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THE LATTICE STRUCTURE OF THE FINITE-DIFFERENCE PRIMITIVE AND VORTICITY EQUATIONS†

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ABSTRACT

The use of central differences on a rectangular net, in solving the primitive or vorticity equations, produces solutions on each of two lattices. By exploring this lattice structure, a formal equivalence is established between the central-difference vorticity and primitive equations. A demonstration is given also that exponential instability previously found to result from certain types of boundary conditions is suppressed by applying these conditions in such a way as to avoid coupling the lattices.

1. INTRODUCTION

In a recent study (Platzman [7]) the primitive equations (that is, the momentum and continuity equations) were integrated numerically in an investigation of surges on Lake Michigan. It was pointed out there that, when central differences are used, the finite-difference primitive equations give rise to solutions on two independent lattices. The purpose of the present note is to demonstrate that a more satisfactory understanding of the analytical properties of these equations is provided by taking account of this lattice structure.*

2. THE WAVE EQUATION

Consider a canal of uniform depth D and uniform cross section, closed by rigid vertical walls at $x=0$ and $x=L$. The differential equation for "tidal" disturbances is the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2},$$

where u is the particle velocity along x, and $c\equiv\sqrt{gD}$.

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*After the substance of the present work was completed, the writer's attention was called to an elegant study by A. Eliassen, in which similar ideas are involved (A. Eliassen, "A Procedure for Numerical Integration of the Primitive Equations of the Two-Parameter Model of the Atmosphere," University of California at Los Angeles, Department of Meteorology, 1956).

Partition the axes uniformly as follows: $t/\Delta t=0, 1, 2, \ldots,$ and $x/\Delta x=0, 1, 2, \ldots, p$, so that $L=p\Delta x$. Using central differences,

$$\delta_t^2 u=\sigma^2\delta_x^2 u, \tag{2.1}$$

where $\sigma\equiv c\Delta t/\Delta x$ and δ denotes the operator

$$\delta_t u(x,t)=u(x,t+\tfrac{1}{2}\Delta t)-u(x,t-\tfrac{1}{2}\Delta t)$$

$$\delta_x u(x,t)=u(x+\tfrac{1}{2}\Delta x,t)-u(x-\tfrac{1}{2}\Delta x,t).$$

Subject to boundary conditions $u(0,t)=0$ and $u(L,t)=0$, we may represent $u(x,t)$ by the discrete Fourier series

$$u(x,t)=\sum_{l=1}^{p-1}u_l(t)\sin k_l x \tag{2.2}$$

$$k_l\equiv l\pi/L \quad (l=1, 2, \ldots, p-1)$$

where k_l is the wave number of the lth mode.

Substitution of (2.2) in (2.1) yields the following difference equation for the expansion coefficient $u_l(t)$:

$$\delta_t^2 u_l=-4\,(\sigma\sin\tfrac{1}{2}k_l\Delta x)^2 u_l.$$

The general solution of this is

$$u_l(t)=\alpha\cos\nu_l t+\beta\sin\nu_l t, \tag{2.3}$$

where α and β are constants of integration, and

$$\sin \tfrac{1}{2}\nu_l\Delta t=\sigma \sin \tfrac{1}{2}k_l\Delta x \qquad (2.4)$$

determines the frequency.

For initial conditions we specify u and $\partial u/\partial t$; in the finite-difference frame, this is equivalent to a specification of $u_l(0)$ and $u_l(\Delta t)$, in terms of which (2.3) becomes

$$u_l(t)=(\csc \nu_l\Delta t)[-u_l(0) \sin \nu_l(t-\Delta t)+u_l(\Delta t) \sin \nu_l t].$$

If we assign to $u_l(\Delta t)$ the value

$$u_l(\Delta t)=u_l(0)+\dot{u}_l(0)\Delta t,$$

where $\dot{u}_l(0)$ is $\partial u_l/\partial t$ at $t=0$, then we get

$$u_l(t)=(\sec \tfrac{1}{2}\nu_l\Delta t)[u_l(0) \cos \nu_l(t-\tfrac{1}{2}\Delta t)+\dot{u}_l(0)(\tfrac{1}{2}\Delta t \csc \tfrac{1}{2}\nu_l\Delta t) \sin \nu_l t].$$

This form shows clearly the amplitude and phase distortion produced by truncation errors (see Platzman [5]), and expresses the solution explicitly in terms of data prescribed along the initial line.

The finite-difference system has $2(p-1)$ degrees of freedom, represented by the $p-1$ independent values of $u(x,0)$ and the $p-1$ independent values of $u(x,\Delta t)$, or by the values of the corresponding Fourier coefficients. These degrees of freedom are provided by the $2(p-1)$ distinct fundamental modes $\exp(\pm i\nu_l t)$ determined by the frequency equation (2.4), of which $p-1$ correspond to wave propagation along positive x, and $p-1$ correspond to backward propagation.

3. THE PRIMITIVE EQUATIONS

The preceding problem may be formulated in terms of the linearized primitive equations

$$\frac{\partial u}{\partial t}=-g\frac{\partial h}{\partial x}$$

$$\frac{\partial h}{\partial t}=-D\frac{\partial u}{\partial x},$$

where h is the free-surface displacement from mean level.

As before, we place u at the points $t/\tfrac{1}{2}\Delta t=0, 2, 4, \ldots,$ and $x/\tfrac{1}{2}\Delta x=0, 2, 4, \ldots, 2p$; further, we place h at the "half-way" points $t/\tfrac{1}{2}\Delta t=1, 3, 5, \ldots,$ and $x/\tfrac{1}{2}\Delta x=1, 3, 5, \ldots, 2p-1$ (see fig. 1a). Then the central difference equations are, in the notation of (2.1),

$$\delta_t u=-\sigma\delta_x h \qquad (3.1a)$$

$$\delta_t h=-\sigma\delta_x u; \qquad (3.1b)$$

in (3.1), u and h have been reduced to dimensionless form through division, respectively, by $\sqrt{gD}$ and D. If one eliminates h from (3.1), by applying δ_t to (3.1a), equation (2.1) is obtained, so that our frequency equation is again (2.4). However, instead of working with the wave-equation solutions, we will obtain the solutions of (3.1) directly without elimination.

