RELATION BETWEEN EQUIVALENT POTENTIAL TEMPERATURE AND WET-BULB POTENTIAL TEMPERATURE

By H. H. BINDON

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The wet-bulb potential temperature and the equivalent potential temperature have both been regularly employed in air-mass analysis as an identification invariant for air undergoing pseudoadiabatic changes; and since there is an underlying relation between these two quantities, it appears worth while to derive the equation for transforming one into the other.

The following notation will be used:

w=mixing ratio in grams of water vapor per gram of dry air;
y=number of grams of liquid water per gram of dry air;
$\xi=w+y$;
T=air temperature;
T_w=wet-bulb temperature;
θ_d=partial potential temperature;
θ_e=equivalent potential temperature (**1**), (**2**);
θ_w=wet-bulb potential temperature;
T_c=temperature at the condensation level;
e_m=water-vapor pressure (since saturation is assumed, this is the maximum vapor pressure);
p_d=partial pressure of dry air;
p_c=pressure at condensation level;
c_p=specific heat of dry air (0.240 cal/gm);
L=latent heat of condensation of water vapor; $L=594.9-0.51t$ (**3**);
c=specific heat of liquid water;
A=reciprocal of the mechanical equivalent of heat, (2.392×10^{-8});
R=gas constant for dry air $(2.87\times10^{6}$ c. g. s. units).

The equation for the equivalent potential temperature as derived by Rossby (**4**) is:

$$\theta_e=\theta_d e^{\frac{L_c w}{c_p T_c}}, \tag{1}$$

where L_c=latent heat of condensation at temperature T_c. The equation for the saturated adiabatic (**5**) is:

$$ln\ p_d-\frac{c_p+\xi c}{AR}ln\ T-\frac{Lw}{ART}=k, \tag{2}$$

where k is a constant.

The partial potential temperature is given by:

$$\frac{c_p}{AR}ln\left(\frac{\theta_d}{T}\right)=ln\left(\frac{1000}{p_d}\right). \tag{3}$$

Substituting this value for $ln\ p_d$, equation (2) becomes:

$$\frac{c_p}{AR}ln\ \theta_d+\frac{\xi c}{AR}ln\ T+\frac{Lw}{ART}=\text{constant}. \tag{4}$$

In pseudoadiabatic ascent, the second term in (4) is neglected, so that the equation for the pseudoadiabatic is given by:

$$ln\ \theta_d+\frac{Lw}{c_pT}=\text{constant}. \tag{5}$$

If the air is not saturated at the beginning, the temperature T in (5) has to be replaced by the condensation temperature T_c while w of course remains the same until condensation sets in. Then:

$$ln\ \theta_d+\frac{Lw}{c_pT_c}=\text{constant}. \tag{5a}$$

According to Rossby's definition (**2**), the equivalent potential temperature is obtained on the assumption that the air mass ascends pseudoadiabatically until the mixing ratio vanishes, $w=0$. The value of θ thus arrived at is the equivalent potential temperature θ_e. It follows that:

$$ln\ \theta_e=\text{constant}, \tag{6}$$

where the constant is the same as in equation (5). Equating these two expressions leads to equation (1). The derivation of this equation has been repeated here, since some of the equations will be used to establish the relation between equivalent potential and wet bulb potential temperature.

The wet bulb potential temperature of a mass of air has been defined by Normand (**6**) as the temperature attained by a mass of air brought adiabatically to saturation and then carried pseudoadiabatically to a pressure of 1,000 mb. In figure 1, the connection between temperature, pressure, mixing ratio, wet-bulb temperature, wet-bulb potential temperature, and equivalent-potential temperature on a tephigram is recapitulated. In the example it is assumed that the air has originally a pressure of 900 mb., a temperature of 10° C. and a relative humidity of 65 percent. Since the saturation mixing ratio at p=900 mb. and T=10° C. is 8.5 gm/kg as shown by the tephigram, the actual mixing ratio is w=5.5 gm/kg. Similarly the maximum vapor pressure is 12.3 mb., and the actual one 8 mb. To find the pressure p_c and temperature T_c at the condensation level, the point p=900 mb., T=10° C. has to be moved along the dry adiabatic to the intersection with the mixing ratio line 5.5 gm/kg. The pressure and temperature are here 820 mb. and 2.3° C. If the particle is moved along the pseudoadiabatic until it reaches its original pressure of 900 mb., its wet-bulb temperature is found to be 6.5° C. By bringing the particle still further down along the pseudoadiabatic until it reaches the pressure 1,000 mb., its wet-bulb potential temperature is found to be 11° C.

