[ \frac{d\xi}{dr} - mg \frac{\xi}{r} \right] + ik\xi = 0.
\]

(48)

Further analysis (Arfken et al. 2013, p. 352; Deal 2018) shows that for integer \( m \) the general solution of Eq. (54) is
\[
\dot{\xi} = A J_m(\beta r) + B Y_m(\beta r),
\]

(55)

where \( A \) and \( B \) are arbitrary constants, and \( J_m(\beta r) \) and \( Y_m(\beta r) \) are, respectively, the Bessel functions of the first and second kind, of order \( m \). These are shown in Fig. 6. The Bessel function of the first kind of order \( m = 0 \), \( J_0(x) \), (red curve in Fig. 6a) qualitatively behaves like a damped cosine wave, and \( J_1(x) \)

(56)

(dashed black curve in Fig. 6a) behaves similarly to a damped sine wave. Since \( Y_m(\beta r) \) is negatively infinite at \( r = 0 \), this solution is rejected, so that
\[
\dot{\xi}(r) = AJ_m(\beta r).
\]

(57)

The eigenvalue \( \beta^2 \) represents the squared radial wavenumber. Because the boundary is rigid, the kinematic boundary condition is simply that the normal velocity at the boundary should vanish, so \( \dot{u}(R) = 0 \). Since the flow is inviscid, the flow parallel to the boundary, \((v, w)\), is free-slip. Using the equation for the radial velocity, Eq. (46), and the solution, Eq. (57), the boundary condition becomes
\[
\dot{u}(R) = -\frac{ig}{k(4\Omega^2 - \xi^2)} \left[ A\beta J_m(\beta r) - A\frac{2\Omega m}{R} J_m(\beta R) \right] = 0.
\]

(58)

Here the chain rule has been used,
\[
\frac{d\xi}{dr} = A \frac{d}{dr} J_m(\beta r) = A\beta J_m(\beta r),
\]

(59)

where the prime denotes the derivative with respect to the argument \( \beta r \). Equation (58) then becomes
\[
\beta R J_m(\beta R) - 2\Omega m J_m(\beta R) = 0,
\]

(60)

which may be written as
\[
\frac{J_m(\beta R)}{J_m(\beta R)} = \frac{2m\Omega}{\beta R g}.
\]

(61)

The goal is to find the \( \beta \) values for which this equation is fulfilled. One then solves Eq. (52) for the wave frequency \( \omega \) (which is defined via \( g \)) and obtains the dispersion relation.

**Note:** Kelvin (Thomson 1880) allowed the boundary to be perturbed, so \( \dot{u}(R) \neq 0 \) in his analysis of the bounded vortex in solid-body rotation.
The ratio of the Bessel functions behaves qualitatively like the (negative of the) tangent function, implying that this equation has a countably infinite set of solutions.

b. Axisymmetric modes in a bounded domain

For $m = 0$, one can find the solution of Eq. (61) right away. In this case the rhs is zero and use can be made of the fact that $J_0(R) = -J_1(R)$ (Arfken et al. 2013, p. 646), resembling the relationship $d/dx(\cos x) = -\sin x$. The fraction on the lhs of Eq. (61) becomes zero only where the numerator $J_0(\beta R)$ is zero [the zeros of the denominator, which make the equation singular, do not coincide with $J_1(\beta R) = 0$]. It follows that the requirement for a solution is

$$J_0(\beta R) = 0.$$  (62)

These roots are tabulated, the first three being $\beta_j R = \alpha_j = 3.8317, 7.0156, 10.1735, j = 1, 2, 3$. The different $\beta_j$ values correspond to the different discrete radial modes of the waves, which will be introduced in the next paragraph. Since $\beta_j$ may be interpreted as radial wavenumber, the root $\alpha_0 = 0$ is not considered here. Now Eq. (52) can be used to determine $g_j(k)$:

$$(a_j^2)^2 = R^2 k^2 \left[ \frac{4 \Omega^2}{g_j^2} - 1 \right].$$  (63)

Solving for the wave frequency, noting that now $g_j(k) = \omega_j(k)$, yields

$$\omega_j(k) = \pm \frac{2\Omega}{\sqrt{(a_j/k R)^2 + 1}},$$  (64)

which is the desired dispersion relation for the axisymmetric Kelvin vortex waves in a fluid in solid-body rotation. The axial (here, vertical) phase speed is given by

$$c_j = \frac{\omega_j(k)}{k} = \pm \frac{2\Omega}{\sqrt{(\alpha_j/R)^2 + k^2}}.$$  (65)

The dispersion relation for the first three radial modes is plotted in Fig. 7. The positive branches pertain to upward propagating waves, and the negative branches to downward propagating waves. Given the symmetry about the horizontal axis, only the positive branches of the phase and group speeds...
are shown. It follows that the lowest-order axisymmetric radial mode, often referred to as the fundamental mode, moves faster than the higher-order modes. Moreover, the phase and group speeds coincide for small $k$, and the longest axisymmetric waves of the fundamental radial mode propagate the fastest.

To gain insight into the structure of the different modes in meridional $(r, z)$ plane, one uses the equations for $\ddot{u}$ [Eq. (57)] and $\ddot{u}$ [Eq. (46)], and we put $w_1 = A$. Since the problem at hand is governed by a linear, homogeneous ODE, the amplitude $w_1$ may be selected arbitrarily. Then, starting with the radial velocity equation, Eq. (46), the solution $\dot{w}(r) = w_1 J_m(\beta r)$ is inserted:

$$\ddot{u} = -i \frac{k}{k^2 \beta^2} \left[ g \beta^2 J_m(\beta r) - \frac{2 \Omega m}{r} w_1 J_m(\beta r) \right].$$

Per Eq. (52) the denominator is equal to $\beta^2 g^2 / k$, and one may write

$$\ddot{u} = i \frac{w_1 k}{\beta} J_m(\beta r).$$

If $m = 0$ this equation becomes

$$\ddot{u} = i \frac{w_1 k}{\beta} J_0(\beta r).$$

Using again $J'_0(\beta r) = -J_1(\beta r)$,

$$\ddot{u} = -i \frac{w_1 k}{\beta} J_1(\beta r) = u_1 J_1(\beta r),$$

with the amplitude

$$u_1 = -i w_1 \frac{k}{\beta} \tag{70}$$

This shows that aside from the phase shift between the $u$ and $w$ waves, the vertical motion has a different magnitude than the radial motion; for instance, for small $k$, $u_1 \ll w_1$, and the perturbation flow is dominated by the vertical perturbation velocity (i.e., the perturbations are manifest mainly as axial jets; the reader may skip ahead to Figs. 17e,f for an example), and for large $k$, the perturbation flow is mainly in the radial direction.

The full equation for the perturbation vertical velocity is

$$w' = w_1 J_0(\beta r)e^{i(kz - \omega t)},$$

or, taking the real part,

$$w' = w_1 J_0(\beta r) \cos(kz - \omega t).$$

For the radial velocity using Eq. (69), one finds

$$u' = w_1 \frac{k}{\beta} J_1(\beta r) \sin(kz - \omega t).$$

Assuming the vortex is bounded by a cylinder with radius $R$, the boundary condition, $u(R) = 0$, is fulfilled only if $\beta_1 = \alpha \sqrt{gR} = 3.8317R/7.0156/R, 10.1735/R, \ldots$. Since $\beta$ may be interpreted as a radial wavenumber, these solutions dictate the radial structure of the wave. For the smallest eigenvalue one obtains a single cell in the radial direction, which is the classic “sausage” mode, shown in Figs. 8a and 8b. For the third eigenvalue, the result is three circulation cells in the radial direction, because the radial velocity distribution now has three roots. This configuration is shown in Figs. 8c and 8d. The higher the order of the solution, the more cells appear in the radial direction. The magnitude of these cells slowly decays with increasing distance from the axis. These different solutions are the radial modes (Alekseenko et al. 2007). Fultz (1959) presents laboratory observations of the fundamental and the second radial modes of these waves. Theoretically, for each azimuthal wavenumber there are infinitely many radial solutions [in reality, the high-order modes tend to be damped by viscous effects (Arendt et al. 1997; Fabre et al. 2006)]. Shapiro (2001a) arrived at solutions for nonlinear axisymmetric centrifugal waves with a similar radial structure as Kelvin vortex waves, although he assumed no radial boundary and the wave perturbations increased in magnitude with $r$.

RESTORING FORCE AND PROPAGATION MECHANISM OF THE AXISYMMETRIC MODE

To gain a better understanding of the propagation mechanism of the axisymmetric wave, it helps to consider the restoring force responsible for the oscillation. Markowski and Richardson (2010, p. 49) show that in the inviscid limit the radial displacement of a ring of parcels is governed by the following linear equation:

$$\frac{D^2 r}{D t^2} + \frac{1}{r} \Gamma^2 \frac{d r}{d t} = 0,$$  \tag{74}

where $\Gamma = \sqrt{V r}$ is proportional to the base-state circulation, which corresponds to the angular momentum. This pertains to the shortwave limit where the perturbation flow is mostly in the radial direction per Eq. (70). Equation (74) is a linear harmonic oscillation equation if $\sqrt{1/r^2}(\Gamma^2/dr) > 0$. In this context, the squared oscillation frequency, $\omega^2 = (1/r^2)(\Gamma^2/dr)$, is referred to as the Rayleigh discriminant (e.g., Chandrasekhar 1961). The restoring forces are the radial pressure gradient force and the centrifugal force. The oscillation frequency $\omega$ is related to the rate at which the ambient pressure gradient changes in the radial direction, which is linked to $\Gamma$ via cyclostrophic balance. A ring displaced in the positive radial direction is losing azimuthal speed due to angular-momentum conservation. Now if the circulation increases outward such that $\omega^2 > 0$, the radially displaced ring will find itself in a stronger pressure gradient than required for cyclostrophic balance. The result is an inward radial acceleration (i.e., stability; e.g., Markowski and Richardson 2010, p. 49). If the Rayleigh discriminant is negative, the displacement is unstable, which results in radial accelerations leading to toroidal circulation cells (e.g., Kundu and Cohen 2008, p. 486). For common vortex profiles, such as the Rankine and Burgers–Rott profiles (e.g., Wood and White 2011), or the Lamb–Oseen vortex (e.g., Fabre et al. 2006), $\Gamma$ increases in the outward direction in the core, so axisymmetric displacements are stable there (and neutral where $\Gamma = \text{const}$ as, e.g., in the periphery of the Rankine vortex).

To understand why the phase speed depends on the axial wavenumber, consider the variation of the width of the vortex (e.g., as measured by the radial displacement of a concentric material surface relative to the base state), which serves as a
measure of vertical vorticity. The narrower the vortex, the larger the vertical vorticity due to angular-momentum conservation. As shown in Fig. 9, which depicts an upward propagating wave, the regions of horizontal convergence near the vortex axis are shifted upward by a quarter wavelength relative to the axial vorticity maxima, implying that the vorticity extrema are propagating upward via vortex stretching and compression (see also Shapiro 2001a; Fabre et al. 2006). The regions of horizontal divergence and convergence are associated with meridional vorticity, which arises from tilting of the axial vorticity into and out of the meridional plane by the axial gradients of azimuthal velocity. The periodic vertical motion induced by the meridional vorticity stretches/compresses the axial vorticity. This is associated with a twisting and untwisting of the vortex lines (Melander and Hussain 1994; Arendt et al. 1997). The longer the wave, the larger are the pressure perturbations that accompany the vertical vorticity extrema [see, e.g., Markowski and Richardson (2010, p. 27) for the relationship between vorticity and pressure] or alternatively, the larger the induced velocity magnitudes (e.g., Dahl 2020) associated with the meridional vorticity extrema. As a consequence, as the vertical wavelength increases there are stronger vertical pressure-gradient accelerations and \( w' \) gradients, resulting in more vigorous stretching/compression of vertical vorticity and hence faster wave motion.

c. Spiral modes in a bounded domain

With \( m \neq 0 \), the wave frequency, using Eq. (52) with \( g = \omega - m\Omega \), is given by
muthal velocity perturbations of axial-vorticity stretching. The blue contours represent the azi-
locations of extrema of axial vorticity in the vortex center as well as
implying, e.g., a maximum of vertical perturbation vorticity near
the vortex axis where

\[ \omega_j(k) = m\Omega \pm \frac{2\Omega}{\sqrt{(\alpha_j/kR)^2 + 1}}. \]  

(75)

When changing from a left-handed to a right-handed mode, i.e., \( m \to -m \) with \( k > 0 \), the sign of \( \omega_j \) changes, but the dispersion behavior remains the same. Changing the sign of \( k \) does not change the wave frequency. Because of this symmetry, we may restrict the analysis without loss of generality to the case of \( k > 0 \) and \( m > 0 \), which describes the left-handed spiral modes (e.g., Saffman 1992, p. 232; Alekseenko et al. 2007). The dispersion relation, with \( \alpha_j \) obtained via Eq. (61), of the first three radial modes of the bending wave is shown in Fig. 10. Also displayed (Fig. 10a) is the graphical solution of Eq. (61). The cograde modes are represented by solid lines and have a positive oscillation frequency and axial phase speed (i.e., they move upward), and vice versa for the retrograde modes (Figs. 10b,c). The axial phase speeds behave like \( k^{-1} \) for small \( k \), and thus attain infinite values in the longwave limit (which results from the fact that the phase lines are almost parallel to the \( z \) axis). The group speeds, Fig. 10d, remain finite and are always slower than the phase speeds.