We first represent $u(x,t)$ in the manner of (2.2), and $h(x,t)$ by the discrete series

$$h(x,t)=\sum_{l=0}^{p-1}h_l(t) \cos k_l x. \qquad (3.2)$$

Substitution in (3.1) yields

$$\delta_t u_l=+2(\sigma \sin \tfrac{1}{2}k_l\Delta x)h_l$$

$$\delta_t h_l=-2(\sigma \sin \tfrac{1}{2}k_l\Delta x)u_l,$$

for which we have the general solution

$$u_l(t)=\alpha \cos \nu_l t+\beta \sin \nu_l t$$

$$h_l(t)=\beta \cos \nu_l t-\alpha \sin \nu_l t,$$

with ν_l as in (2.4). If, in addition to the initial distribution of u, one specifies the initial distribution of h (instead of the initial acceleration $\partial u/\partial t$), the solution is

$$u_l(t)=(\sec \tfrac{1}{2}\nu_l\Delta t)[u_l(0) \cos \nu_l(t-\tfrac{1}{2}\Delta t)+h_l(\tfrac{1}{2}\Delta t) \sin \nu_l t] \qquad (3.3a)$$

$$h_l(t)=(\sec \tfrac{1}{2}\nu_l\Delta t)[h_l(\tfrac{1}{2}\Delta t) \cos \nu_l t-u_l(0) \sin \nu_l(t-\tfrac{1}{2}\Delta t)]. \qquad (3.3b)$$

Note that the initial values of h are those assigned on the line $t=\tfrac{1}{2}\Delta t$, because in the lattice of figure 1a, values of h are not given on $t=0$.

In figure 1b is shown the lattice which is initiated by assigning h on the line $t=0$ and u on $t=\tfrac{1}{2}\Delta t$. The solution in this case is

$$u_l(t)=(\sec \tfrac{1}{2}\nu_l\Delta t)[u_l(\tfrac{1}{2}\Delta t) \cos \nu_l t+h_l(0) \sin \nu_l(t-\tfrac{1}{2}\Delta t)] \qquad (3.4a)$$

$$h_l(t)=(\sec \tfrac{1}{2}\nu_l\Delta t)[h_l(0) \cos \nu_l(t-\tfrac{1}{2}\Delta t)-u_l(\tfrac{1}{2}\Delta t) \sin \nu_l t]. \qquad (3.4b)$$

The lattices of figures 1a and 1b differ only by a phase shift of $\tfrac{1}{2}\Delta t$.

The equations (3.1) involve $p-1$ values of u (exclusive of the boundary values, which are fixed at zero), and p values of h; hence, there are $2p-1$ degrees of freedom in the initial conditions. In order to obtain an equal number of fundamental modes for (3.1), one must include the root $l=0$, $\nu_l=0$ of the frequency equation (2.4); the corresponding mode is $u=0$, $h=h_0=$constant, which evidently satisfies (3.1) and boundary conditions. Note also that if (3.1b) is summed over all values of x where h is defined, the right side sums to zero, with the help of the boundary conditions, so that the x-sum of h is independent of t. (This is merely a statement of the integral form of the continuity equation.) From (3.2), we see that the x-sum of h is equal to ph_0; hence, again, $h_0=$constant.

To insure that the finite-difference solution is stable, it is sufficient to choose Δt and Δx so that $\sigma\equiv c\Delta t/\Delta x<1$, since then the frequency ν_l determined by (2.4) must be real. This is the von Neumann stability condition, and is also the well-known restriction on the difference equations for hyperbolic systems, which insures that the solu-