The equivalent potential temperature is obtained by following the pseudoadiabatic from the condensation point to a pressure as low as is possible on the thermodynamic chart used. On the tephigram used here, for example, the pseudoadiabatic is quite closely tangential to the dry adiabatic at a pressure of 380 mb. Strictly speaking, the particle should be brought to the pressure zero to ensure that all the water vapor is precipitated. However, it is obviously sufficient in the example chosen here to bring the particle only to 380 mb. pressure. From this position the particle has to be moved dry adiabatically to the standard pressure 1,000 mb. The temperature reached here is the equivalent potential temperature. In the present example it is 34° C.

Instead of using this graphical method for determining the equivalent potential temperature which corresponds to a given wet-bulb potential temperature, a formula can be derived to express the relation between the two quantities. To obtain the wet-bulb potential temperature on the thermodynamic chart the air mass had to be moved *pseudo*adiabatically from the condensation point p_c, T_c, to the pressure 1,000 mb., since the wet bulb is being con-

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sidered. At this pressure of 1,000 mb. it assumes the wet-bulb potential temperature θ_w while its mixing ratio is given as the saturation mixing ratio at the temperature θ_w, $w(\theta_w)$. Therefore, equation (5) can be applied:

$$ln\ \theta_w+\frac{Lw(\theta_w)}{c_p\theta_w}=\text{constant.} \qquad (5b)$$

The constants on the right of (5b) and (5a) are equal since the points p_c, T_c, and 1,000 mb., θ_w, lie on the same pseudoadiabatic. The constant is furthermore, according to equation (6), equal to $ln\ \theta_e$ so that

$$ln\ \theta_e=ln\ \theta_w+\frac{Lw(\theta_w)}{c_p\theta_w},$$

or

$$\theta_e=\theta_w e^{\frac{Lw(\theta_w)}{c_p\theta_w}}. \qquad (7)$$

(It should be noted that the saturation mixing ratio $w(\theta_w)$ has to be taken at a pressure of 1,000 mb., as is obvious from the preceding derivation.) By substituting for the mixing ratio the vapor pressure e_m, which is given to an accuracy sufficient for our purpose by

$$w=0.622\frac{e_m}{1000}, \qquad (8)$$

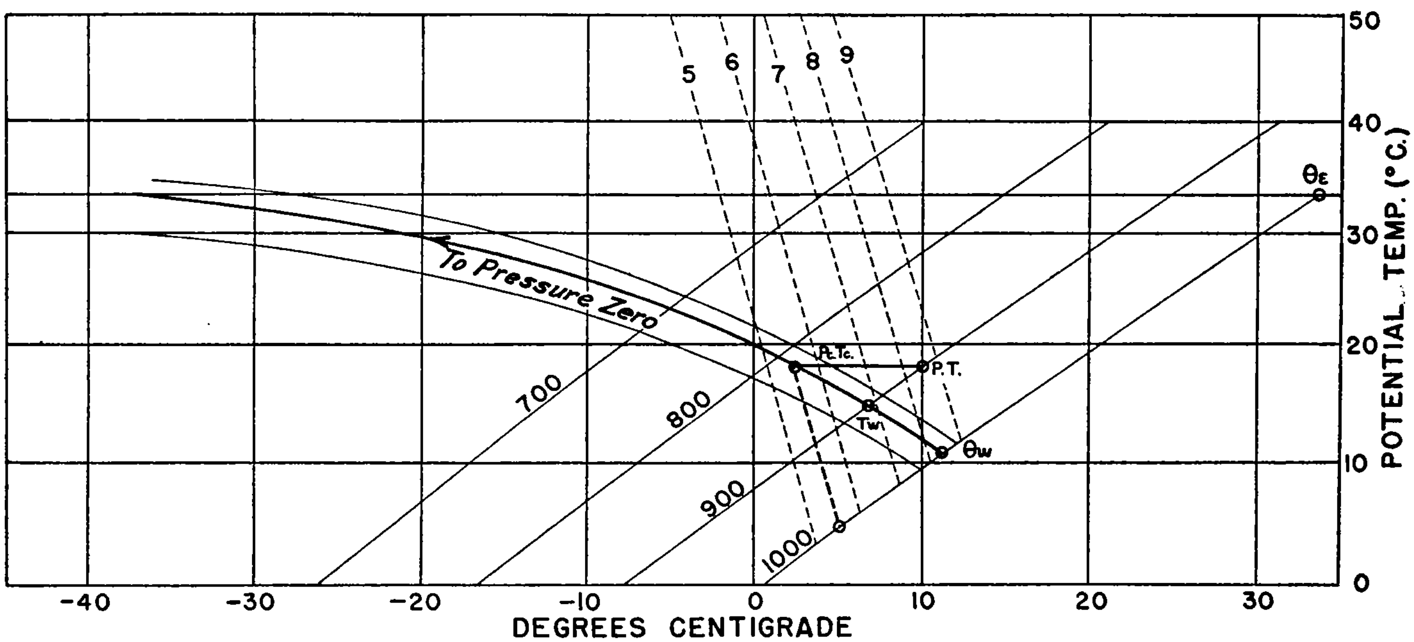

FIGURE 1.