The different modes, like in the axisymmetric case, correspond to the radial structure. Insight into the horizontal structure of these waves may be obtained using the expressions for \( \tilde{u} \) and \( \tilde{v} \) [Eqs. (46) and (47)], and inserting the solution for \( w(r) = w_1J_1(\beta r) \). This gives, using again Eq. (52):

\[
\tilde{u}'(r, z, \theta, t) = -w_1 \frac{k}{g\beta} \left[ g\beta J_1(\beta r) - \frac{2\Omega m}{r} J_1(\beta r) \right] \times \sin(kz + m\theta - \omega t),
\]

(76)

and

\[
v'(r, z, \theta, t) = w_1 \frac{k}{g\beta} \left[ 2\Omega \beta J_1'(\beta r) - \frac{mg}{r} J_1(\beta r) \right] \cos(kz + m\theta - \omega t).
\]

(77)

To obtain the total flow, \( V(r) = \Omega r \) must be added to the last equation. These flow fields are plotted in Figs. 11 and 12. For \( m = 1 \), the perturbation flow (Fig. 11a) features a dipole, and the total flow (Fig. 11b) exhibits a displacement of the vortex center. For higher-order radial modes, the perturbation flow becomes more structured in the radial direction, but the total flow is still characterized mainly by a shift of the vortex center (Figs. 11c,d). To demonstrate the richness of structure of the higher-order modes, Fig. 12 shows the \( m = 4 \) spiral mode for the fundamental (Figs. 12a,b) and third (Figs. 12c,d) radial modes.

**PROPAGATION MECHANISM OF THE SPIRAL MODE**

To gain an understanding of the propagation of these modes, the velocity and the pressure fields of the fundamental spiral mode (\( m = 1 \)) are shown in Fig. 13. The fields are displayed on an unrolled cylindrical surface, i.e., in the \((\theta, z)\) plane and for a small axial wavenumber. Like in the axisymmetric case, positive axial vorticity perturbations are associated with a negative pressure perturbation, and negative axial vorticity perturbations are associated with a positive pressure perturbation, so \( p' \sim -\zeta' \), where \( \zeta' \) is the axial vorticity perturbation (Fabre et al. 2006). It is apparent that there is axial stretching a quarter wavelength above the perturbation vorticity maxima for the cograde (upward propagating) mode (Fig. 13a). For the retrograde mode, (Fig. 13b), the maximum stretching occurs below the perturbation vorticity maxima. Similarly to the axisymmetric mode, these waves thus propagate due to stretching and compression of axial perturbation vorticity.

To summarize this section, each of the azimuthal modes (i.e., axisymmetric and spiral modes) have an infinite number of radial modes, which are characterized by an increasing number of circulation cells in the radial direction with increasing order of the radial mode. The perturbation flow of the fundamental mode has only one circulation cell in radial direction. Moreover, it was highlighted that the longwave limit of the axisymmetric fundamental mode moves the fastest. A better approximation of a tornado-like flow is a vortex not bounded by cylindrical walls, which will be considered next.

**6. Rankine vortex**

*a. General solution and boundary conditions*

For the Rankine vortex, one combines the solution for solid-body rotation of the previous section with an irrotational outer region. As mentioned in section 4c, the inner solution is designated by the suffix “1” and the outer solution is designated by the suffix “2.” For now, it is assumed that \( W_1 = W_2 = W \), such that \( g_1 = g_2 \).

For the outer region, the irrotational azimuthal wind profile is given by (e.g., Kundu and Cohen 2008, p. 70):

\[
V(r) = \frac{\Omega R^2}{r},
\]

(78)
where \( R \) may be interpreted as core radius. Inserting this into the equation for the radial velocity, Eq. (24) with \( \beta \) reducing to \(-g^2\) per Eq. (25), leads to

\[
\frac{\vartheta}{\omega} = -\frac{g}{k} \left[ \frac{d\dot{\vartheta}}{dr} \right] \Rightarrow -i \frac{d\dot{\vartheta}}{k dr}.
\]

For the azimuthal velocity, Eq. (26), one finds

\[
\dot{u} = \frac{m\dot{\varphi}}{-g^2 k r} \left( -g^2 \right) - \frac{m\dot{\varphi}}{kr}.
\]

Like before, these expressions are inserted into the continuity equation, Eq. (22):

\[
\frac{d^2\dot{\vartheta}}{dr^2} + \frac{1}{r} \frac{d\dot{\vartheta}}{dr} \left[ k^2 + \frac{m^2}{r^2} \right] = 0.
\]

This equation is similar to Eq. (53) and becomes the modified Bessel differential equation upon substituting \( r = x/k \). The modified Bessel ODE equals the regular Bessel ODE if the argument is imaginary (Arfken et al. 2013, 680–683). If \( m \) is an integer, the solution is a linear combination of the Bessel functions.
encountered in the previous section but with imaginary argument (Arfken et al. 2013, 680–683). The solution of Eq. (81) becomes
\[ w(r) = A I_m(kr) + B K_m(kr). \] (82)
Here again \( A \) and \( B \) are arbitrary constants, and \( I_m(kr) \) and \( K_m(kr) \) are the modified Bessel functions of the first and second kind, respectively. These functions loosely resemble exponential functions, becoming infinite for large arguments in case of \( I_m \) and for small arguments in case of \( K_m \), as shown in Fig. 14. In the present case, the \( I_m \) solution is unphysical because the perturbations are required to be finite at radial infinity, so only the modified Bessel function of the second kind, \( K_m \), is retained. Then,
\[ \hat{w}(r) = BK_m(kr). \] (83)
It follows that Kelvin vortex waves are primarily core waves that rapidly decay with increasing radius in the outer region. The last step is to match the boundary displacement and the total pressure at the boundary between the inner and outer regions, as discussed in section 4c.

The kinematic condition, Eq. (33), with \( g_1 = g_2 \) simply enforces continuity of the normal velocities on each side of the boundary. Using Eq. (67) for the inner side, one finds that
The solution at the outer side of the boundary, per Eqs. (79) and (83), becomes
\[ \hat{u}_2(R) = iA \frac{k}{g\beta^2} \left[ g\beta J_m'(\beta R) - \frac{\Omega m}{R} J_m(\beta R) \right]. \] (84)

The kinematic boundary condition is thus
\[ \hat{u}_2(R) = -i \frac{k}{\beta^2} KBK_m'(kR) = -iBK_m'(kR). \] (85)

The kinematic boundary condition is thus
\[ A \frac{k}{g\beta^2} \left[ g\beta J_m'(\beta R) - \frac{\Omega m}{R} J_m(\beta R) \right] = -BK_m'(kR). \] (86)

Considering the dynamic boundary condition, the base-state pressure in the inner region is given by [appendix B, Eq. (B6)]
\[ P_1(r) = P_e + \frac{\rho}{2} \Omega^2 [r^2 - 2R^2], \] (87)

so that at \( r = R \),
\[ \frac{dP_1}{dr} \bigg|_{r=R} = \rho \Omega^2 R. \] (88)

In the outer region [appendix B, Eq. (B6)],
\[ P_2(r) = P_e - \frac{\rho}{2} \Omega^2 \frac{R^4}{r^2}, \] (89)

and at \( r = R \),
\[ \frac{dP_2}{dr} \bigg|_{r=R} = \rho \Omega^2 \frac{R^4}{r^3} \bigg|_{r=R} = \rho \Omega^2 R. \] (90)

This is equal to Eq. (88) and in this case the dynamic boundary condition, Eq. (41), reduces to
\[ p_1'(R) = p_2'(R). \] (91)
From Eq. (21) it follows that \( \dot{p} = (\rho g/r)\dot{w} \), so the dynamic boundary condition is equivalent to equating \( \dot{w} \) on each side of the boundary as was done by Kelvin (Thomson 1880). So,

\[
g \frac{\rho}{k} A J_m (\beta R) = g \frac{\rho}{k} B K_m (KR),
\]

or simply

\[
A J_m (\beta R) = B K_m (KR).
\]

Equations (86) and (93) form a linear system of equations, which has a nontrivial solution if the determinant of the coefficient matrix vanishes:

\[
\begin{vmatrix}
\frac{k}{gR^2} [g \beta J_m (\beta R) - 2 \frac{\Omega m}{R} J_m (\beta R)] & -K'_m (KR) \\
J_m (\beta R) & -K'_m (KR)
\end{vmatrix} = 0.
\]

The requirement for a nontrivial solution is thus

\[
\frac{k}{gR^2} [g \beta J_m (\beta R) - 2 \frac{\Omega m}{R} J_m (\beta R)] [-K'_m (KR)] - K'_m (KR) J_m (\beta R) = 0.
\]

Dividing through by \( J_m (\beta R) K'_m (KR) \) and some minor rearrangement gives the desired expression:

\[
\frac{1}{\beta R J_m (\beta R)} - 2 \frac{\Omega m}{gR^2} = \frac{1}{kR K'_m (KR)}.
\]

This is the dispersion relation derived 140 years ago by Kelvin [his Eq. (50)], and it must be solved for \( \beta \), which yields \( \omega \) via Eq. (52). A few words about the stability of the waves described by this dispersion relation are in order. For unstable growth, the equation must be fulfilled for an imaginary wave frequency \( \omega_i \), making \( g \) imaginary. However, the rhs of Eq. (96) is always real, so the imaginary part of the lhs must also be zero. Kelvin vortex waves in a Rankine vortex with uniform or vanishing axial base-state flow are thus stable. The solutions of this transcendental equation may be carried out graphically or numerically. Before proceeding with the numerical solutions, an important and in Kelvin’s words “curiously interesting” limit is considered.
b. Axisymmetric modes in the Rankine vortex: Longwave limit \((k < 1)\)

For the longwave limit of the axisymmetric waves \((m = 0)\), finding the solutions of Eq. (96) is straightforward. Figure 15 shows the rhs as blue horizontal lines for several choices of \(k\). The smaller \(k\), the larger the \(y\) intercept of the line, so for \(k \to 0\), the rhs \(\to \infty\). The lhs is also plotted, revealing singularities where \(J_0\) has its roots. As suggested by the graphic, for small \(k\) (long axial wavelength), the intersection of the lhs and rhs approaches the roots of \(J_0\) (shown as dashed vertical lines) as the rhs increases. See also Saffman (1992, p. 231).

\[
\omega_j(k) = \pm \frac{2\Omega}{\sqrt{\alpha_j^2(1/kR)^2 + 1}}.
\]

For small \(k\) one may use a Taylor-series expansion around \(k = 0\), which gives

\[
\omega_j(k) \approx \omega_j(0) \pm \left[ \frac{2\Omega}{(\alpha_j^2/R^2 + k^2)} \right]^{1/2} \frac{\alpha_j^2}{R^2} k.
\]

Since \(\omega_j(0) = 0\), this yields

\[
\omega_j \approx \pm \frac{2\Omega R}{\alpha_j} k.
\]

The axial phase speed, which for small \(k\) coincides with the group speed, is thus

\[
c_j = \frac{\omega_j}{k} = \pm \frac{d\omega_j}{dk} = \pm \frac{2\Omega R}{\alpha_j} = \pm \frac{2V_{\text{max}}}{\alpha_j},
\]

where \(V_{\text{max}} = V(R)\) is the maximum azimuthal base-state wind. For the smallest root, \(\alpha_1 \approx 2.4048\), corresponding to the fundamental radial mode,

\[
c_1 \approx \pm 0.83V_{\text{max}}.
\]

This means that in the longwave limit this mode propagates vertically at about 83% of the azimuthal velocity \(V\) at the radius of maximum winds \((\text{RMW})\), so the wave speed increases with the swirl velocity. Moreover, like in the bounded case discussed in section 5, the fundamental mode propagates the fastest (smallest \(\alpha_j\)). More generally (i.e., outside the limit of \(k < 1\)), the phase speed must be determined numerically, as will be done next.

c. Axisymmetric modes in the Rankine vortex: Beyond the longwave limit

To drop the restriction of small wavenumbers, one needs to find the intersection of the lhs and rhs of Eq. (96) numerically, an example of which is shown in Fig. 16a. The dispersion relations for axisymmetric waves in the Rankine vortex are shown in Figs. 16b–16d, revealing that the waves qualitatively behave like those in the bounded vortex case (section 5), the main difference being that in the longwave limit the waves in the Rankine vortex move faster by about 60% (cf. Figs. 7c and 16c for \(k = 0\)). Incidentally, these solutions are reminiscent of Love wave\(^6\) dispersion involving the tangent function, which behaves similarly to the ratio of Bessel functions.

To obtain the flow structure, one needs to match the inner solution based on the numerically determined \(\beta R\) value, with the outer solution at the RMW using either the kinematic [Eq. (33)] or the dynamic [Eq. (41)] boundary condition. This way the constant \(w_2 \approx \beta\) in Eq. (83) may be expressed in terms of \(w_1 = A\). For the pure Rankine-vortex case the dynamic boundary condition, \(\tilde{\rho}_1(R) = \tilde{\rho}_2(R)\) [Eq. (91)], is equivalent to \(\tilde{w}_1(R) = \tilde{w}_2(R)\) per Eq. (21). Thus, equating Eqs. (57) and (83) at \(r = R\), gives

\[
w_2 = w_1 \frac{J_1(\beta R)}{K_1(\beta R)}.
\]

As \(w_1\) may be chosen arbitrarily, this completely determines the inner and outer solutions. The results for the fundamental mode are shown for decreasing axial wavenumbers in Fig. 17. It is evident that the circulation cell widens with decreasing wavenumber, transitioning from a Couette-type solution that mainly affects the core, to an “open cell” solution where the outer part of the cell extends to radial infinity in the longwave limit. In addition, the flow becomes dominated by the axial perturbation velocity (Figs. 17e,f) as already inferred for the Couette solution. Regarding higher-order radial modes, the flow pattern resembles that of the Couette flow (Figs. 8c,d), except that the outermost cell intersects the RMW, “opening up” like the fundamental mode as the axial wavenumber decreases (not shown).

d. Spiral modes in the Rankine vortex

The spiral modes in the Rankine vortex behave similarly to those in a vortex bounded by a rigid cylindrical wall (section 5).