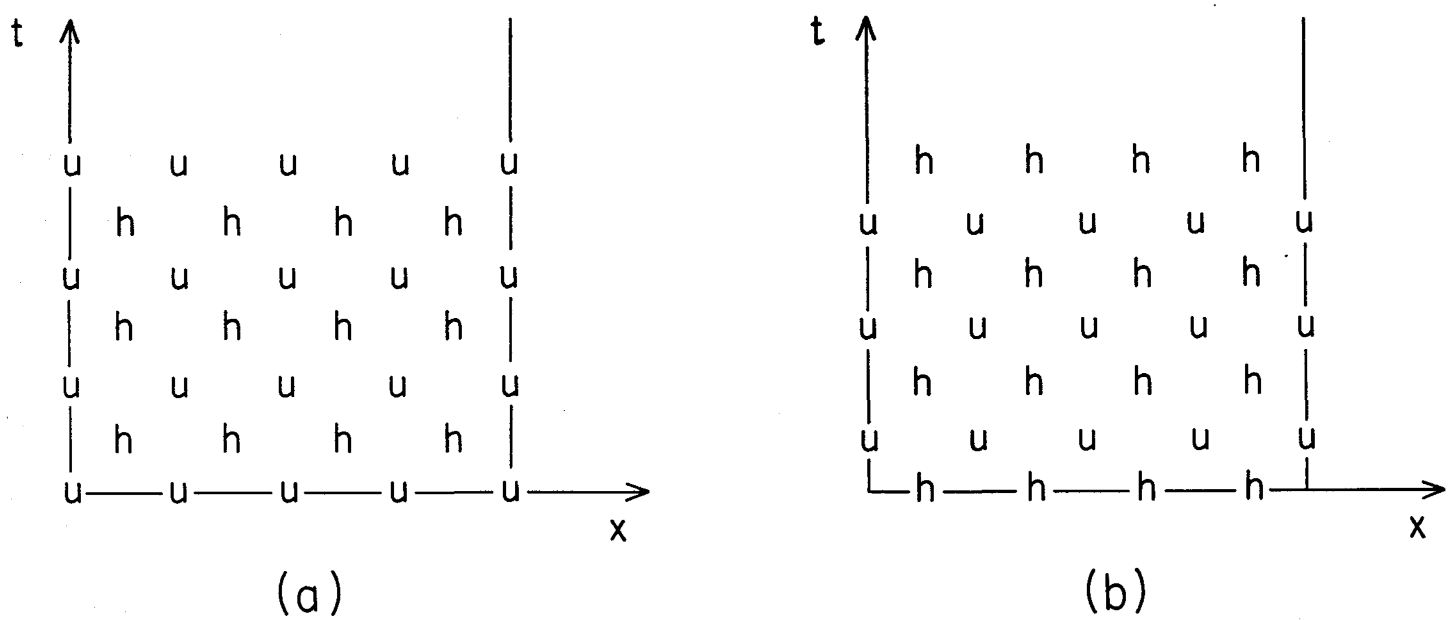

FIGURE 1.—Two lattices for solution of the primitive equations.

tions of the difference equation and the solution of the differential equation coalesce in the limit Δt, $\Delta x \rightarrow 0$.*

Taking this limit in (3.3) or (3.4)

$$u_l(t)=u_l(0)\cos\nu_l t+h_l(0)\sin\nu_l t \tag{3.5a}$$

$$h_l(t)=h_l(0)\cos\nu_l t-u_l(0)\sin\nu_l t, \tag{3.5b}$$

the frequency equation being simply $\nu_l=ck_l$.

Suppose now that we specify as initial conditions the values of $u(x,0)$ at $x/\Delta x=0, 2, 4, \ldots, 2p$, and on the same line $t=0$ the values of $h(x,0)$ at the intermediate points $x/\frac{1}{2}\Delta x=1, 3, 5, \ldots, 2p-1$. To initiate the lattice of figure 1a, we must move from $h(x,0)$ to $h(x,\frac{1}{2}\Delta t)$; if we do this by the uncentered step

$$h(x,\tfrac{1}{2}\Delta t)=\bar{h}(x,0)-\tfrac{1}{2}\sigma\delta_x u(x,0),$$

the expansion coefficients $h_l(\frac{1}{2}\Delta t)$ will be

$$h_l(\tfrac{1}{2}\Delta t)=\bar{h}_l(0)-u_l(0)\sin\tfrac{1}{2}\nu_l\Delta t.$$

(In these and in subsequent relations the bar is used to identify quantities associated with the lattice of figure 1b.) Substitution in (3.3) yields

$$u_l(t)=u_l(0)\cos\nu_l t+\bar{h}_l(0)(\sec\tfrac{1}{2}\nu_l\Delta t)\sin\nu_l t \tag{3.6a}$$

$$h_l(t)=\bar{h}_l(0)(\sec\tfrac{1}{2}\nu_l\Delta t)\cos\nu_l t-u_l(0)\sin\nu_l t. \tag{3.6b}$$

Similarly, to initiate the lattice of figure 1b, we must move from $u(x, 0)$ to $u(x, \frac{1}{2}\Delta t)$; proceeding as above, we find

$$\bar{u}_l(t)=u_l(0)(\sec\tfrac{1}{2}\nu_l\Delta t)\cos\nu_l t+\bar{h}_l(0)\sin\nu_l t \tag{3.7a}$$

$$\bar{h}_l(t)=\bar{h}_l(0)\cos\nu_l t-u_l(0)(\sec\tfrac{1}{2}\nu_l\Delta t)\sin\nu_l t. \tag{3.7b}$$

In (3.6) the initial data $u_l(0)$ are *concordant*, in the sense that they belong to the lattice on which the solution is represented; on the other hand, $\bar{h}_l(0)$ in (3.6) are *discordant* data. Similarly, with respect to the lattice on which (3.7) represents a solution, the $u_l(0)$ are discordant and $\bar{h}_l(0)$ are concordant data.

Using the exact solution (3.5) for comparison, one may characterize (3.6) and (3.7) by saying that, if all initial data originate on the same line ($t=0$), *the concordant data are propagated without distortion* (of amplitude or phase) while the discordant data suffer an amplitude distortion $\sec\frac{1}{2}\nu_l\Delta t$.

It is important to bear in mind that in the solution (3.6), on the lattice of figure 1a, the values of $u_l(t)$ are to be taken only for $t/\frac{1}{2}\Delta t=0, 2, 4, \ldots$, while the values of $h_l(t)$ are for $t/\frac{1}{2}\Delta t=1, 3, 5, \ldots$; the converse is true of (3.7), which applies to the lattice of figure 1b. One can, of course, represent the two solutions in a single formula. This can be done with the aid of the functions

$$F_l(t)\equiv\tfrac{1}{2}[1+(-)^n]+\tfrac{1}{2}[1-(-)^n]\sec\tfrac{1}{2}\nu_l\Delta t$$

$$G_l(t)\equiv\tfrac{1}{2}[1-(-)^n]+\tfrac{1}{2}[1+(-)^n]\sec\tfrac{1}{2}\nu_l\Delta t,$$

where $n\equiv t/\frac{1}{2}\Delta t$; evidently, when n is even, $F_l=1$ and $G_l=\sec\frac{1}{2}\nu_l\Delta t$, and when n is odd, $F_l=\sec\frac{1}{2}\nu_l\Delta t$ and $G_l=1$. On this understanding, the solutions (3.6) and (3.7) are contained in the single representation

$$u_l(0)F_l(t)\cos\nu_l t+\bar{h}_l(0)G_l(t)\sin\nu_l t \tag{3.8a}$$

$$\bar{h}_l(0)F_l(t)\cos\nu_l t-u_l(0)G_l(t)\sin\nu_l t, \tag{3.8b}$$

*Convergence (coalescence) may be obtained for special choices of initial data even though $\sigma>1$ (see, for example, Platzman [5]). This may seem paradoxical because it suggests that in the limit, the domain of dependence does not span the interval intercepted by two characteristics; but the special choices in question all involve analytic initial data, capable of continuation. On the other hand, if one adopts the definition of convergence employed by Lax and Richtmyer [3], the condition $\sigma<1$ is also necessary (as well as sufficient), since they require convergence for *arbitrary* initial data. In their basic study, Lax and Richtmyer have established equivalence between stability and convergence, under very general conditions.