equation (7) may be transformed into

$$\theta_e=\theta_w e^{\frac{Le_m(0.622)}{c_p 1000\theta_w}}. \qquad (9)$$

Since the exponent of the exponential function is small, it may be developed into series, and terms of a higher order than the first omitted without serious loss of accuracy in most cases.

Equation (7) becomes thus

$$\theta_e=\theta_w+\frac{Lw(\theta_w)}{c_p}. \qquad (7a)$$

$Lw(\theta_w)$ is the latent heat contained in the air due to the presence of water vapor saturated at the temperature θ_w. Thus equation (7a) states that the equivalent potential temperature is equal to the wet-bulb potential temperature plus the increase which could be caused by the latent heat of the saturated water vapor contained in the air at the wet-bulb potential temperature. Equation (9) can be written

$$\theta_e=\theta_w+\frac{(0.622)Le_m(\theta_w)}{c_p\ 1000}. \qquad (9a)$$

This relation may also be obtained directly from the psychrometer equation (7) $e_m(T_w)-e=Bp(T-T_w)$, where e is the actual vapor pressure of the air, and

$$B=\left(1-\frac{e_m(T_w)}{p}\right)\frac{c_p}{L(T_w)(0.622)}.$$

If the air of pressure p, temperature T, and vapor pressure e is brought pseudoadiabatically to an infinitely small pressure, so that $e=0$ since all the water vapor condenses and falls out, and is then carried adiabatically to the standard pressure of 1,000 mb., T changes into θ_e while T_w becomes θ_w. Thus the psychrometer equation changes into

$$e_m(\theta_w)=B(1000)(\theta_e-\theta_w),\ \text{or}$$

$$\theta_e=\theta_w+\frac{e_m(\theta_w)}{B(1000)};$$

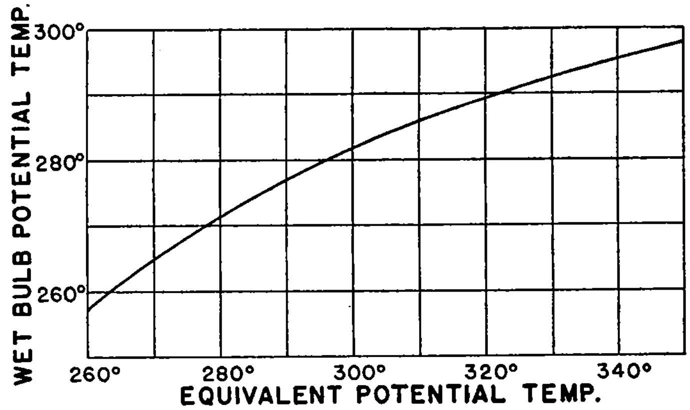

FIGURE 2.—Relation between equivalent potential and wet-bulb potential temperature.

since in B the expression $\frac{e_m(\theta_w)}{p}$ is very much less than unity, this is practically identical with (9a).

From equation (9) the following numerical relation between the equivalent potential temperature θ_e and the wet-bulb temperature θ_w is found:

θ_e	260°	270°	280°	290°	300°	310°	320°	330°	340°	350°
θ_w	257.2	264.6	271.3	276.8	281.7	285.9	289.4	292.5	295.3	297.8

The relation is represented graphically in figure 2.

Thus the use of θ_e or θ_w to identify a mass of moist air undergoing pseudoadiabatic changes is a matter of personal choice. The wet-bulb potential temperature appears to have a certain conceptual advantage since it does not require the air to be taken to zero pressure for extraction of its precipitable water. Moreover, the wet-bulb potential temperature is closely related to the wet bulb temperature, a quantity which can be easily measured.

REFERENCES

(1) F. Linke, *Met. Zeit.* 55, 345–350 (1938).
(2) C.-G. Rossby, *M. I. T. Met. Papers,* Vol. 1, No. 3 (1932).
(3) D. Brunt, *Physical and Dynamical Meteorology,* Cambridge, p. 406 (1934).
(4) *l. c.* (2) equation 26, page 10.
(5) *l. c.* (3) equation 11, page 54.
(6) Normand, *Mem. Indian Met. Dept.,* No. 23 (1921).
(7) *l. c.* (3), equations 38, 39, page 82.