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\(^6\) Love waves are one type of waves propagating at the surface of elastic solids and they feature prominently in seismology because these waves tend to cause the most damage during earthquakes as they are associated with lateral displacements of the surface (Stein and Wyssession 2003, p. 91).
The dispersion relation for $\omega > 0$ (cograde modes) is shown in Fig. 18, and for $\omega < 0$ (retrograde/countergrade modes) in Fig. 19. The only fundamental difference to the bounded vortex is that now a countergrade mode appears (dashed black curves in Fig. 19). Both the phase and group speeds of this mode approach zero for small wavelengths (large $k$, Fig. 19).

The structure of the fundamental cograde modes for $m = 1$ and $m = 2$ is shown in Fig. 20, resembling the corresponding modes in the Couette flow, although the perturbation flow generally extends beyond the RMW, outside of which the waves rapidly decay. For higher-order radial modes, additional cells appear in the perturbation flow, just like in the case of the bounded vortex (not shown). The velocity perturbations are discontinuous across at $r = R$ as indicated by the kinks in the streamlines. This is a result of the discontinuous radial derivative of the base-state azimuthal velocity (i.e., the “cusp” in the velocity profile) of the Rankine vortex and is not seen in more realistic, smooth profiles (see, e.g., Walko and Gall 1984).\(^7\)

Figures 21a and 21b show the second radial retrograde mode for $m = 1$, which will briefly be returned to in the next section. The wave shown in Figs. 21c and 21d is the countergrade mode, which is interesting because it has unidirectional perturbation flow in the vortex core, rather than the cellular structure exhibited by the other modes. This mode shifts the entire core without any structural change. This is especially true for small $k$, where the perturbation velocity is not only unidirectional but

\(^7\)The total azimuthal flow, however, is continuous across the perturbed boundary.
FIG. 17. Wave structure in the meridional plane of an axisymmetric Kelvin vortex waves in a Rankine vortex. (a) Perturbation streamlines and (b) perturbation flow field for \( k = 5 \text{ m}^{-1} \). (c),(d) As in (a) and (b), but for \( k = 1 \text{ m}^{-1} \). (e),(f) As in (a) and (b), but for \( k = 0.05 \text{ m}^{-1} \). The red curve shows the instantaneous radial displacement of a material surface at a mean radial location \( r = R \) (dashed black line). Here \( \Omega = 1 \text{ s}^{-1} \), \( R = 1 \text{ m} \), and the amplitude \( \eta_1 \) was set to \( 1 \text{ m s}^{-1} \).
also uniform (the flow depicted in Figs. 21c and 21d exhibits a shear vorticity couplet, which disappears for $k \to 0$; not shown). This mode is impossible in a vortex bounded by a rigid cylindrical wall, and it is called “displacement” mode (Fabre et al. 2006) or “structureless” mode (Roy and Subramanian 2014), because of the absence of circulation cells. Kelvin (Thomson 1880) and Leibovich et al. (1986) refer to this wave as “slow wave.” In the longwave limit, where the perturbation flow is completely structureless, the motion of the perturbation may be understood as the combined effects of downstream advection by the mean flow and upstream advection due to flow induced by the vorticity perturbations. The latter take the shape of a thin vortex sheet as explained by (Batchelor 2002, p. 533) or Roy and Subramanian (2014). The induced flow associated with this vortex sheet accounts for the retrograde motion (Fig. 22a). Alternatively, this wave behaves like a helical vortex filament (Fuentes 2018). The self-induced motion of the helix acts to advect the helix downward (Fig. 22b), resulting in clockwise motion of the perturbations on a horizontal plane. These explanations rely on the vortex core being shifted as a whole while the axial vorticity remains unchanged—for shorter wavelengths, this is no longer the case and the propagation effect discussed in section 5c cannot be neglected.

The results presented so far portray the rich structure of Kelvin vortex waves, especially the infinite number of radial modes for each of the azimuthal modes (i.e., axisymmetric and spiral modes). Each of these modes obeys its own dispersion relation, and the mode with the fastest axial propagation speed is the axisymmetric fundamental mode in the longwave limit.
In this limit, phase and group speeds of the axisymmetric modes coincide. These results were obtained for a bounded vortex in solid body rotation as well as for a Rankine vortex with uniform (or vanishing) axial base state flow \( W \). Note that some details regarding the wave structure (e.g., the dependence of the radial structure on the axial wavenumber) are different for non-Rankine profiles (e.g., Fabre et al. 2006). In the following, \( W \) is allowed to have different values in the inner and outer domains, which allows for unstable growth of vortex waves.

7. Vortex instabilities

a. Rankine vortex with axial jet in its core

Now the same scenario as in the previous section is considered, but there is a uniform upward axial base-state flow in the core \( r < R \) and no base-state axial flow in the irrotational outer region of the vortex (i.e., a so-called “plug-flow” \( W \) profile is assumed). This flow loosely mimics a tornado away from the lower boundary, and it will be shown that this flow has unstable solutions, which will be discussed in the context of vortex breakdown in section 8. This scenario was investigated in detail by Loiseleux et al. (1998).

The only difference to the treatment in the previous section is that in the inner domain

\[
\dot{g}_1 = \omega - \Omega m - Wk, \tag{103}
\]

and consequently, \( \beta \) [Eq. (52)] is defined using \( \dot{g}_1 \); in the outer domain

\[
\dot{g}_2 = \omega - \Omega m, \tag{104}
\]
signifying that there is a uniform updraft only in the vortex core. The kinematic boundary condition [Eq. (33)], analogously to Eq. (86) with \( g_1 \neq g_2 \) becomes
\[
\frac{iA}{g_1^2 \beta^2} \left[ g_1 \beta J_m'(\beta R) - 2 \frac{\Omega m}{R} J_m(\beta R) \right] = - \frac{iB}{g_2} K_m(kR). \tag{105}
\]
The dynamic boundary condition, Eq. (41) again reduces to
\[
\rho'(R) = \rho_1'(R), \tag{106}
\]
with which \( \rho = (\rho g/k) \omega' \) [Eq. (21)] becomes
\[
A \frac{g_2}{k} J_m'(\beta R) = B \frac{g_2}{k} K_m(kR). \tag{107}
\]
This leads to the following requirement for the existence of a nontrivial solution:
\[
\frac{k}{g_1^2 \beta^2} \left[ g_1 \beta J_m'(\beta R) - 2 \frac{\Omega m}{R} J_m(\beta R) \right] \frac{K_m'(kR)}{g_2} = 0. \tag{108}
\]
Following the same steps as in the previous section, but with \( g_1 \neq g_2 \) yields the following condition for the existence of a nontrivial solution:
\[
D = g_2^2 \left[ \beta R J_m'(\beta R) \frac{2\Omega m}{g_1} - \frac{R^2 \beta^2 \beta^2 K_m'(kR)}{kR} \right] = 0. \tag{109}
\]

FIG. 20. Structure in the horizontal plane of cograde, fundamental Kelvin vortex wave modes in a Rankine vortex. (a) Perturbation streamlines for \( m = 1 \), (b) total flow field; here, \( k = 1.0 \text{ m}^{-1} \). (c),(d) As in (a) and (b), but for the \( m = 2 \) mode. Here \( \Omega = 1 \text{ s}^{-1} \) and \( R = 1 \text{ m} \) (dashed black line), and the amplitude \( w_1 \) was set to 1.25 m s\(^{-1}\).
This is a complex transcendental equation, whose solution, \( \omega(k) \), must be found numerically. Specifically, one needs to determine the values of the complex \( \omega \) for which the equation is fulfilled. Following Loiseleur et al. (1998) one seeks the condition for which both the real and imaginary parts of \( D \) are zero. This is done by interpreting \( D_r = \text{Re}[D] \) and \( D_i = \text{Im}[D] \) as 2D functions of both \( \omega_r \) and \( \omega_i \) for given \( k \), \( m \), and \( \Omega \). So, \( D = D(\omega_r, k; m, \Omega) \). The frequencies enter the equation via \( g_1 = \omega_r + i\omega_i - \Omega m - Wk \) and \( g_2 = \omega_r + i\omega_i - \Omega m \). The values of \( D_r(\omega_r, \omega_i) \) and \( D_i(\omega_r, \omega_i) \) are assigned on a 2D grid with real and imaginary \( \omega \) as axes, respectively. This gives two fields, \( D_r \) and \( D_i \), and the \((\omega_r, \omega_i)\) pair where \( D_r = D_i = 0 \) (i.e., where the zero contours of each field intersect) is the desired solution. This procedure needs to be repeated for each axial and azimuthal wavenumber. Examples of the solutions for \( m = \pm 1 \) on the complex plane are included as an online supplement.

1) **STABLE MODES**

For the stable modes, the graphical/numerical procedure used for the pure Rankine vortex in section 6 may be used because \( \omega \) and hence \( D \) are real. An example for \( m = 1 \) is shown in Fig. 23, relative to a reference frame rotating, and rising, with the vortex core. The result bears close resemblance to the dispersion relations shown in Figs. 18 and 19, suggesting that these stable waves correspond to the stable Kelvin modes in the Rankine vortex scenario without gradients in axial base-state flow.

2) **UNSTABLE MODES**

The growth rates for the spiral modes, \( m = \pm 1 \), are shown in Fig. 24. The \( m = -1 \) mode grows faster than the \( m = +1 \)
mode, a result briefly returned to in section 8. To see how the instability arises consider the shortwave limit, \( \frac{kR}{C^2} \ll 1 \), where the dispersion relation, Eq.(109) was shown by Loiseleux et al. (1998) and Alekseenko et al. (2007, p. 177) to reduce to

\[
\omega = \frac{kW}{2} \pm i \frac{kW}{2},
\]

(110)

which is precisely the dispersion relation for flows exhibiting Kelvin–Helmholtz (KH) instability (e.g., Batchelor 2002, p. 511; Kundu and Cohen 2008, p. 493). This suggests that the unstable modes in the shortwave limit are a result of a KH instability due to the radial \( W \) gradient. For longer wavelengths, an additional effect emerges. Figure 23 shows the intrinsic frequency of the unstable \( m = +1 \) mode (thick blue line), which intersects the frequency curves of the stable modes. At the intersections, a resonance between the stable retrograde modes and the unstable mode occurs. As a consequence, the unstable mode inherits the radial complexity of the stable mode it resonates with. The unstable wave, shown in Fig. 25 at the resonance wavenumber \( k \approx 1.1 \) (Fig. 23), features a spiral structure, which becomes increasingly “curled up” as the wavenumber decreases, reflecting the increased radial complexity for small wavenumbers. The azimuth with the most radial structure (red arrow in Fig. 25) has the same number and placement of extrema of the eigenfunctions as the stable mode (Fig. 21a; see also Loiseleux et al. 1998). The spiral structure results from the wave frequency being complex, thus rendering \( b \) and \( J_m(bR) \) complex as well, leading to a radially dependent phase shift.

The normal velocities across the perturbed boundary are discontinuous because of the different \( W \) in each region (only the displacement is required to be continuous across the boundary). A 3D animation of a pressure surface of this wave is available as online supplement. The main insight from this analysis is that the instability in this scenario results from a
coupling of resonance and KH instability due to the updraft gradient (Loiseleux et al. 1998). A common scenario in which an unstable vortex configuration such as the one discussed in this subsection arises is during vortex breakdown within the region of the breakdown bubble, as will be discussed in section 8.

b. Cylindrical vortex sheet

As a final scenario, the case of a cylindrical vortex sheet is considered, with downward axial flow in the otherwise resting vortex core (i.e., there is no base-state azimuthal velocity in the core). The outer region is characterized by an irrotational vortex and upward axial velocity. This case is a generalization of Kelvin’s (Thomson 1880) “hollow vortex” scenario in which there is no fluid altogether in the inner region of the vortex (see also Alekseenko et al. 2007, p. 175). Here the vortex is not truly hollow but the base state has no swirl velocity in the vortex core. This flow configuration crudely mimics a tornado flow after a two-celled structure (section 9) has developed, with the effect of the lower boundary being ignored. This scenario was discussed by Rotunno (1978) to provide a simplified model of multiple-vortex development, and his results also directly follow from Kelvin’s approach.