in which (3.8a) and (3.8b) represent, respectively, $u_l(t)$ and $\overline{h}_l(t)$ when $t/\frac{1}{2}\Delta t$ is even, or $\overline{u}_l(t)$ and $h_l(t)$ when $t/\frac{1}{2}\Delta t$ is odd. This solution exhibits a form which is characteristic of all double-lattice representations, namely, a "carrier" wave of period Δt, upon which the true solution is modulated. The carrier wave is imposed by the functions $F_l(t)$ and $G_l(t)$.

In connection with numerical computation, since truncation errors are determined primarily by the size of the mesh (Δx, Δt), and since the two lattices of figure 1 have the same mesh size, little is gained by computing on both lattices. Indeed, the double-lattice representation (3.8) evidently is more involved than (3.6) or (3.7), and tends to obscure the simpler features of the latter.

A final comment will be helpful here, concerning a relaxation of the end conditions $u=0$. Suppose that at the ends $x=0$ and $x=L$ the canal is not closed, but that the flow there can be regulated independently of the motion in the canal; let $u(0,t)$ and $u(L,t)$ be the end velocities thus imposed. The representation (2.2) may now be amended to read

$$u(x,t)=\sum_{l=1}^{p-1}u_l(t)\sin k_l x+u(0,t)+[u(L,t)-u(0,t)]x/L,$$

while (3.2) is unaltered. Substitution in (3.1) leads to an inhomogeneous system for $u_l(t)$ and $h_l(t)$. The fundamental modes are identical to those obtained above, but a particular solution must be added now to incorporate the forcing which comes from the ends. The integral form of the continuity equation is

$$\delta_t\sum_{x(h)}h=-\sigma[u(L,t)-u(0,t)],$$

obtained by summing (3.1b) over all x-points where h is defined. Clearly, this states merely that the change of mean level is determined by the net volume transport through the ends.

In particular, if $u(0,t)$ and $u(L,t)$ are constants (independent of t) we have the following particular solution

$$u=u(0,t)+[u(L,t)-u(0,t)]x/L$$

$$h=-[u(L,t)-u(0,t)]ct/L$$

(Since x and t enter here linearly, this is a solution with zero truncation error.) The fact that h is a linear function of t obviously results from the constant net volume transport imposed at the ends. Mathematically, this may be interpreted as a resonant solution of the inhomogeneous system for u_l and h_l coming from the mode corresponding to $l=0$, the frequency of which is zero and coincident therefore with the zero frequency inherent in the constant values of $u(0,t)$ and $u(L,t)$.*

4. THE VORTICITY EQUATION

The linearized one-dimensional vorticity equation is

$$\frac{\partial\zeta}{\partial t}+U\frac{\partial\zeta}{\partial x}+\beta v=0$$

$$\zeta=\nabla^2\psi,\ v=\partial\psi/\partial x,$$

where β is the Rossby parameter and U is the zonal wind component, assumed uniform and constant.

For the present purpose it suffices to take the case $\beta=0$:

$$\frac{\partial\zeta}{\partial t}=-U\frac{\partial\zeta}{\partial x}. \tag{4.1}$$

Using central differences

$$\delta_t\zeta=-\sigma\delta_x\zeta, \tag{4.2}$$

where $\sigma\equiv U\Delta t/\Delta x$ and δ_t, δ_x are the difference operators defined previously, as in (2.1). To aid in identifying the lattice structure of (4.2), and for comparison with the primitive equations discussed in the preceding section, we will denote vorticity values at the points $x/\frac{1}{2}\Delta x=1, 3, 5, \ldots, 2p-1$ by the symbol ω and those at the points $x/\frac{1}{2}\Delta x=0, 2, 4, \ldots, 2p$, by the symbol ζ. On reference to the lattices of figure 2, we find that equation (4.2) is equivalent to the system

$$\delta_t\zeta=-\sigma\delta_x\omega \tag{4.3a}$$

$$\delta_t\omega=-\sigma\delta_x\zeta. \tag{4.3b}$$

The structure of these equations is identical to that of the primitive equations (3.1), with ζ playing the role of u and ω that of h. The significance of this equivalence is that (4.3) must admit of both forward and backward wave propagation (as in the primitive equations), whereas the parent differential equation (the vorticity equation) permits propagation in one direction only; in other words, (4.3) has the properties of a continuous hyperbolic system with two families of characteristics, while only one family is allowed by the vorticity equation. The intrusion of this virtual set of characteristics, and the associated truncation errors, are the penalties imposed by the use of central differences.

The preceding remarks can be made more specific, as follows. If one eliminates ω from (4.3), the result is

$$\delta_t^2\zeta=\sigma^2\delta_x^2\zeta, \tag{4.4}$$

which is the wave equation (2.1). In other words, the ζ-field on the lattices of figure 2 satisfies the finite-difference wave equation. The corresponding differential equation may be obtained merely by differentiating (4.1) with respect to t, and eliminating the resulting mixed derivative by using (4.1) again; one finds

$$\frac{\partial^2\zeta}{\partial t^2}=U^2\frac{\partial^2\zeta}{\partial x^2}. \tag{4.5}$$

We now ask the question: how do the solutions of (4.5) compare with those of (4.1)?

To examine this question, one must be very explicit about the boundary and initial conditions. For (4.5), we

*A thorough discussion of resonance in the application of end conditions to the finite-difference vorticity equation has been undertaken by Birchfield [1].