TROPICAL DISTURBANCES OF SEPTEMBER 1940

By J. H. GALLENNE

[Weather Bureau, Washington, Nov. 2, 1940]

August 30–September 3.—The first indications of probable origin of this hurricane appeared on the morning of August 30, as a mild depression central about 225 miles off the Florida east coast. A slow progressive movement toward the north-northwest with rapid development, was indicated by the report of an unidentified vessel near 32°12′ N., and 72°24′ W., at 5 p. m. of that day, which recorded an east-southeast wind, force 10, with barometer reading of 978.7 millibars (28.90 inches). During the next day the disturbance was attended by severe squalls and strong shifting gales over a large area and by winds of hurricane strength near its center. Shortly after the morning observation of September 1, the course of the disturbance seems to have changed from north-northwest to north-northeast.

The American S. S. *Dungannon* reported that she encountered north-northeast winds, force 10, at 8 a. m. of September 1, near 35°50′ N., and 73°45′ W., with pressure reading 993 millibars (29.32 inches), and that the wind shifted to northwest and increased to force 12 shortly thereafter. During the evening of the same day, the tanker *Franklin K. Lane,* on a voyage from New York to Corpus Christi, reported that she met an east-southeast hurricane which shifted to west-northwest near 38°17′ N., and 70° 32′ W. She also reported that a pressure reading of 965.1 millibars (28.50 inches) was noted during the passage of the hurricane. This is the lowest barometer reading of record in connection with the disturbance. Several other vessels reported winds of force 8 or higher, on the 1st. (See Table of Ocean Gales and Storms on page 255 in this REVIEW.)

The disturbance was centered at 7:30 a. m., September 2, about 75 miles east-northeast of Nantucket, Mass., moving rapidly north-northeastward. The Weather Bureau office, Nantucket, Mass., recorded a maximum velocity, for a 5-minute period, of 57 miles an hour, from the northeast and an extreme velocity of 65 miles an hour on September 2. This exceeds all previous September wind records at that station.

The storm moved inland a short distance to the northwest of Yarmouth, Nova Scotia, with rapidly diminishing intensity, during the evening of the 2d, and apparently dissipated in the region north of Anticosti Island, Quebec, on September 3.

Timely warnings and advisories were issued from the forecast center at Washington, D. C., covering the movement of this hurricane.

September 11–18.—On the 7:30 a. m. chart of September 11, 1940, there were some indications of a disturbance of slight intensity about 250 miles northeast of St. Thomas, V. I., moving in a west-northwesterly direction. During that afternoon, an unidentified vessel near latitude 20° N., and longitude 64°30′ W., reported cloudy weather, northwest wind, force 6, with a barometric pressure reading 1,007 millibars (29.74 inches).

The depression developed very rapidly during the 12th, causing moderate gales over a large area to the right of its path. At 7:30 p. m. the center was near 22°30′ N., and 68° W., from which point it continued to move in a west-northwesterly direction until the following morning. During the 13th it curved to the north and northeast attended by strong gales and continued falling pressure.

At the morning observation of September 14 the storm was central about 475 miles east-northeast of Nassau, moving at a rate of about 12 to 14 miles an hour. An observation from the S. S. *Borinquen* indicated that the disturbance developed to full hurricane strength during that day. The vessel met a north-northeast wind, force 12, at 5 p. m., near latitude 30°24′ N., and longitude 71° W., with barometer reading 988.3 millibars (29.19 inches). Her daily journal from local noon to midnight of September 14 reads: "Overcast, heavy rain, ship hove to; vessel laboring and shipping water." The S. S. *Coamo* also became involved in the hurricane on the 14th, reporting that she encountered an east-northeast wind, force 11, at 11 p. m., near 30°14′ N., and 72° W.

For the next 48 hours the storm moved rapidly in a north-by-east direction and was centered near 39°30′ N., and 68° W., at 7:30 a. m. of September 16, attended by moderate to heavy rain, in the vicinity of Nantucket, Mass., and by gales over a very wide ocean area.

Scattered ship reports indicate that thick weather, with rough seas, and heavy rain squalls were associated with the disturbance as it moved inland during the evening of September 18, a short distance north of Cape Race, Newfoundland.

Although this storm developed full hurricane force, no reports have been received of loss of life or property damage, probably due to the fact that it remained well at sea during practically its entire passage.