1) INNER REGION

In the inner domain, \( V_1 = \Omega r = 0 \). The axial base-state velocity \( W_1 < 0 \) and \( |W_1| = W \). This gives

\[
g_1 = \omega + Wk, \tag{111}
\]

and for the radial velocity, Eq. (24), one finds with \( V = 0 \)

\[
\hat{u}_1 = \frac{i \, dw_1}{k \, dr}. \tag{112}
\]

The azimuthal velocity, Eq. (26), becomes

\[
\hat{v}_1 = \frac{m \hat{w}_1}{r k}. \tag{113}
\]

These happen to be the same expressions found for the irrotational outer region of the Rankine vortex. Inserting these into the mass continuity equation, Eq. (22), thus again gives the modified Bessel differential equation for \( \hat{w}_1 \), which is solved by

\[
\hat{w}_1(r) = A I_m(kr) + B K_m(kr), \tag{114}
\]

where \( A \) and \( B \) are again arbitrary constants. Because the solution is required to be finite in the core, this time only the first term on the rhs of Eq. (114) is retained:

\[
\hat{w}_1(r) = A I_m(kr). \tag{115}
\]

Thus, per Eq. (112),

\[
\hat{u}_1(r) = -i A I_m(kr) \tag{116}
\]

and per Eq. (21),

\[
\hat{p}_1(r) = A \frac{\rho \hat{w}_1}{k} I_m(kr). \tag{117}
\]

2) OUTER REGION

Here, \( V_2(r) = \Omega R^2/r \), and \( W_2 > 0 \), leading to

\[
g_2 = \omega - m \Omega - Wk, \tag{118}
\]

and again, like for the outer region of the Rankine vortex in section 6, one finds for the radial and azimuthal velocities, respectively,

\[
\hat{u}_2 = -i \frac{d \hat{w}_2}{k \, dr} \tag{119}
\]

and

\[
\hat{v}_2 = \frac{m \hat{w}_2}{r k}. \tag{120}
\]

The resulting solution for \( \hat{w}_2 \) is thus again
\[ \dot{w}_2(r) = BK_m(kr), \]

implying
\[ \dot{u}_2(r) = -iBK'_m(kr) \]

and
\[ \dot{\rho}_2(r) = B\frac{\bar{g}_2\rho}{k}K_m(kr). \]

3) MATCHING CONDITION

The kinematic boundary condition, Eq. (33), reads
\[ \frac{-i\Lambda I_m(kR)}{g_1} = \frac{-iBK'_m(kR)}{g_2}. \]

For the dynamic boundary condition, first note that \( P_1 = \text{const.} \), so \( dP_1/dr = 0 \). In the irrotational outer region, as shown in appendix B, one again finds
\[ P_2(r) = P = \frac{\rho \Omega^2 R^2}{2}, \]

and consequently
\[ \frac{dP_2}{dr} |_{r=R} = \rho \Omega^2 R. \]

Then, using the dynamic boundary condition [Eq. (41)],
\[ Ag_1 I_m(kR) = Bg_2 K_m(kR) + A \frac{k\Omega^2 R}{g_1} I'_m(kR). \]

Like before, the boundary conditions [Eqs. (124) and (127)] constitute a linear homogeneous system of equations, and for a nontrivial solution to exist, the determinant of the coefficient matrix needs to vanish, which leads to the desired dispersion relation:
\[ \frac{k\Omega^2 I'_m(kR)K'_m(kR)}{g_1g_2 I_m(kR)K_m(kR)} - \frac{g_1 I'_m(kR)}{g_2 I_m(kR)} - \frac{g_1 K'_m(kR)}{g_2 K_m(kR)} = 0. \]

This equation is equivalent to Rotunno’s Eq. (2.20) (Rotunno 1978), which is again a complex transcendental equation, which has been solved in the same manner as Eq. (109) in the previous subsection. The result is that the cylindrical vortex sheet with differential axial flow is unstable for all azimuthal modes across a wide range of axial wavelengths [the waves are unstable for all wavelengths for \( m > 1 \); Rotunno (1978)]. The growth rates for the \( m = 2 \) and \( m = 3 \) modes are shown in Fig. 26. The graphical solution of Eq. (128) is included as online supplement. The unstable modes of the cylindrical vortex sheet for \( \Omega = 1 \, \text{s}^{-1}, R = 1 \, \text{m}, \) and \( W = 1 \, \text{m/s}^{-1} \), which acts to deform the radius of maximum winds into an ellipse (shown in red in Fig. 27; this ellipse may ultimately close off into two separate vortices as the instability progresses). A 3D animation of this wave is included as an online supplement. Because of the \( W \) gradient across the boundary, the normal velocities are again discontinuous like in the previous case. Much like the waves along a rectilinear vortex sheet may be interpreted as barotropic Rossby waves (Hoskins et al. 1985), these Kelvin vortex waves may be interpreted as vortex Rossby waves (e.g., Montgomery and Kallenbach 1997).

To explain why the \( m = 2 \) mode is observed to be the most unstable one in tornado vortex chambers during multiple vortex development (e.g., Walko and Gall 1984), Rotunno (1978) suggested that it fulfills the compromise between (i) being too small in scale and thus being dampened by diffusive processes, and (ii) being too large to fall outside the maximum growth regime for finite-thickness shear layers [the fastest growing perturbation has a wavelength on the order of the thickness of the shear layer (Rotunno 1978)]. Although it is clear that more complex models are necessary to obtain a realistic representation of the unstable modes occurring in tornadoes, the analyses of the unstable modes highlight that it is not only the radial gradient of the azimuthal flow that is responsible for the energy growth of the perturbations like in parallel shear flows. Instead additional gradients, such as the radial variation of the axial base-state flow, are also important.

Kelvin’s approach allows for a large number of additional solutions, which may be analyzed in the same manner as those presented here [e.g., the axial flow may be varied in a systematic manner as done by, e.g., Loiseleux et al. (1998) and Gallaire and Chomaz (2003), or additional radial profiles of the azimuthal flow may be evaluated].

\[ \text{FIG. 26. Growth rate of the } m = 2 \text{ (red) and } m = 3 \text{ (black) unstable modes of the cylindrical vortex sheet for } \Omega = 1 \, \text{s}^{-1}, R = 1 \, \text{m}, \text{ and } W = 1 \, \text{m/s}^{-1}. \]
c. Unstable waves in realistic tornado-like vortices

The previous analyses provide some insight into the structure and stability of Kelvin vortex waves, but admittedly, these infinitely extended vortices are not quite suitable to model a tornado, where at low levels there are large gradients of the radial inflow as well as of the azimuthal and vertical velocity components (e.g., Snow 1982). To address this problem, Walko and Gall (1984) performed a linear stability analysis of a 3D numerical simulation of vortex chamber experiments, and found that the azimuthal wavenumber exhibiting the fastest growth increases with increasing swirl ratio (see next section), just like in the vortex chamber experiments. In addition, they found that the unstable modes for smaller swirl ratios gained their energy from radial gradients of vertical velocity, while for larger swirls, the radial gradients of azimuthal velocity became more relevant. However, Walko and Gall (1984) used a free-slip lower boundary, prompting Nolan (2012) to revisit the problem, analyzing simulations with a no-slip lower boundary. His work is arguably the most comprehensive analysis of flow instabilities associated with tornadoes to date. In particular, he investigated a one-celled vortex, a two-celled vortex, and a drowned vortex jump (see section 8). Which velocity gradient contributes the most to the energy exchange depends on the vortex structure. Perhaps surprisingly, the primary source was the vertical variation of the vertical wind, a contribution neglected in the analytical results above. Additional energy sources for the perturbations in the cases he considered are the radial shear of the vertical wind and the vertical shear of the azimuthal wind, the latter being particularly relevant for the two-celled vortex. This highlights the limitations of the analytical models discussed in this paper. Nolan (2012) found that unstable axisymmetric modes lead to oscillations of the vortex core, in some respects resembling those observed by, e.g., Bluestein et al. (2003) in a real tornado. The types of waves discussed by Nolan (2012) are much more complex than those included in Kelvin’s equations, but Kelvin’s treatment may be considered a starting point to understand the more complex wave structures emerging in 3D full-physics simulations.

To summarize this section, the main finding is that vortex waves may become unstable and that the instability arises from radial gradients of $W$ in the Rankine-vortex case, and from both the radial gradients of $V$ and $W$ in the case of the “two-celled” vortex modeled using a cylindrical vortex sheet. Multiple-vortex formation in tornadoes is thus more complex than the traditional shear instability. In fact, in more realistic tornado flow models with a no-slip lower boundary, radial and vertical gradients of the horizontal and vertical flow components all contribute to the growth of the perturbations. The main concepts discussed so far, i.e., the propagation characteristics of stable vortex waves as well as vortex instabilities may be leveraged to explain what may be one of the most complex phenomena in vortex dynamics: Vortex breakdown, which will be discussed next.

8. Vortex breakdown

As an application of the results obtained so far, vortex breakdown is considered, which often occurs in vortices with nonzero axial flow such as tornadoes. An example of a vortex breakdown is shown in Fig. 28. This phenomenon is characterized by an abrupt widening of the vortex core, and a stagnation or reversal of the axial velocity in the downstream direction (e.g., Leibovich 1978). A feature of vortex breakdown is the axisymmetric “breakdown bubble,” in which flow stagnation and reversal occurs. In the vicinity of the breakdown bubble, the flow tends to become unstable with respect to helical disturbances, as also seen in Fig. 28. This instability may lead to the so-called spiral breakdown (Lambourne and Brayer 1961). There are several additional forms of vortex breakdown, including the conical breakdown (Sarpkaya 1995), as reviewed by, e.g., Khoo et al. (1997) or Luca-Negro and O’Doherty (2001), with axisymmetric breakdown (also called bubble breakdown) and spiral breakdown being the most common

![FIG. 27. As in Fig. 25, but for the unstable $m = 2$ cylindrical vortex sheet case with uniform rising motion in the periphery and uniform sinking motion in the interior of the vortex. Here $\Omega = 1 \text{s}^{-1}$, $R = 1 \text{m}$, $W = 1 \text{m} \text{s}^{-1}$, $\omega = 0.909 + 1.691 \text{i} \text{s}^{-1}$, and $k = 1 \text{m}^{-1}$. The red ellipse in (b) represents the deformed boundary originally at $r = R$ (black dashed line). The amplitude $w_1$ was set to $0.2 \text{m} \text{s}^{-1}$. An animation of this solution is available as an online supplement.]
modes. In the atmosphere, bubble breakdown has been observed in waterspouts (e.g., Grotjahn 2000, his Fig. 3). In this section the main focus is on the axisymmetric breakdown, while the processes leading to the different breakdown modes will briefly be discussed at the end of this section. The focus here is on the role of Kelvin vortex waves in the onset of vortex breakdown.

Vortex breakdown has been reviewed by several authors, e.g., Hall (1972), Leibovich (1978), Escudier (1988), Keller (1995), or Lucca-Negro and O’Doherty (2001). These reviews provide a comprehensive picture of the observations of vortex breakdown, but generally emphasize the lack of a theoretical framework to describe the phenomenon. However, there has been a lot of theoretical development since these reviews were written, allowing for a relatively robust picture of the basic breakdown mechanisms to emerge. In particular, Ruith et al. (2003) offer a thorough treatment of vortex breakdown, and they are able to explain many of the observed structures by analyzing direct numerical simulations.

a. Hydraulic jump analogy

The basic concept describing the onset of vortex breakdown may be understood as an axisymmetric analog of a hydraulic jump. Perhaps the easiest way to illustrate the hydraulic jump is to consider a channel with water entering through an open end while the other end is closed, as shown in Fig. 29.10 As the water spreads downstream, it slows down somewhat due to surface drag, and as it reaches the end wall (Fig. 29a) it will slosh backward (upstream) as shown in Fig. 29b. This is accomplished by a surface gravity wave propagating in the upstream direction. The fastest surface waves are those with the longest wavelength (e.g., Kundu and Cohen 2008, p. 223), and in the longwave limit the phase speed approaches \( c = \sqrt{G\bar{h}} \), where \( G \) is the gravitational acceleration, and \( \bar{h} \) is the base-state water depth (e.g., Holton and Hakim 2013, p. 141). If the flow speed \( U \) at the inlet is greater than \( c \), the wave will encounter increasingly opposing flow and is unable to propagate upstream beyond the critical location where \( U = c \). At that point, the waves coming from the closed end will “pile up” and create a steep, stationary turbulent water front (Fig. 29c). This jump is analogous to a shockwave, where upstream conditions are supersonic and downstream conditions are subsonic (Anderson 2002). For channel flows, instead of the Mach number, one considers the Froude number, which is the ratio between the stream velocity and the wave speed in the longwave limit. Upstream of the jump the flow is supercritical \((F > 1)\) and in the wake of the jump the flow is subcritical \((F < 1)\). The jump is located where the flow is critical \((F = 1)\). Squire (1960) developed a theory that treats the abrupt widening of the vortex analogously to the hydraulic jump. If the axial flow \( W \) is faster than the upstream propagation speed of the fastest axisymmetric Kelvin mode, the flow is supercritical. If downstream the flow should become subcritical, it is possible for waves to travel upstream, up to the point where \( W = c \) and a jump (i.e., vortex breakdown) occurs. The result is a stationary soliton-like wave that constitutes the breakdown bubble (Leibovich and Randall 1973; Randall and Leibovich 1973).

While this review focuses on waves, and while the “wave theory” of vortex breakdown offers a physical mechanism by which the jump may be accomplished, for quantitative predictions regarding the magnitude of the jump, conservation properties across the jump must be invoked (e.g., Benjamin 1962; Fiedler and Rotunno 1986; Lewellen and Lewellen 2007). In this context, the flows upstream and downstream of the jump are “conjugate” to one another. Moise (2020a) offers an evaluation of the validity of the conjugate flow theories by Benjamin (1962) and Wang and Rusak (1997).