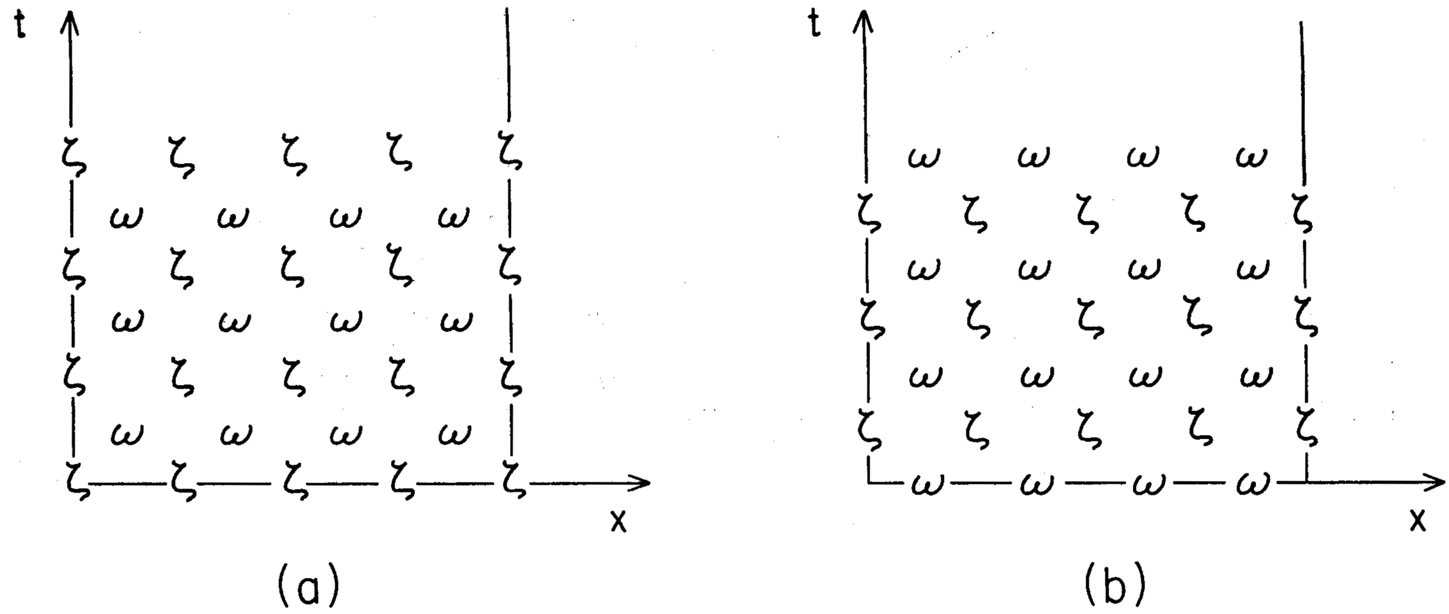

FIGURE 2.—Two lattices for solution of the vorticity equation.

impose ζ and $\partial\zeta/\partial t$ on the line $t=0$, and $\zeta=0$ on the ends $x=0$ and $x=L$ for all t; the latter condition corresponds to total reflection. For (4.1), we impose ζ on the line $t=0$ and $\zeta=0$ on the end $x=0$; the latter condition presupposes $U>0$ (which we assume henceforth), so that $x=0$ is a point of inflow for the vorticity. The solution of (4.1) is then, with reference to figure 3,

$$\zeta(x,t)=\zeta(x-Ut,0) \text{ inside } A_0B_0A_1$$

$$\zeta(x,t)=0 \text{ outside } A_0B_0A_1.$$

Thus, the initial values of ζ on A_0B_0, as well as the inflow values of ζ on the line $A_0B_1A_2 \ldots$ are simply propagated without change along the characteristics $x-Ut=$constant, which are the only characteristics admitted by (4.1).

To obtain an explicit and appropriate solution of (4.5), we will assume that $\partial\zeta/\partial t$ *on the initial line only* is specified in accordance with (4.1). This means that *initially*, the motion is organized strictly as a forward wave propagation. A disturbance at any point on the initial line, such as D (fig. 3), therefore moves out along the forward characteristic $x-Ut=$constant, until the boundary at $x=L$ is reached. There, total reflection occurs and the disturbance returns along the backward characteristic $x+Ut=$constant. It follows that in the triangle $A_0B_0C_0$ the solutions of (4.1) and (4.5) coincide, since no point in this triangle is accessible by reflection; in particular, $\zeta(P)=\zeta(D)$. On the other hand, every point in the triangle $C_0B_0A_1$, is accessible from two points on the initial line; for example, $\zeta(Q)=\zeta(D)-\zeta(E)$. In this triangle therefore, the solutions of (4.1) and (4.5) do not coincide, since (4.1) requires $\zeta(Q)=\zeta(D)$. Evidently, (4.5) permits the point Q to be affected by the reflected disturbance originating at E on the initial line. Similarly, in the parallelogram $C_0A_1C_1B_1$, we have $\zeta(R)=-\zeta(D)$ from (4.4) and $\zeta(R)=0$ from (4.1), while in the triangle $B_1C_1A_2$ we have $\zeta(S)=\zeta(E)-\zeta(D)$ from (4.5) and $\zeta(S)=0$ from (4.1).

The preceding discussion shows that only in the triangle $A_0B_0C_0$ do the solutions of (4.5) and (4.1) coincide.* The finite-difference equations (4.3) will, of course, exhibit the properties of (4.5). We turn now to a consideration of these equations.