Squire’s (1960) result for the Rankine vortex may be obtained directly from Kelvin’s approach by considering axisymmetric perturbations \((m = 0)\) and assuming uniform axial velocity \( W \) inside and outside the vortex core. The solution is
precisely that found in section 6 [Eq. (99)], except that now $g = \omega - Wk$ because of the Doppler shift associated with the axial base-state flow. This solution pertains to the fastest axisymmetric mode, which has the least radial structure and the longest axial wavelength (i.e., the fundamental mode in the longwave limit in which $k \to 0$, corresponding to $\alpha = \beta R = 2.4048$). To apply this solution to determine flow criticality analogous to Froude or Mach numbers, Eq. (99) thus, is used:

$$\omega = g + Wk = \pm \frac{2kV_{\text{max}}}{\alpha} + Wk. \tag{129}$$

The phase and group speeds are identical (which is true for arbitrary velocity profiles in the longwave limit; Leibovich 1979), and relative to a stationary observer these speeds may be expressed as

$$c = \pm \frac{2V_{\text{max}}}{\alpha} + W. \tag{130}$$

The jump occurs when the upward vertical flow matches the downward wave speed, which implies that $c = 0$, again relative to a stationary observer. Then we have

$$W = \frac{2V_{\text{max}}}{\alpha}, \tag{131}$$

or

$$\frac{V_{\text{max}}}{W} = \frac{\alpha}{2}. \tag{132}$$

This expression has far-reaching consequences, because it connects the flow criticality to a quantity known as swirl parameter or swirl ratio (e.g., Loiseleux et al. 1998),\textsuperscript{11} which is given by

$$S_l = \frac{\Omega R}{W} = \frac{V_{\text{max}}}{W}. \tag{133}$$

As defined here, and following, e.g., Squire (1960), the swirl ratio is defined locally, and it is allowed to vary along the vortex axis. The critical local swirl ratio is consequently given by

$$S_{l\text{crit}} = \frac{V_{\text{max}}}{W} = \frac{\alpha}{2} = \frac{2.4048}{2} \approx 1.2. \tag{134}$$

This is the result obtained by Squire (1960), and it follows directly from Kelvin’s approach [the phase speed of this longwave limit is discussed in Kelvin’s paper as an example (Thomson 1880, p. 166)]. In this context, the local swirl ratio describes the relative importance of downward propagation of the wave, which depends on $V_{\text{max}}$, and upward advection of the wave, associated with $W$. The larger $S_l$, the more important downward propagation becomes relative to upward advection. The characteristic number describing the criticality

\textsuperscript{11}The swirl parameter is also referred to as swirl number, or simply swirl. It is the inverse of the so-called vortex Rossby number.
of the flow (in analogy to the Mach or Froude numbers), is given by
\[ N = \frac{W}{C} = \frac{\alpha}{2V_{\text{max}}} \frac{W}{C}, \]  
(135)
where \( C = 2V_{\text{max}}/\alpha \) is the wave speed relative to the rising air. Since \( V_{\text{max}}/W = \tan \delta \), where \( \delta \) is the "swirl angle," it follows that for \( N = 1, \delta \approx 50^\circ \) (Lutg 1989). Consequently, supercritical conditions exist in regions where \( N > 1, S_I < 1.2, \) or \( \delta < 50^\circ \). For vortex profiles different than the Rankine vortex, Squire (1960) found similar values for the critical swirl ratio. To determine the criticality of the vortex flow at a given location in the axial direction, one may calculate \( S_I \) at each location along the axial direction and determine where \( S_I = S_I^{\text{crit}} \) or \( N = 1 \). Applying the above criteria to determine flow criticality requires knowledge of the group speed of the fastest axisymmetric Kelvin mode— to avoid explicit calculation of the group speed, Benjamin (1962) proposed a different (albeit equivalent) approach, which is briefly addressed next.

b. Connection to Benjamin’s criticality test

Most treatments of vortex breakdown do not take the detour via Kelvin’s time-dependent solutions discussed above, but standing (or stationary) wave solutions are considered. A brief introduction into the treatment is presented here, mainly to demonstrate that this treatment is consistent with Kelvin’s approach. The existence of free, standing vortex waves may be used to infer subcriticality (Benjamin 1962), and many features of the breakdown bubble itself may be described as a solitary wave (Leibovich and Randall 1973; Randall and Leibovich 1973; Darmofal and Murman 1994; Alekseenko and Shkort 1997; Alekseenko et al. 2007, p. 215). The starting point is the inviscid 3D equation of motion in vector form (Markowski and Richardson 2010, p. 21):
\[ \frac{\partial \mathbf{v}}{\partial t} + \nabla \times \mathbf{v} = -\frac{1}{\rho} \nabla p, \]  
(136)
where \( U \) is the 3D velocity magnitude and \( p \) is the perturbation pressure. Now \( \rho \) is again taken to be constant and the flow is assumed to be stationary, leading to
\[ \mathbf{v} \times \omega = \nabla \times \left( \frac{p}{\rho} + \frac{U^2}{2} \right) = \nabla H, \]  
(137)
where \( H = p/\rho + (1/2)U^2 \) is the “total head” (i.e., total energy). In cylindrical coordinates, and assuming axisymmetry (\( \partial / \partial \theta = 0 \)), we have
\[ \begin{bmatrix} \nu \xi - \nu \eta \\ \nu \xi - \nu \xi \\ \nu \xi - \nu \xi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \partial H / \partial \zeta \end{bmatrix}, \]  
(138)
where \( (\xi, \eta, \xi) \) are the radial, azimuthal, and axial (or in this case, vertical) vorticity components, respectively. Although any of the components of Eq. (138) may be considered, the easiest way to proceed is to take the \( \xi \) component and solve it for the azimuthal vorticity \( \eta \).
\[ \eta = \frac{1}{u} \partial H / \partial \zeta + \frac{\nu}{u} \xi. \]  
(139)
As shown in appendix C, this is equivalent to
\[ \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial \zeta^2} = r \frac{dH}{d\phi} \frac{dI}{d\phi} = 0, \]  
(140)
where \( \phi \) is Stokes’ streamfunction in the \((r, \zeta)\) plane and \( I = (1/2)\Gamma^2 \) with \( \Gamma = ru \) being proportional to the circulation. Equation (140) is referred to as Squire—Long or Bragg-Hawthorne equation (Bragg and Hawthorne 1950; Long 1953; Squire 1956). Benjamin (1962) used the linearized version of this equation in a rather elegant manner. The linearized equation is given by (see appendix C):
\[ \frac{\partial^2 \phi'}{\partial r^2} + \frac{1}{r} \frac{\partial \phi'}{\partial r} + \frac{\partial^2 \phi'}{\partial \zeta^2} + \left[ \frac{1}{rW} \frac{dW}{dr} - \frac{1}{W} \frac{dW}{dr^2} + \frac{2}{W} \frac{dI}{dr} \right] \phi' = 0, \]  
(141)
where \( \phi' \) is the perturbation streamfunction and \( I = (1/2)\Gamma^2 \) with \( \Gamma = ru \) being proportional to the base-state circulation. This form of the equation was also used by Rotunno (1979). Assuming a pipe flow (i.e., a flow bounded by cylindrical walls), the boundary conditions are given by \( \phi'(0, \zeta) = \phi'(r, \zeta) = 0 \) (Benjamin 1962). These conditions, assuming nontrivial \( \phi' \), imply that the axial perturbation velocity is nonzero and has opposite signs at the axis and cylindrical boundary, respectively, consistent with the fundamental mode of the axisymmetric wave shown in Figs. 8a and 8b. Equation (141) is dynamically equivalent to the Bessel ODEs governing the radial structure of the Kelvin vortex waves that were found above, except that this equation is valid for arbitrary velocity profiles, but only for stationary flows.

Benjamin (1962) argued that a flow can only sustain stationary free waves if the flow is subcritical. In supercritical flows, even the fastest upstream propagating waves are advected downstream, and no stationary pattern can arise. Criticality may then be diagnosed by assuming a solution in the form of \( \phi(r, \zeta) = \phi(r) \exp(\gamma \zeta) \). A stationary wave is possible only if a solution for \( \gamma^2 < 0 \) exists. Then \( \gamma = ik \) and \( \text{Re}\{\phi'\} \sim \cos(kz) \). The existence of an imaginary \( \gamma \) is thus equivalent to subcriticality. The “critical” regime where \( \gamma \) changes sign is characterized by \( k = 0 \), which is just the longwave limit used above. To determine criticality, one first inserts the assumed solution for \( \phi' \) in Eq. (141), where the \( \zeta \) derivative is simply \( \gamma \phi' \). But since \( \gamma^2 = 0 \) in the critical regime, the \( \zeta \) derivative vanishes. This leads to
\[ \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial \zeta^2} + \left[ \frac{1}{rW} \frac{dW}{dr} - \frac{1}{W} \frac{dW}{dr^2} + \frac{2}{W} \frac{dI}{dr} \right] \phi = 0, \]  
(142)
where \( \phi' \) (or) serves as a criticality test function. To see how the criticality test works, consider the simple example of waves in a cylindrical container discussed in section 5. For all axial wavenumbers of the fundamental axisymmetric mode, it was found that \( \alpha_1 = 3.8317 \). Assuming solid-body rotation and \( W = \text{const} \), it follows that \( dI/dr = 2W^2 r \), and Eq. (142) becomes
\[ \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \kappa^2 \phi = 0, \]  
(143)
where
\[ \kappa^2 = \frac{4S^2}{R^2}. \] (144)

The general solution of this equation is
\[ \phi_s(r) = ArJ_1(\kappa r) + BrY_1(\kappa r). \] (145)

Because \( rY_1(\kappa r) \) does not fulfill the boundary condition \( \phi_s(0) = 0 \), the second term in Eq. (145) is omitted and the general solution reads
\[ \phi_s(r) = ArJ_1(\kappa r). \] (146)

The boundary conditions for the standing wave solution, \( \phi_s(0) = \phi_s(R) = 0 \) are fulfilled if \( S = S^\text{crit} = \alpha/2 = 3.8317/2 \approx 1.92 \). This is shown in Fig. 30 (red curve). The coefficient \( \kappa \) functions as radial wavenumber and it follows that if \( \kappa \) increases, the solution will oscillate faster and consequently have a zero in the open interval \((0, R)\) (dashed black curve in Fig. 30). Increasing \( \kappa \) means that \( S \) increases, which entails subcriticality. To test for criticality, for a given \( S \) one uses only the boundary condition \( \phi_s(0) = 0 \) and if the solution has a zero in \((0, R)\), the flow is subcritical. If \( S \) is smaller than \( S^\text{crit} \) it follows that \( \phi_s \) has no zero in \((0, R)\) and the flow is supercritical (solid black curve in Fig. 30). Formally, this behavior is encapsulated in Sturm’s comparison theorem (Benjamin 1962), and it is consistent with the above argument using Kelvin’s solution.

As is apparent from this example, it is not guaranteed that the criterion employed by Benjamin (1962) works for vortices not bounded by a cylindrical wall. In the relevant long-wave limit of waves in the Rankine vortex (Figs. 17c,f), the boundary condition \( \phi_s(R) = 0 \) is not fulfilled because \( \omega(R) = 0 \). In this case, the requirement for subcriticality of \( \phi_s \) having a zero in the interval \((0, R)\), may lead to an inaccurate assessment of criticality. Still, the outer boundary condition for unbounded vortices is usually taken to be \( \phi_s(R) = 0 \), where \( R \) is the edge of the vortex core (the RMW in case of the Rankine vortex), e.g., Ruth et al. (2003), Oberleithner et al. (2012), or Moise and Mathew (2019). There does not appear to be a rigorous justification for this approach except that the waves only propagate in the core, so only the core solution is relevant (e.g., Moise and Mathew 2019), which is intuitively appealing but incompatible with the boundary condition for the long waves in the Rankine vortex. For instance, applying the pipe-flow boundary conditions to the Rankine vortex will underestimate the velocity of the fastest axisymmetric mode (cf. Figs. 7c and 16c for \( k = 0 \)). When diagnosing criticality using Benjamin’s equation, Eq. (142), it is thus important to verify that the boundary conditions are compatible with the fastest waves occurring in the given vortex.

Interestingly, in some numerical simulations the relevant waves thought to be responsible for vortex breakdown are not observed (e.g., Moise 2020a), possibly because of strong viscous or numerical damping in these simulations. It is thus not clear how effective waves are in these cases in propagating information upstream, and instability mechanisms are being investigated as possible trigger for vortex breakdown (Vaniwerschot 2017) in these cases. An additional approach to axisymmetric vortex breakdown, based on vorticity dynamics, has been proposed by Brown and Lopez (1989).

c. Beyond the axisymmetric breakdown

So far, only the axisymmetric breakdown has been examined. As mentioned, however, vortex breakdown may take many additional forms, and to understand, e.g., the spiral breakdown, one needs to consider the stability of the flow around the breakdown bubble. To analyze this instability, the concept of absolute and convective instability (AI/CI) has been invoked (Huerre and Monkewitz 1990). If the flow is convectively unstable, a growing disturbance will be swept downstream, and the flow will recover to its base state configuration. (The term “convective” here refers to what is commonly called “advective” in meteorology.) If the flow is absolutely unstable, the growing disturbance may contaminate the entire flow (e.g., Huerre and Monkewitz 1990; Ruth et al. 2003).

Loiseux et al. (1998) investigated whether the onset of vortex breakdown could be linked to the transition from CI to AI, but without considering the effect of the breakdown bubble. The idea was that AI/CI may be interpreted as a natural extension of the subcritical/supercritical regimes, with AI corresponding to subcriticality, and CI to supercriticality. Loiseux et al. (1998) demonstrated that the Rankine vortex profile with an axial jet in its core (section 7a) becomes absolutely linearly unstable for negative helical modes \((m < 0)\) once the swirl ratio exceeds a certain threshold, with \( |m| \) of the most absolutely unstable mode increasing as the swirl ratio is increased.12 While that study highlights the development of AI.

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12 Although this is consistent with the example presented in section 7a, where it was found that negative azimuthal modes have larger growth rates than positive azimuthal modes, a more detailed analysis is needed to establish the existence of absolute instability (Loiseux et al. 1998).
above a critical swirl, a direct physical link between AI and the onset of vortex breakdown has not materialized (Loiseleux et al. 2000), arguably because the effect of the breakdown bubble was neglected.

Ruith et al. (2003) built on the observation that the breakdown bubble in many ways behaves like a solid bluff object inserted in a swirling flow. Such an object modifies the flow so that, e.g., helical instabilities may arise in the wake of the object (Monkewitz 1988). To analyze these instabilities, the base-state flow needs to be allowed to vary at least slowly in the axial direction. In the “traditional” stability analyses (including those presented in section 7), the base-state flow is assumed to be strictly parallel. The original approach to account for a weakly nonparallel base state, reviewed by Huerre and Monkewitz (1990) or Wu et al. (2006, chapter 9), uses the traditional stability analysis, which, however, is performed at several locations along the vortex axis. That is, the type of ID eigenvalue problems considered in section 7 are solved at different axial locations [Theofilis (2003, 2011) reviews more recent approaches to analyzing these instabilities]. The upshot is that in such weakly nonparallel base states, convectively and absolutely unstable regions may coexist side-by-side in the axial direction. If sufficiently large, a region of absolute instability may give rise to a “global mode,” which imparts its frequency to all downstream locations (e.g., Drazin 1974; Monkewitz 1988; Huerre and Monkewitz 1990; Pier and Peake 2015).

Near the breakdown bubble two regions of absolute instability are usually identified: One within the recirculation region of the breakdown bubble, and one in the wake of the bubble (Liang and Maxworthy 2005; Gallaire et al. 2006; Oberleithner et al. 2012; Qadri et al. 2013; Rukes et al. 2017; Müller et al. 2020). Which of these regions gives rise to the global helical instability, i.e., where the wave maker is located (e.g., Rukes et al. 2017), is still a topic of ongoing research. For instance, Ruith et al. (2003) and Gallaire et al. (2006) inferred that spiral breakdown results from a “steep” nonlinear global spiral mode originating in the AI region in the wake of the breakdown bubble. This steep global mode is also referred to as “elephant mode” (Pier and Huerre 2001; Meliga et al. 2012). In contrast, Qadri et al. (2013) and Rukes et al. (2017) highlighted the relevance of the upstream AI pocket within the recirculation bubble (rather than the AI region in its wake) to explain spiral breakdown, which in their analysis results from a linear global helical mode. The unstable spiral mode often absorbs the breakdown bubble, such that the bubble is no longer visible as separate entity (Ruith et al. 2003). The breakdown mode involving a spiral tail in the wake of the breakdown bubble (as in Fig. 28) has been shown to be associated with a linear global helical instability (Moise 2020b).

The spiral instability often becomes visible in the wake of the still descending (i.e., upstream propagating) breakdown bubble, as was already reported by Maxworthy et al. (1985) and Alekseenko and Shtork (1997). A particularly impressive example of a tornado exhibiting a descending breakdown bubble with a spiral instability is shown in Fig. 31. The breakdown rapidly descended to the surface, leading to a possible drowned vortex jump (see next section) with a spiral instability, which is shown in Fig. 32. This evolution of the axisymmetric bubble breakdown into a, e.g., spiral breakdown mode is referred to as mode selection (Ruith et al. 2003). The breakdown mode also depends on the Reynolds number: The larger the importance of viscosity, the less likely it is for AI to be achieved, and the breakdown bubble tends to remain intact (Ruith et al. 2003).

In summary, axisymmetric vortex breakdown may be explained as axisymmetric analog of the hydraulic jump, where criticality is defined via the propagation speed of the fastest waves (in case of vortex breakdown, these are the long-wavelength fundamental modes of axisymmetric Kelvin vortex waves). In the region of the breakdown bubble the flow may become unstable to helical disturbances, and these unstable helical modes may absorb the breakdown bubble, leading to spiral breakdown. The initial breakdown mechanism is thus related to flow criticality, while the breakdown mode selection is tied to flow instabilities.

9. Discussion

a. Limitations—and utility—of Kelvin’s analysis

Given the simplicity of the base states, Kelvin’s solutions are a somewhat crude model of real tornadoes. The most obvious limitation is the absence of the lower boundary. This no-slip boundary gives rise to the corner flow, which is likely to influence wave propagation and especially the instability of the vortex by introducing vertical and radial gradients of all velocity components (e.g., Nolan 2012). However, although in the idealized analytical treatment presented here the velocity gradients responsible for the growth of the perturbations are different than in more advanced treatments (e.g., Nolan 2012), the fact that, e.g., multiple-vortex formation results from the unstable growth of centrifugal waves is captured even in the idealized treatment. Another assumption in Kelvin’s treatment is that the base-state flow is axisymmetric, which is not valid for translating tornadoes (Lewellen et al. 2000). Moreover, the base-state flow in real-world vortices is not generally parallel (i.e., the vortex widens in the axial direction), even away from, or in the absence of, a lower boundary (requiring the consideration of global instabilities, as mentioned in the previous section). In addition, the neglect of viscous effects in Kelvin’s model results in an unrealistic radial structure of the base-state vorticity, which partly affects the wave structure and propagation characteristics compared to a viscous model (e.g., Fabre et al. 2006). Further, the flow is neutrally stratified, which is an assumption probably not met in the immediate environment of tornadoes. In fact, the stratification of the fluid surrounding the vortex may result in unstable modes as discussed by, e.g., Billant and Le Dizes (2009).

In addition to these limitations regarding mainly the base state, the instabilities discussed pertain to the exponential growth of discrete normal modes. Unfortunately, there is no guarantee that the flow is stable even if none of the normal

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13 This nomenclature was inspired by Saint-Exupéry (1946).
modes have a positive growth rate. The reason is that finite-amplitude perturbations may, for some flows, undergo transient growth as the individually decaying eigenfunctions temporarily overlap so as to allow for growth (Wu et al. 2006, p. 461; Schmid 2007). The reader is referred to Drazin and Reid (1981), Schmid et al. (1993), Arendt et al. (1997), Pradeep and Hussain (2006), or Roy and Subramanian (2014) for discussions on nonmodal disturbances. Finally, the analysis presented in this paper is strictly valid only for infinitesimal perturbations, a feature generally shared by all linear analyses. Once nonlinearities become important, the perturbations may interact with each other, which is excluded from the linear analysis. One example in which this interaction is relevant in tornadoes is demonstrated by Orf (2019), who shows how helical vortex filaments merge during tornadogenesis.

Despite these limitations, Kelvin’s approach provides detailed insight into the many wave types ubiquitous in concentrated vortices. Moreover, his results may be considered a first-order description of the processes relevant for understanding vortex breakdown and the development of subvortices due to instabilities. The analysis of absolute/convective instability (e.g., Loiseleux et al. 1998; Gallaire and Chomaz 2003) utilizes dispersion relations such as those derived in the present paper. More complex base states that allow for global modes are analyzed by obtaining the eigenvalues and eigenfunctions directly from the linearized set of equations, such as Eqs. (19)–(22), but the stability analyses in such flows are often still based on local AI/CI analyses. The analytical results also are helpful when testing the implementation of numerical solvers designed for more complicated base-state flows. Kelvin’s results thus not only provide a basic understanding of waves and instabilities in concentrated vortices, but his results also serve as starting point for addressing more complex scenarios.

b. Relation between tornado vortex behavior, swirl ratio, and Kelvin vortex waves

The results obtained in this paper help understand the behavior of tornado-like vortices in vortex chambers (e.g., Ward 1972; Church et al. 1977; Snow 1982; Church and Snow 1993), specifically the development of the “two-celled” vortex structure which descends to the surface as the vortex chamber swirl ratio is increased (see also Markowski and Richardson 2010, p. 291). This progression is shown schematically in Fig. 33. In such laboratory experiments the swirl ratio is defined as

$$S = \frac{R_u \Gamma}{2Q},$$

(147)

where $R_u$ is the radius of the updraft hole, $\Gamma$ is the circulation at the edge of the updraft hole, and $Q$ is the total volume flux into the chamber (Church et al. 1977). This definition may be cast in a form corresponding to the one in Eq. (133) because the volume influx is equal to the efflux, so $Q = \pi R^2_u W$ (Fiedler and Rotunno 1986), where $W$ is the averaged vertical velocity at some height in the chamber; also $\Gamma = 2\pi R_u V_w$, where $V_w$ is the azimuthal velocity at the edge of the updraft hole, so the rhs of Eq. (147) equals $V_w W$. While this expression formally corresponds to that of Eq. (133), these two parameters measure different aspects of the vortex. The quantity $S$ characterizes the vortex as a whole, which is why usually swirl ratios pertain to input parameters (e.g., Ruith et al. 2003; Moise and Mathew 2019) or are designed such that they remain constant along the vortex axis (Oberleithner et al. 2012). On the other hand, $S_f$ characterizes flow criticality locally along the vortex axis, i.e., it does not characterize the vortex as a whole. The benefit of considering $S_f$ is that it directly pertains to the local wave propagation along the vortex axis. The values of $S$ and $S_f$ may differ from each other (much like the corner-flow swirl ratio defined by Lewellen et al. (2000) does not necessarily attain the same values as the “outer” swirl ratio $S_f$). However, the local swirl ratio, $S_f$ co-varies with $S$ if surface roughness and chamber geometry remain unchanged (Lewellen et al. 2000), which is assumed here, and if the flow is supercritical.

Figure 33a shows the configuration for zero swirl, in which the flow converges toward the center while rising. Since $S = 0$, the local swirl, $S_f$ likewise is zero and hence the flow is supercritical everywhere. A stagnation high and flow separation occur near the center of the domain. A slight increase in $S$ leads to an increase also of $S_f$ along the vortex axis, resulting in concentrated rotation only near the top of the domain (not shown), while the convergence near the ground still dominates the rotation. This results in a high pressure perturbation, which is related to the dominance of deformation (splat) forcing relative to rotation (spin) forcing (e.g., Markowski and Richardson 2010, p. 27) and the flow still separates near the surface and no intense vortex develops at the ground.
As $S$ is further increased (Fig. 33b) by increasing $G$, the pressure at the surface decreases, and the strong rotation spreads to the surface. At the same time, the flow-straightening honeycomb baffle at the top of the domain leads to a decrease in axial vorticity and hence an adverse pressure gradient arises, which causes the flow near the top to become subcritical (Davies-Jones 1976). The result is a vortex breakdown, where $S_l = S_c$ near the top of the domain. As the flow is supercritical below the vortex breakdown location, the implication is that $S_l$ increases with height below the breakdown point. Continuing to increase $S$ leads to larger $S_l$ below the breakdown point, thus driving the vortex toward subcriticality. With $S_l$ increasing with height, subcriticality is achieved at successively lower altitudes and the vortex breakdown descends toward the surface as $S$ increases. This behavior continues until the breakdown reaches the surface (Figs. 33c,d), and practically the entire vortex is subcritical (except very close to the ground). The situation in which the breakdown is located just above the surface is referred to as a “drowned vortex jump” [in analogy to a drowned hydraulic jump; Maxworthy (1972)].

Beyond this point, Kelvin's results cannot comfortably be applied because the effects of the lower boundary become important, which is absent in Kelvin’s analysis. Once the vortex breakdown has reached the surface, the result is a two-celled vortex structure with rising motion in the vortex periphery and with often turbulent, generally downward motion in the interior region. This scenario crudely resembles the cylindrical vortex sheet considered by Rotunno (1978), which leads to instabilities and multivortex development as discussed in section 7.

ON THE ONSET OF VORTEX BREAKDOWN IN THE ABSENCE OF THE FLOW-STRAIGHTENING BAFFLE

In the absence of the lower endwall and the baffle at the top of the domain, vortex breakdown often occurs due to the effects of viscous or turbulent diffusion: As vorticity diffuses outward, the vorticity weakens in the downstream direction and an axial adverse pressure gradient arises, causing the flow sufficiently far downstream to become subcritical (i.e., the local swirl ratio increases in the axial direction). However, this does not work if the “system” swirl ratio (which characterizes the vortex as a whole such as $S$) is too small, in which case the local swirl ratio decreases in the axial direction, and the flow remains supercritical, as demonstrated by, e.g., Ruith et al. (2003), Oberleithner et al. (2012), and Moise and Mathew (2019) using Benjamin's (1962) criticality test. Presumably this behavior arises because the azimuthal flow is more strongly affected by diffusion than the axial flow for small swirl ratios. The result is that vortex breakdown in the absence of a top boundary sets in intermittently in pockets of subcriticality and eventually becomes established once a certain value of $S$ is exceeded (Oberleithner et al. 2012). In tornadoes embedded in supercells, the onset of vortex breakdown is probably complicated by the pressure perturbations associated with the mesocyclone aloft (e.g., Markowski and Richardson 2014; Coffer and Parker 2017), as well as pressure perturbations associated with internal surges (e.g., Skinner et al. 2015; Schenkman et al. 2016), which could influence the criticality of the tornado vortex by affecting the axial pressure gradient. Moreover, the presence of the lower endwall likely affects criticality and hence the onset of vortex breakdown close to the ground.