In an earlier study of equation (4.2) the writer showed that an arbitrary imposition of ζ on the boundaries leads to a "neutral" prediction system; that is, a system whose modes are neither damped nor amplified, apart from resonance (Platzman [6]). In particular, if one takes $\zeta=0$ at the ends, then the problem posed by (4.3) is identical to that of "tidal" motion in a closed canal, the solutions of which have been given in detail in the preceding sections; hence, if in the latter solutions we replace u by ζ and h by ω, we obtain the solution of (4.3). From (2.2) and (3.2) we have

$$\zeta(x,t)=\sum_{l=1}^{p-1}\zeta_l(t)\sin k_l x \tag{4.6a}$$

$$\omega(x,t)=\sum_{l=0}^{p-1}\omega_l(t)\cos k_l x; \tag{4.6b}$$

if all initial conditions are specified on the line $t=0$, we get solutions of the form (3.6a) and (3.6b), or (3.7a) and (3.7b), for the coefficients $\zeta_l(t)$ and $\omega_l(t)$, respectively.

It is pertinent to note that in the limit Δx, $\Delta t\rightarrow 0$ the finite-difference solutions ζ and ω determined in the manner just described will coalesce with the continuous solutions of the equivalent primitive (or wave) equations,

*In the particular case considered, with $\zeta=0$ on the inflow boundary, we have coincidence of the two solutions also in the triangles $A_0C_0B_1$, $C_1A_1B_2$, and so on, where both solutions give $\zeta=0$.

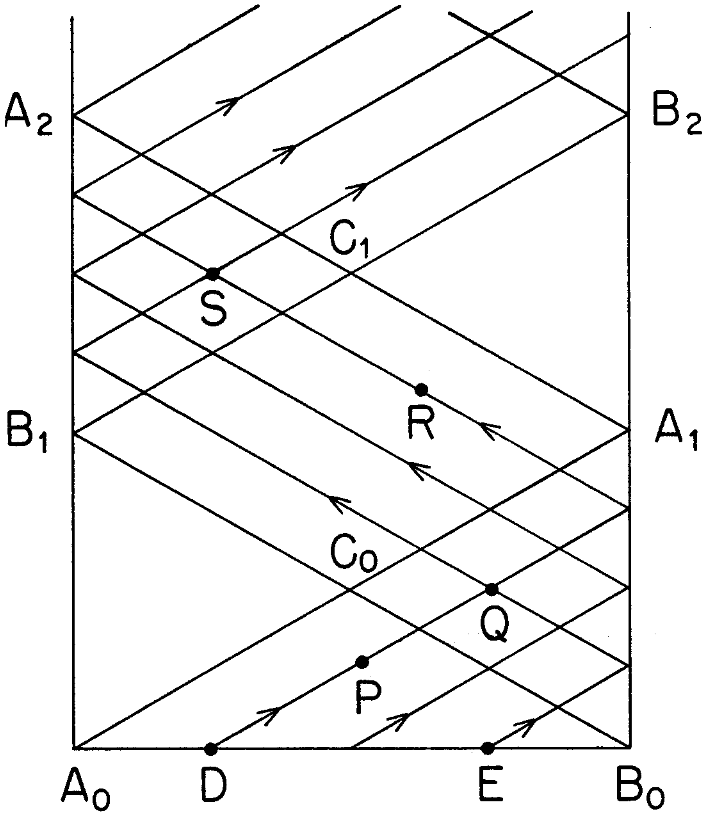

FIGURE 3.—Characteristic diagram for comparison of equations (4.1) and (4.3).

but will not coalesce with the continuous solution of the vorticity equation, and in particular, the fields ζ and ω will not themselves coalesce. This anomalous behavior of the central-difference vorticity equation is readily understood from figure 3 and the related discussion of (4.5) given above.

One way to circumvent the difficulties encountered in the central-difference vorticity equation is to use one-sided differences. In (4.1), we may take forward differences in t and (if $U>0$) backward differences in x:

$$\zeta(x,t+\Delta t)-\zeta(x,t)=\sigma[\zeta(x,t)-\zeta(x-\Delta x,t)],$$

where $\sigma\equiv U\Delta t/\Delta x$. This makes

$$\zeta(x,t+\Delta t)=(1-\sigma)\zeta(x,t)+\sigma\zeta(x-\Delta x,t).$$

The solution of this equation is obtained most readily by induction. If $\zeta(0,t)=0$, one finds

$$\zeta(x,t)=\sum_{k=0}^{n}\binom{n}{k}\sigma^{n-k}(1-\sigma)^k\zeta(x-n\Delta x+k\Delta x,0), \quad (4.7)$$

where $n\equiv t/\Delta t$, on the understanding that in the summation, $\zeta(x,0)=0$ where $x<0$.

The use of one-sided differences can be extended quite naturally to the two-dimensional vorticity equation, and in this context has been reported by Bolin [2] as having been tested by the Stockholm group. For computational stability one must make $\sigma\leq 1$; hence, if variations of U are considered, one normally will have $\sigma<<1$ over most of the field, in which case by inspection of (4.7) it is clear that the solution is highly damped.*

5. THE STABILITY OF BOUNDARY CONDITIONS

In the earlier study the writer [6] proved that instabilities are introduced by boundary conditions in which the vorticity at an outflow point is determined by "zero- or first-order extrapolation" from the concurrent vorticities at neighboring interior points. The boundary conditions of "zero-order extrapolation" (case II in the previous work) may be expressed

$$\begin{aligned}\zeta(L,t)&=\bar{\omega}(L-\tfrac{1}{2}\Delta x,t)\\ \bar{\zeta}(L,t)&=\omega(L-\tfrac{1}{2}\Delta x,t),\end{aligned} \quad (5.1)$$

where the bar signifies elements of the lattice of figure 2b. Since the two lattices were not considered separately, these conditions evidently introduce a coupling between the lattices; the result is a weak but exponential instability in the fundamental modes.

We will show now that if the outflow conditions just described are applied in such a way that the lattices are not coupled, then the instability is removed.† The appropriate revision is

$$\zeta(L,\ t)=\zeta(L-\Delta x,\ t). \quad (5.2)$$

Our proof (of stability) will consist of a construction of the fundamental modes. We begin with the expansions

$$\zeta(x,\ t)=\sum_{l=1}^{p-1}\zeta_l(t)\ \sin\ \kappa_l x \quad (5.3a)$$

$$\omega(x,\ t)=\sum_{l=1}^{p-1}\omega_l(t)\ \cos\ \kappa_l x+\Omega, \quad (5.3b)$$

which are similar to (4.6). Irrespective of the values assigned to κ_l, these satisfy the basic equations (4.3) provided $\zeta_l(t)$ and $\omega_l(t)$ are suitable linear combinations of cos $\nu_l t$ and sin $\nu_l t$, where ν_l is the frequency determined by (2.4). Further, (5.3a) satisfies the inflow condition $\zeta(0,t)=0$. In (5.3b), Ω is an absolute constant.

We now effect the determination of κ_l by forcing (5.3a) to comply with the outflow condition (5.2); this makes

$$\sin\ \kappa_l L=\sin\ \kappa_l(L-\Delta x),$$

the roots of which are given by

$$\kappa_l=(l-\tfrac{1}{2})\pi/(L-\tfrac{1}{2}\Delta x)$$
$$(l=1,\ 2,\ 3,\ \ldots,\ p-1). \quad (5.4)$$

There remains only the demonstration that, with κ_l as in (5.4), the representations in (5.3) are complete. This

* In a thorough study of the finite-difference primitive and wave equations, Mihaljan [4] has pointed out that when $\sigma=1$ in (4.7), there are no truncation errors.

†For the conditions of "first-order extrapolation" (case III in the previous work), uncoupling the lattices does not completely remove the instability, but eliminates its exponential character.

hinges upon the following identities:

$$\left\{\begin{matrix}\sum\limits_{x(\zeta)} \sin \kappa_l x(\zeta) \sin \kappa_{l'} x(\zeta) \\ \sum\limits_{x(\omega)} \cos \kappa_l x(\omega) \cos \kappa_{l'} x(\omega)\end{matrix}\right\} = \left\{\begin{matrix}0 & \text{if } l\neq l' \\ \frac{1}{2}(p-\frac{1}{2}) & \text{if } l=l'\end{matrix}\right\}$$

$$\left\{\begin{matrix}\sum\limits_{l=1}^{p-1} \sin \kappa_l x(\zeta) \sin \kappa_l x'(\zeta) \\ \sum\limits_{l=1}^{p-1} \cos \kappa_l x(\omega) \cos \kappa_l x'(\omega)\end{matrix}\right\} = \left\{\begin{matrix}0 & \text{if } x\neq x' \\ \frac{1}{2}(p-\frac{1}{2}) & \text{if } x=x'\end{matrix}\right\}$$

where $x(\zeta)$ and $x(\omega)$ signify values of x at ζ-points and at ω-points, respectively, the x-sums being taken over internal points only. The first set of identities leads to the inversion

$$\zeta_l(t)=2(p-\tfrac{1}{2})^{-1}\sum_{x(\zeta)}\zeta(x,t) \sin \kappa_l x(\zeta)$$

$$\omega_l(t)=2(p-\tfrac{1}{2})^{-1}\sum_{x(\omega)}[\omega(x,t)-\Omega] \cos \kappa_l x(\omega),$$

and the second set insures that when returned to (5.3), the latter expressions of $\zeta_l(t)$ and $\omega_l(t)$ do, in fact, reproduce $\zeta(x,t)$ and $\omega(x,t)$.

For the determination of Ω in (5.3b), we note that, in view of (5.4), $\cos \kappa_l(L-\frac{1}{2}\Delta x)=0$; hence, from (5.3b),

$$\Omega=\omega(L-\tfrac{1}{2}\Delta x,t),$$

which means that at the point adjacent to the outflow end, ω is constant in this scheme. This completes the construction of the fundamental modes associated with the "uncoupled" outflow condition (5.2); evidently, these modes are neutral.

An alternative proof of stability can be given in terms of the "semi-discrete" equation

$$\frac{\partial^2\zeta}{\partial t^2}=\left(\frac{U}{\Delta x}\right)^2\delta_x{}^2\zeta, \tag{5.5}$$

obtained from (4.4) by replacing the space derivative, but not the time derivative, by a difference quotient (see Platzman [6]). Let $\vec{\zeta}(t)$ denote the column vector formed from the $p-1$ values of $\zeta(x,t)$ at internal points, and define the matrix

$$\mathbf{A}=\begin{bmatrix} 2 & -1 & 0 & \cdots & & & 0 \\ -1 & 2 & -1 & & & & \cdot \\ 0 & -1 & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & 0 \\ \cdot & & & & -1 & 2 & -1 \\ 0 & \cdots & & & 0 & -1 & 1 \end{bmatrix}.$$

Then, with the boundary condition (5.2) at $x=L$ and $\zeta(0,t)=0$ at $x=0$, the equation satisfied by $\vec{\zeta}(t)$ is

$$\frac{\partial^2\vec{\zeta}}{\partial t^2}=-\left(\frac{U}{\Delta x}\right)^2\mathbf{A}\vec{\zeta}. \tag{5.6}$$

This has fundamental modes exp $(\pm i\mathbf{C}t)$, where

$$\mathbf{C}^2=(U/\Delta x)^2\,\mathbf{A}.$$

Clearly, the semi-discrete system is neutral if and only if the latent roots of $\mathbf{A}$ are real and non-negative.