10. Conclusions

In 1880, Lord Kelvin derived the equations governing the dynamics of centrifugal waves in columnar vortices (Thomson 1880). His derivations have been retraced in the present paper while adding the almost trivial generalization of allowing for a
piecewise constant base-state axial flow. Aside from vortex flows bounded by cylindrical walls and the Rankine vortex, unstable flows were also considered (a Rankine vortex profile with rising motion in its core and a cylindrical vortex sheet with sinking motion in the interior, and rising motion in the exterior regions). These flow patterns loosely model concentrated atmospheric vortices such as tornadoes, dust devils, or fire devils away from the lower boundary. The resulting wave solutions are extremely rich in structure and dynamics, some of which are presented in detail in this paper. The unstable solutions have been invoked to explain multiple-vortex development in tornadoes, although the absence of the lower boundary precludes a direct application to tornadoes. These unstable solutions rely on the radial gradient of the axial, and additionally in some cases of the azimuthal base-state flow, suggesting that these instabilities are more complex than the shear instability in rectilinear shear flows, a feature shared by real tornadoes as well. Moreover, Kelvin’s solutions provide a basic explanation for the vortex breakdown phenomenon, which is treated in analogy to the hydraulic jump except that instead of the fastest surface gravity wave mode one considers the fastest axisymmetric Kelvin vortex wave mode. For the Rankine vortex, Kelvin’s analysis directly leads to the same vortex breakdown criterion as that derived by Squire (1960) and Benjamin (1962). The mode selection of vortex breakdown (whether, e.g., an axisymmetric or a spiral breakdown develops) involves flow instabilities triggered in the region of the breakdown bubble. State-of-the-art analyses of such instabilities go beyond the formalism provided by Kelvin, but his treatment may be considered a starting point for such analyses.

Although oscillations of the radius of maximum winds have been observed in actual tornadoes using mobile Doppler radars, quantitative measurements of the structure of these waves and of related phenomena such as the vortex breakdown, to the author’s knowledge, have not been successful yet. Future efforts will hopefully provide such data, including measurements of the corner flow region, which would be useful in assessing the realism of the mathematical and numerical models, and for designing realistic base states. Theoretical analyses should include the lower boundary and its effects on wave dynamics, structure, and propagation, and
hence flow criticality. Application of global instability concepts could yield insights into the mode selection in vortex breakdown scenarios near the surface, multivortex development, and vortex intensification.

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Data availability statement. Python scripts used to generate the solutions and the plots presented in this paper are available from the author’s GitHub repository: https://github.com/joda80/vortex_waves/.

APPENDIX A

Equations for the Radial and Azimuthal Velocity Amplitudes

The analysis starts with the amplitude equations, Eqs. (19)–(22):

\[
\begin{align*}
\dot{\psi} + \frac{2}{\rho} \left[ \frac{V}{r} + \frac{dV}{dr} \right] \psi &= \frac{i g \psi}{\rho}, \\
\dot{\psi} &= \frac{i g \psi}{\rho}, \\
\dot{g} &= \frac{k g}{\rho}, \\
\frac{d \psi}{dr} + \frac{1}{\rho} \left( \frac{d \psi}{dr} + \frac{i m \psi}{\rho} \right) + ik \psi &= 0,
\end{align*}
\]  
(A1)  
(A2)  
(A3)  
(A4)

where \( g = \omega - m V/r - Wk \). Beginning with the radial-velocity equation, Eq. (A1), the goal is to eliminate \( \psi \) and \( \dot{\psi} \). The first step is to eliminate \( \dot{\psi} \) by taking the radial derivative of Eq. (A3). With

\[
\dot{\psi} = \frac{g \psi}{k},
\]  
(A5)

we see that

\[
\begin{align*}
\dot{\psi} &= \frac{g \psi}{k} \\
\frac{d \psi}{dr} &= \frac{g \psi}{k} - \frac{1}{\rho} \left( \frac{d \psi}{dr} + \frac{i m \psi}{\rho} \right) + ik \psi.
\end{align*}
\]  
(A6)

Inserting this in the rhs of Eq. (A1) gives

\[
\begin{align*}
\dot{\psi} &= \frac{g \psi}{k} - \frac{1}{\rho} \left( \frac{d \psi}{dr} + \frac{i m \psi}{\rho} \right) + ik \psi.
\end{align*}
\]  
(A7)

From Eq. (A2),

\[
\begin{align*}
\dot{\psi} &= \frac{g \psi}{k} - \frac{1}{\rho} \left( \frac{d \psi}{dr} + \frac{i m \psi}{\rho} \right) + ik \psi.
\end{align*}
\]  
(A8)

This is inserted in the lhs of Eq. (A7), and one finds that

\[
\begin{align*}
\dot{\psi} &= \frac{g \psi}{k} - \frac{1}{\rho} \left( \frac{d \psi}{dr} + \frac{i m \psi}{\rho} \right) + ik \psi.
\end{align*}
\]  
(A9)

Finally, using Eq. (A5) again to express \( \dot{\psi} \) on the lhs in terms of \( \psi \) one obtains

\[
\begin{align*}
\dot{\psi} &= \frac{g \psi}{k} - \frac{1}{\rho} \left( \frac{d \psi}{dr} + \frac{i m \psi}{\rho} \right) + ik \psi.
\end{align*}
\]  
(A10)

Collecting terms gives

\[
\begin{align*}
\dot{\psi} &= \frac{g \psi}{k} - \frac{1}{\rho} \left( \frac{d \psi}{dr} + \frac{i m \psi}{\rho} \right) + ik \psi.
\end{align*}
\]  
(A11)

or

\[
\begin{align*}
\dot{\psi} &= \frac{g \psi}{k} - \frac{1}{\rho} \left( \frac{d \psi}{dr} + \frac{i m \psi}{\rho} \right) + ik \psi.
\end{align*}
\]  
(A12)

Solving for \( \dot{\psi} \) finally yields the desired equation:

\[
\begin{align*}
\dot{\psi} &= \frac{g \psi}{k} - \frac{1}{\rho} \left( \frac{d \psi}{dr} + \frac{i m \psi}{\rho} \right) + ik \psi.
\end{align*}
\]  
(A13)

with

\[
\begin{align*}
\dot{\psi} &= \frac{g \psi}{k} - \frac{1}{\rho} \left( \frac{d \psi}{dr} + \frac{i m \psi}{\rho} \right) + ik \psi.
\end{align*}
\]  
(A14)

This is the desired equation for the radial dependence of the radial velocity amplitude. To obtain the equation for the azimuthal velocity amplitude, Eq. (A2) is used and solved for \( \dot{\psi} \):

\[
\begin{align*}
\dot{\psi} &= \frac{i g \psi}{\rho} + \frac{1}{\rho} \left( \frac{d \psi}{dr} + \frac{i m \psi}{\rho} \right).
\end{align*}
\]  
(A15)
where for notational ease \( \gamma = (V/r + dV/dr) \). This we insert into Eq. (A7), giving

\[
i \frac{\hat{v}}{\gamma} + i\frac{\omega}{\gamma} g + \frac{2}{r} v + \frac{V}{k} \hat{w} + \frac{mV}{k^2} \hat{w} - \frac{m}{k^2} \frac{dV}{dr} \hat{w} = 0. \tag{A16}
\]

Last, one uses Eq. (A5) again to eliminate \( \hat{p} \) and after collecting terms this leads to

\[
\hat{v} \left[ 2 - \frac{g^2}{\gamma} \right] = \frac{g}{k} \frac{d\hat{w}}{dr} \frac{mV}{k} \hat{w} - \frac{m}{k} \frac{dV}{dr} \hat{w} - \frac{g^2}{\gamma} \hat{w}. \tag{A17}
\]

or

\[
\hat{v} \left[ 2 - \frac{g^2}{\gamma} \right] = \frac{g}{k} \frac{d\hat{w}}{dr} \frac{mV}{k} \hat{w} + \frac{m}{k} \frac{dV}{dr} \hat{w} - \frac{g^2}{\gamma} \hat{w}. \tag{A18}
\]

The last term on the rhs may be rewritten remembering that \( \gamma = (V/r + dV/dr) \):

\[
\frac{dV}{dr} - \frac{g^2}{\gamma} = \frac{1}{\gamma} \left[ \frac{V}{r} \frac{dV}{dr} - \frac{V^2}{r^2} \right] - \frac{1}{\gamma} \left[ \frac{V}{r} \frac{dV}{dr} - \frac{V^2}{r^2} \right] + \frac{1}{\gamma} \left[ \frac{V}{r} \frac{dV}{dr} - \frac{V^2}{r^2} \right]. \tag{A19}
\]

The lhs of Eq. (A18) may also be written more compactly:

\[
\hat{v} \left[ 2 - \frac{g^2}{\gamma} \right] = \frac{1}{\gamma} \left[ \frac{V}{r} \frac{dV}{dr} - \frac{V^2}{r^2} \right] = \frac{d}{\gamma}. \tag{A20}
\]

With this, Eq. (A18) becomes

\[
\frac{d}{\gamma} \hat{v} = \frac{1}{\gamma} \left[ \frac{V}{r} \frac{dV}{dr} - \frac{V^2}{r^2} \right] + \frac{g}{k} \frac{d\hat{w}}{dr} \frac{m}{k} \hat{w}. \tag{A21}
\]

Solving for \( \hat{v} \) finally gives the desired equation:

\[
\hat{v} = \frac{1}{m} \left[ \frac{V}{r} \frac{dV}{dr} - \frac{V^2}{r^2} \right] \hat{w} + \frac{g}{k} \frac{d\hat{w}}{dr} \frac{m}{k} \hat{w}. \tag{A22}
\]

**APPENDIX B**

**Base-State Pressure Distributions**

The swirling base-state flow is in cyclostrophic balance, in which the radial pressure gradient force balances the centrifugal force. So,

\[
\frac{1}{\rho} \frac{dP}{dr} = -\frac{V(r)^2}{r}. \tag{B1}
\]

Integrating radially gives the pressure

\[
\int_{r_0}^{r} \frac{dP}{dr'} dr' = P(r) - P_\infty = -\rho \int_{r_0}^{r} \frac{V(r')^2}{r'} dr'. \tag{B2}
\]

\[
P(r') = P_\infty + \rho \int_{r}^{R} \frac{V(r')^2}{r'} dr'. \tag{B3}
\]

where \( r' \) is the dummy integration variable and \( P_\infty \) is the pressure at radial infinity.

a. **Pressure in a vortex in solid-body rotation in a domain bounded by a cylinder**

Here \( V(r) = \Omega r \) and \( r \leq R \), where \( R \) is the radius of the cylinder. Then, Eq. (B3) becomes

\[
P(r) = P(R) + \rho \int_{r}^{R} \Omega^2 \frac{V(r')^2}{r'} dr' = P(R) + \rho \Omega^2 \int_{r}^{R} \frac{r'dr'}{r}.
\]

\[
P(r) = P(R) + \rho \Omega^2 \left[ \frac{1}{2} r^2 \right]_r^R = P(R) + \rho \Omega^2 \left[ r^2 - R^2 \right]. \quad \tag{B4}
\]

b. **Pressure in an irrotational vortex**

If the vortex is irrotational, we have \( V(r) = \Omega R^2/r \) (e.g., Kundu and Cohen 2008, p. 70). The integration is started at some nonzero radius \( r \) (else \( V \) become infinite). Inserting the irrotational velocity profile in Eq. (B3), we find that

\[
P(r) = P_\infty + \rho \Omega^2 R^2 \int_{r}^{R} \frac{r'dr'}{r} = P_\infty + \rho \Omega^2 R^2 \left[ \frac{1}{2} r^2 \right]_r^R
\]

\[
= P_\infty - \rho \Omega^2 R^2 \frac{1}{2r^2}. \quad \tag{B5}
\]

c. **Pressure in a Rankine vortex**

In the case of a Rankine vortex, the inner solution is given by Eq. (B4) but using Eq. (B5) for \( P(R) \). The outer solution is simply the irrotational solution. This gives

\[
P(r) = \begin{cases} 
P_\infty - \rho \Omega^2 R^2 + \rho \Omega^2 r^2, & \text{if } r \leq R \\
P_\infty - \rho \Omega^2 R^2/r^2, & \text{if } r > R.
\end{cases}
\]

**APPENDIX C**

**The Squire-Long Equation and Its Linearization**

Following Batchelor (2002, p. 544), the starting point is Eq. (139):

\[
\eta = \frac{1}{u} \frac{\partial H}{\partial \xi} + \frac{v}{u} \xi. \quad \tag{C1}
\]

To obtain the Squire–Long equation, it is customary to express the velocity in terms of Stokes’s streamfunction \( \phi \), which ensures mass continuity. Given the radial symmetry of this
problem, this streamfunction may be thought of as a quasi-
cylindrical stream surface. The velocity components in the
meridional plane are then given by
\[
u = \frac{1}{r} \frac{\partial \phi}{\partial z}
\]
(C2)
and
\[
w = \frac{1}{r} \frac{\partial \phi}{\partial r}.
\]
(C3)
This use of \(\phi\) ensures that the mass flux is only a function of \(\phi\).
Any quantity that is conserved following the flow will always
be attached to the same streamsurface. Hence, conserved
quantities only depend on \(\phi\). Since total energy is conserved,
\(\partial H/\partial t = 0\), which implies that \(H = H(\phi)\). The goal is to ex-
press \(\eta\) as a function of \(\phi\). Since the flow is axisymmetric, \(\phi = \phi(r, z)\), and using the chain rule it follows that
\[
\frac{\partial H[\phi(r, z)]}{\partial r} = \frac{dH(\phi)}{d\phi} \frac{\partial \phi(r, z)}{\partial r},
\]
(C4)
and
\[
\frac{\partial H[\phi(r, z)]}{\partial z} = \frac{dH(\phi)}{d\phi} \frac{\partial \phi(r, z)}{\partial z}.
\]
(C5)
With this, Eq. (C1) may be written as
\[
\eta = \frac{1}{u} \frac{dH}{d\phi} \frac{\partial \phi}{\partial z} + \frac{v}{u} \frac{\partial \eta}{\partial z} = \frac{1}{u} \frac{\partial \phi}{\partial z} + \frac{v}{u} \frac{\partial \eta}{\partial z}.
\]
(C6)
Using Eq. (C2) immediately gives
\[
\eta = -\frac{dH}{d\phi} + \frac{v}{u} \xi.
\]
(C7)
Now the second term needs to be expressed in terms of \(\phi\). Using the definition for the radial vorticity as well as the fact
that the flow is axisymmetric, one obtains
\[
\frac{v}{u} \xi = \frac{1}{u} \frac{\partial w}{\partial \phi} - \frac{\partial \nu}{\partial \phi} = \frac{v}{u} \frac{\partial \nu}{\partial \phi}.
\]
(C8)
To express the rhs of this equation in terms of the conserved
county, \(\Gamma = rv\), it is noted that \(r\) and \(z\) are independent of each
other, so \(r\) may be pulled into the \(z\) derivative. Then,
\[
\frac{v}{u} \xi = -\frac{rv}{(ru)^2} \frac{\partial \Gamma}{\partial \phi} = \frac{1}{r u^2} \frac{\partial \Gamma}{\partial \phi}.
\]
(C9)
or
\[
\frac{v}{u} \xi = -\frac{1}{r u^2} \frac{\partial \Gamma}{\partial \phi} = \frac{1}{r u^2} \frac{\partial \Gamma}{\partial \phi} = \frac{1}{r u^2} \frac{\partial \Gamma}{\partial \phi}.
\]
(C10)
where the product rule has been used in the last step. With \(I =
(1/2)\Gamma^2\), applying the chain rule, and using Eq. (C2), one finds that
\[
\frac{v}{u} \xi = -\frac{1}{r u^2} \frac{\partial \Gamma}{\partial \phi} = \frac{1}{r u^2} \frac{\partial \Gamma}{\partial \phi} = \frac{1}{r u^2} \frac{\partial \Gamma}{\partial \phi}.
\]
(C11)

So,
\[
\eta = \frac{dH}{d\phi} + \frac{1}{r} \frac{dI}{d\phi}.
\]
(C12)
The azimuthal vorticity may be written in terms of the
streamfunction:
\[
\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} = \frac{1}{r} \frac{\partial \phi}{\partial z} - \frac{\partial \xi}{\partial r} = -\frac{1}{r} \frac{\partial \phi}{\partial z} - \frac{1}{r^2} \frac{\partial \phi}{\partial r}.
\]
(C13)
Equating this expression with Eq. (C12) gives
\[
\frac{\partial^2 \phi}{\partial r^2} \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = \frac{dH}{d\phi} - \frac{dI}{d\phi}.
\]
(C14)
This is the starting point for Squire’s (1960) and Benjamin’s (1962)
discussion, and the equation is often referred to as Squire–Long
or Bragg–Hawthorne equation (Bragg and Hawthorne 1950; Long
1953; Squire 1956). An alternative derivation of this equation may
be found in Shapiro (2001b). To linearize this equation, starting with the rhs, consider
\[
H(r, z) = T(r) + H'(r, z),
\]
(C15)
where \(T(r)\) is the energy of the base-state flow, and \(H'\) is the
perturbation energy. Since \(H\) is conserved, it may be expressed
in terms of \(T\),
\[
H(r, z) = T(r) - \frac{dT(r)}{dr} r'(r, z),
\]
(C16)
where \(T = \rho c_1 + 1/2(V^2 + W^2)\) and \(r'\) is the material
displacement [this is analogous to the description of the mixing-
length hypothesis (e.g., Holton and Hakim 2013, p. 265)].
Using the base-state streamfunction \(\Phi(r)\) as independent variable in lieu of \(r\),
\[
H(\Phi, z) = T(\Phi) - \frac{dT(\Phi)}{d\Phi} \Phi'(\Phi, z).
\]
(C17)
Here, analogously to the reasoning above, \((d\Phi/dr)' = -\phi'\),
where \(\phi'\) is the perturbation streamfunction. Then,
\[
H(\Phi, z) = T(\Phi) + \frac{dT(\Phi)}{d\Phi} \phi'(\Phi, z).
\]
(C18)
The \(\phi\) derivatives on the rhs of Eq. (C14) are further approxi-
mated by interpreting \(\phi = \Phi + \phi'\) as the function \(\phi = \phi(\Phi, \phi')\).
Thus, the chain rule may be used and for an arbitrary func-
tion \(f(\phi)\):
\[
\frac{df(\phi(\Phi, \phi'))}{d\Phi} = \frac{df}{d\phi} \frac{d\phi}{d\Phi} \Phi' + \phi' = \frac{df}{d\phi} \left[ 1 + \frac{d\phi'}{d\Phi} \right].
\]
(C19)
Solving for \(df/d\phi:\)
\[
\frac{df}{d\phi} \left[ 1 + \frac{d\phi'}{d\Phi} \right].
\]
(C20)
Using this result along with Eq. (C18) and with \( f = H \):

\[
\frac{dH(\phi)}{db} = \frac{\partial H(\Phi, z)}{\partial \Phi} \approx \left[ 1 + \frac{\partial \phi(\Phi, z)}{\partial \Phi} \right] \left[ \frac{\partial \Pi(\Phi)}{\partial \Phi} \right] + \frac{\partial^2 \Pi(\Phi)}{\partial \Phi^2} \phi(\Phi, z) + \frac{\partial \Pi(\Phi)}{\partial \Phi} \frac{\partial \phi(\Phi, z)}{\partial \Phi}.
\]  

(C21)

or (omitting function arguments for brevity):

\[
\frac{dH}{db} \left[ 1 + \frac{\partial \phi}{\partial \Phi} \right] \approx \frac{d\Pi}{d\Phi} + \frac{d^2 \Pi}{d\Phi^2} \phi + \frac{d\Pi}{d\Phi} \frac{\partial \phi}{\partial \Phi}.
\]  

(C22)

With \( H = \Pi + H' \) on the lhs:

\[
\frac{dH}{db} + \frac{\partial H'}{\partial \Phi} \frac{\partial \phi}{\partial \Phi} + \frac{\partial H'}{\partial \phi} \frac{\partial \phi}{\partial \Phi} \approx \frac{d\Pi}{d\Phi} + \frac{d^2 \Pi}{d\Phi^2} \phi + \frac{d\Pi}{d\Phi} \frac{\partial \phi}{\partial \Phi}.
\]  

(C23)

The third term on the lhs of quadratic order in the perturbations and is thus neglected. The \( \phi \) derivative in the second term on the lhs may be written in terms of \( \Phi \), noting that \( \Phi(\phi, \psi) = \phi - \psi \). Again, using the chain rule:

\[
\frac{\partial \Pi(\Phi(\phi, \psi))}{\partial \phi} = \frac{d\Pi}{d\Phi} \left[ 1 - \frac{\partial \phi}{\partial \Phi} \right].
\]  

(C24)

Then the second term on the lhs of Eq. (C23) becomes

\[
\frac{\partial \Pi}{\partial \phi} \frac{\partial \phi}{\partial \Phi} \approx \frac{d\Pi}{d\Phi} \left[ 1 - \frac{\partial \phi}{\partial \Phi} \right] \frac{\partial \phi}{\partial \Phi}.
\]  

(C25)

Retaining only the linear terms in \( \psi \):

\[
\frac{\partial \Pi}{\partial \phi} \frac{\partial \phi}{\partial \Phi} \approx \frac{d\Pi}{d\Phi} \frac{\partial \phi}{\partial \Phi}.
\]  

(C26)

With this, Eq. (C23) becomes

\[
\frac{dH}{db} + \frac{\partial \Pi}{\partial \phi} \frac{\partial \phi}{\partial \Phi} \approx \frac{d\Pi}{d\Phi} + \frac{d^2 \Pi}{d\Phi^2} \phi + \frac{d\Pi}{d\Phi} \frac{\partial \phi}{\partial \Phi}.
\]  

(C27)

The second term on the lhs is equal to the last term on the rhs, so that

\[
\frac{dH}{db} \approx \frac{d\Pi}{d\Phi} + \frac{d^2 \Pi}{d\Phi^2} \phi.
\]  

(C28)

Alternatively,

\[
\frac{dH}{db} \approx \frac{d\Pi}{d\Phi} + \frac{d^2 \Pi}{d\Phi^2} \phi.
\]  

(C29)

where \( 7 = 1/2 T^2 \). Equation (C14) thus becomes

\[
\frac{\partial \phi}{\partial r} - \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial z^2} = \rho^2 \left[ \frac{d\Pi}{d\Phi} + \frac{d^2 \Pi}{d\Phi^2} \phi \right] - \frac{d\Pi}{d\Phi} + \frac{d^2 \Pi}{d\Phi^2} \phi.
\]  

(C30)

Decomposing \( \phi \) on the lhs into its base state and perturbation parts:

\[
\frac{\partial \phi}{\partial r} = \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial z^2} + \rho^2 \left[ \frac{d\Pi}{d\Phi} + \frac{d^2 \Pi}{d\Phi^2} \phi \right] - \frac{d\Pi}{d\Phi} + \frac{d^2 \Pi}{d\Phi^2} \phi.
\]  

(C31)

This equation is separately fulfilled for the base-state variables and for the perturbation variables. For the perturbations, we see that

\[
\frac{\partial^2 \phi'}{\partial r^2} - \frac{\partial \phi'}{\partial r} + \frac{\partial^2 \phi'}{\partial z^2} = \rho^2 \left[ \frac{d\Pi}{d\Phi} + \frac{d^2 \Pi}{d\Phi^2} \phi' \right] - \frac{d\Pi}{d\Phi} + \frac{d^2 \Pi}{d\Phi^2} \phi'.
\]  

(C32)

Last, the rhs is expressed as a function of \( r \). Starting with the first term:

\[
\frac{d\Pi}{d\Phi} = \frac{d \Pi}{d \rho} \left[ \frac{\rho}{r^2} + \frac{1}{2} \frac{W^2}{r^2} \right].
\]  

(C33)

Using \( W = (1/r) \partial \phi / \partial r \):

\[
\frac{df}{d \Phi} \approx \frac{df}{d \rho} \left[ \frac{\rho}{r^2} + \frac{1}{2} \frac{W^2}{r^2} \right].
\]  

(C34)

With \( f = P + I/2 r^2 + 1/2 W^2 \),

\[
\frac{d\Pi}{d\Phi} = \frac{dP}{d\rho r} + \frac{1}{r^2} + \frac{1}{r^2} \frac{dI}{d\rho} - \frac{2I}{r^2}.
\]  

(C36)

The first and last terms cancel on the rhs of Eq. (C36) because of cyclostrophic balance:

\[
\frac{dP}{d\rho} = \rho^2 = \frac{2I}{r^3},
\]  

(C37)

giving

\[
\frac{d\Pi}{d\Phi} = \frac{dW}{r} + \frac{dI}{r^2} - \frac{2I}{r^2}.
\]  

(C38)

The second derivative then reads

\[
\frac{d^2 \Pi}{d\Phi^2} = -\frac{1}{r^2} \frac{dW}{d\rho r} + \frac{1}{r^2} \frac{d^2 W}{d\rho^2} + \frac{1}{r^2} \frac{dI}{d\rho^2} - \frac{3}{r^2} \frac{dI}{d\rho} + \frac{1}{r^2} \frac{dW}{d\rho}.
\]  

(C39)

This can be seen by expanding the variables with respect to a small parameter \( \epsilon \), such that, e.g., \( \phi = \epsilon \phi_0 + \epsilon^2 \phi_1 + \cdots \) \( \Phi = \epsilon \phi_0 \), so \( \phi' = \epsilon \phi_1 \). Equation (C31) must be fulfilled for arbitrary \( \epsilon \), specifically if \( \epsilon \to 0 \), implying that only those terms that scale with \( \epsilon \) approach zero at the same rate. Thus, the equation is fulfilled separately for each order of \( \epsilon \).
For the $\overline{T}$ derivatives, one finds

$$\frac{d\overline{T}}{dW} = \frac{1}{W} \frac{dW}{d\overline{T}}$$

(C40)

and

$$\frac{d^2\overline{T}}{dW^2} = \frac{1}{W^2} \frac{dW}{d\overline{T}} \frac{dW}{d\overline{T}} + \frac{1}{W} \frac{dW}{d\overline{T}}$$

(C41)

Using Eqs. (C39) and (C41) the coefficient on the rhs of Eq. (C32), becomes

$$\frac{r^2}{W^2} \frac{d\overline{T}}{dW} \cdot \frac{d\overline{T}}{dW} - \frac{1}{W} \frac{dW}{d\overline{T}} \frac{dW}{d\overline{T}} + \frac{2}{W^3} \frac{dW}{d\overline{T}}$$

(C42)

so that the linearized Squire–Long equation may be written as

$$\frac{\partial \overline{\phi}^2}{\partial r^2} - \frac{1}{r} \frac{\partial \overline{\phi}^2}{\partial r} + \frac{1}{W^2} \frac{dW}{d\overline{T}} \frac{dW}{d\overline{T}} + \frac{2}{W^3} \frac{dW}{d\overline{T}} \frac{dW}{d\overline{T}} \overline{\phi} = 0$$

(C43)

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