Now consider the fully-discrete system formed from (5.6):

$$\delta_t{}^2\vec{\zeta}=-\sigma^2\mathbf{A}\vec{\zeta}.$$

This has fundamental modes exp $(\pm i\mathbf{D}t)$, where

$$4\ (\sin \tfrac{1}{2}\mathbf{D}\Delta t)^2=\sigma^2\mathbf{A}=(\Delta t)^2\mathbf{C}^2. \tag{5.7}$$

Evidently, the fully-discrete system is neutral only if the semi-discrete system is neutral. We now make the observation that the semi-discrete system (5.5) or (5.6) is precisely that which describes the transverse vibrations of a discretely-loaded, stretched string with one end fixed and one end perfectly free (with respect to transverse displacements). The mass of the string is assumed negligible, and the loading consists of $p-1$ equal loads each of which is concentrated at an interior ζ-point $x/\Delta x=1, 2, 3, \ldots, p-1$. The end at $x=0$ is fixed, while at $x=L$ we have a free end which is not loaded; the condition (5.2) then expresses the requirement that the lateral component of tension must vanish at $x=L$. Since the modes for the loaded string are neutral (as is well known), it follows that the latent roots of $\mathbf{C}$ are real; hence, from (5.7) we see that (for sufficiently small Δt) the latent roots of $\mathbf{D}$ also are real.

The preceding analogue permits some insight into the nature of the instability associated with the "coupled" outflow condition. In the semi-discrete system, we have in general

$$\frac{\partial}{\partial t}\omega\left(L-\frac{1}{2}\Delta x,t\right)=-\left(\frac{U}{\Delta x}\right)[\zeta(L,t)-\zeta(L-\Delta x,t)];$$

but the "coupled" condition is

$$\zeta(L,t)=\omega(L-\tfrac{1}{2}\Delta x,t),$$

so we get the following end condition for ζ:

$$\frac{\partial}{\partial t}\zeta(L,t)=-\left(\frac{U}{\Delta x}\right)[\zeta(L,t)-\zeta(L-\Delta x,t)].$$

For the stretched string, loaded in the manner previously described, with a free end at $x=L$, this condition may be interpreted as expressing the balance (at $x=L$) between the lateral component of tension and a frictional resistance proportional to the displacement velocity of the end of the string along its support. Such a system clearly is

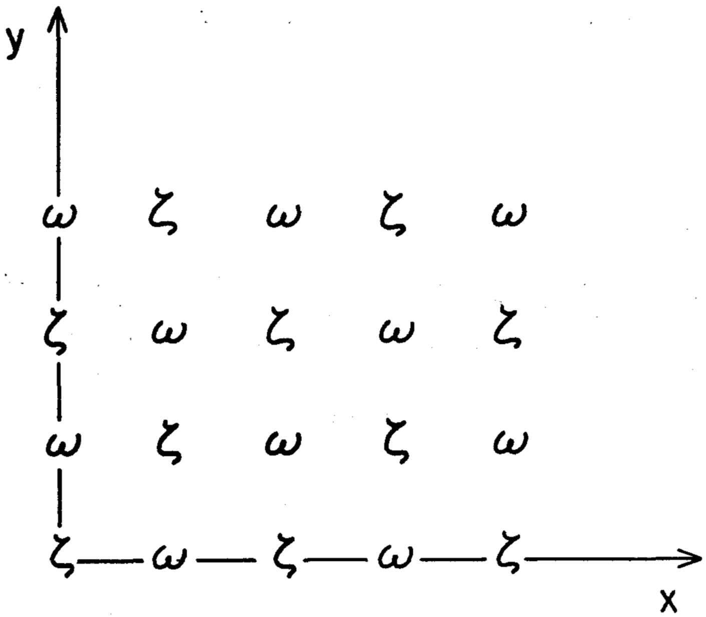

FIGURE 4.—The distribution of ζ- and ω-vorticities in the x, y-plane.

damped. The modes of the corresponding fully-discrete system therefore are amplified.

6. THE 2-DIMENSIONAL VORTICITY EQUATION

In figure 4 is shown the appropriate spatial distribution of ζ- and ω-vorticities for the two-dimensional vorticity equation

$$\frac{\partial \zeta}{\partial t} = -u\frac{\partial \zeta}{\partial x} - v\frac{\partial \zeta}{\partial y}. \tag{6.1}$$

The ω-vorticities are, as before, out of phase with the ζ-vorticities, by an interval $\frac{1}{2}\Delta t$ in time and $\frac{1}{2}\Delta s$ in space (note that figure 4 gives the distribution of these variables with respect to x and y, but not t). One may therefore identify two distinct lattices: one in which the ζ- and ω-fields are defined respectively at $t/\frac{1}{2}\Delta t = 0, 2, 4, \ldots$ and $1, 3, 5, \ldots$; and the other in which the ζ- and ω-fields are defined respectively at $t/\frac{1}{2}\Delta t = 1, 3, 5, \ldots$ and $0, 2, 4, \ldots$. In spite of the nonlinearity of (6.1), it is possible to formulate the difference equations in such a way that the two lattices remain permanently uncoupled, as will now be explained.

Assuming the velocity components to be derivable from a stream function, equation (6.1) may be written as a pair of equations

$$\frac{\partial \zeta}{\partial t} = \frac{\partial(\omega, \chi)}{\partial(x, y)} \tag{6.2a}$$

$$\frac{\partial \omega}{\partial t} = \frac{\partial(\zeta, \psi)}{\partial(x, y)}, \tag{6.2b}$$

where χ is the stream function for the ω-vorticities, and ψ is the stream function for the ζ-vorticities:

$$\omega = \nabla^2\chi;\ \zeta = \nabla^2\psi. \tag{6.3}$$

The procedure for replacement of (6.2) by appropriate difference equations is rather apparent and need not be elaborated here. In connection with (6.3) however, two procedures may be mentioned. If the x, y-axes and the interval $\frac{1}{2}\Delta s$ are used in the finite-difference Laplacian (to obtain the usual five-point formula), then concurrent values of ζ, ω, ψ, and χ are required, and in this way the two lattices would be coupled every time a relaxation is performed (that is, when (6.3) is solved). On the other hand, if axes inclined 45 degrees to the x, y-axes are used, together with an interval $\frac{1}{2}\Delta s\sqrt{2}$, then coupling of the lattices is avoided.

Numerical solutions of the barotropic vorticity equation have been obtained in the manner just described, in connection with an investigation of the prediction of hurricane movement, and will be reported elsewhere by Birchfield.

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