An Upper Bound on Transport Processes in Turbulent Thermohaline Convection

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ABSTRACT

A simple variational approach to turbulent transfer problems is applied to the analytical prediction of upper bounds on the vertical transport of heat and solute in bounded systems with negative vertical gradients of temperature and solute concentration. The results are consistent with previous studies of the special case of thermal convection. It is found that the presence of solute inhibits transport and that for sufficiently high concentration gradients no convection is possible.

1. Introduction

Knowledge of the transport processes involved in a fluid system stratified by both temperature and species gradients is important in many areas of geophysics. An important example of such a stratification is thermohaline convection which involves the combined effect of vertical temperature and concentration gradients on the convective motion of a stratified fluid.

The present paper is concerned with the analytical prediction of an upper bound on the vertical fluxes of heat and solute in a steady-state turbulent thermohaline-convection system. The analysis also serves as an example of an application of the variational approach to turbulent flow problems.

2. The variational approach

This variational approach originates in a theoretical presentation by Malkus (1954) who was concerned with turbulent thermal convection between horizontal planes. Malkus' 1954 paper and further studies by Howard (1963), Herring (1963, 1964), Coles (1965) and Nickerson (1969) have all indicated that certain turbulent flow systems in some way tend to approach a maximum (extremal) state.

The success of the variational approach depends on the proper selection of a system characteristic for which an extreme condition is sought. A rigorous bound on the flow quantities may be sought for any given postulate but only for those postulates which resemble the actual system will the extreme solution have any meaningful interpretation. A bounding process (postulate) is then sought which most closely resembles the actual flow situation. At present, this selection may only be based on the success of previous investigators who have applied the variational approach to similar fluid dynamic problems.

In thermal convection, the postulate that the flow field tends to maximize heat transport is supported by the success of those analytical techniques employing this assumption (Malkus, 1954; Howard, 1963; Herring, 1964; Howard, 1968). Applying the variational approach in a manner similar to the method to be outlined here, Howard (1968) maximized heat transport and obtained a power-law expression for the Nusselt number \( \text{Nu} \) of the form

\[
\text{Nu} - 1 = \left( \frac{8}{11} \right) \left( \frac{3}{11} \right) \frac{5 \nu^2}{16} \text{Ra}^{3/8},
\]

where the numerical constant is 0.198 and \( \text{Ra} \) is the thermal Rayleigh number. A similar approach has been used by Malkus (1968) in the study of Couette flow. In this study, Malkus maximized turbulent shear stress to obtain his "shear stress Nusselt number"

\[
\text{Nu}_s - 1 = \left[ \frac{4}{7} \right] \left( \frac{4}{7} \right) \frac{5 \nu^2}{16} \text{Re}^{1/4},
\]

where \( \text{Nu}_s \) is the ratio of the actual momentum transfer to the molecular momentum transfer with identical boundary conditions, and \( \text{Re} \) is the Reynolds number. This result bounds momentum transfer in the "Blasius regime" but quantitatively is high by a factor of 4.

The success of these applications of the variational approach encourages the application to the similar flow situation of thermohaline convection. The expected benefits of such an analysis may be summarized in the following ways:

1) If solutions are found to the posed problem, they are rigorous upper bounds to the actual flow situation.
2) The resulting solutions of previous studies are simple functional forms which may easily be compared

\[1\] Unpublished paper referred to by Malkus (1968).
to experimental data; similar power-law forms are expected for thermohaline convection.

3) The bounding process does resemble reality when the appropriate postulate is selected. For thermohaline convection, the selected postulates are maximum heat and solute transport, consistent with the successful previous studies of thermal convection.

3. Governing equations and integral constraints

The thermohaline convection system to be considered consists of rigid parallel horizontal planes separated by a vertical distance $L$. The boundaries are maintained at fixed values of temperature and solute concentration where the higher values of each are at the lower boundary.

The governing equations, including the Boussinesq approximation to the Navier-Stokes equation, may be written

$$\nabla \cdot \mathbf{U} = 0,$$

$$\left( \frac{\partial}{\partial t} - \nu \nabla^2 \right) \mathbf{U} = - (\mathbf{U} \cdot \nabla) \mathbf{U} + \frac{1}{\rho_m} \nabla p + g \kappa \left[ \beta \tau (T - T_m) - \beta_C (C - C_m) \right].$$

$$\left( \frac{\partial}{\partial t} - D_v \nabla^2 \right) T = - \mathbf{U} \cdot \nabla T,$$

$$\left( \frac{\partial}{\partial t} - D_c \nabla^2 \right) C = - \mathbf{U} \cdot \nabla C,$$

$$\rho = \rho_m \left[ 1 - \beta \tau (T - T_m) + \beta_C (C - C_m) \right],$$

where $\mathbf{U}$ is the velocity vector [$\mathbf{U} = (U, V, W)$], $p$ the pressure, $k$ the unit vector in the vertically upward direction (the positive $z$ direction), $g$ the acceleration of gravity, $\nu$ the molecular kinematic viscosity, $D_v$ the molecular diffusivity for the dissolved material of concentration $C$ per unit volume, $D_t$ the molecular thermal diffusivity, $T$ the absolute temperature, and $\beta_T$ and $\beta_C$ the coefficients of density change produced by a unit change in temperature and concentration, respectively. Finally, $T_m$, $C_m$ and $\rho_m$ are mean temperature, concentration and density.

The boundary conditions for the system may be stated as

$$\frac{\partial W}{\partial z} = 0, \quad T = T_1, \quad C = C_1; \quad \text{at } z = 0,$$

$$\frac{\partial W}{\partial z} = 0, \quad T = T_2, \quad C = C_2; \quad \text{at } z = L$$

where

$$C_1 \geq C_2, \quad T_1 \geq T_2.$$

The equations which will be used in the present analysis are in the form of integral equations derivable from the Boussinesq equations and boundary conditions. These integral equations are simple extensions of the integral constraints derived by Howard (1963) for thermal convection but with the added effect of a concentration gradient included.

Briefly, the derivation of the integral equations involves decomposing $\mathbf{U}$, $T$ and $C$ into time-averaged and fluctuating components, forming moments of the governing equations (3)-(5) with $\mathbf{U}$ and the fluctuating parts of $T$ and $C$, respectively, and volume-averaging these moment equations over the entire enclosed space. Two assumptions, those of statistical steady-state conditions and horizontal homogeneity, are made to simplify the resulting integral equations. A statistical steady-state condition implies that all ensemble averaged quantities at a point are independent of time and hence that time averages are equivalent to ensemble averages. The assumption of horizontal homogeneity requires that the time-mean flow properties are independent of position in horizontal planes. Time-averaged quantities are thus functions only of the vertical coordinate $z$.

Viewed in another way, homogeneity in horizontal planes implies a statistically ergodic process, where the instantaneous horizontal average is equal to the time mean average at a point in that horizontal plane, i.e.,

$$\bar{A} = \frac{1}{S} \int_{x,y} A(x,y,z,t) \, dx \, dy = \frac{1}{T} \int_0^T \bar{A}(x_0,y_0,z,t + \tau) \, d\tau$$

where $x_0$, $y_0$ are arbitrary.

Details of the integral equation derivations are outlined by Howard (1963). For thermohaline convection the resulting integral equations or integral constraints are (Lindberg, 1970)

$$g_{m} \langle u(\beta_T \theta - \beta_C c) \rangle = \mu \langle \nabla U^2 \rangle,$$

$$\frac{1}{D_T} [\langle (\omega \theta)^2 \rangle - \langle \omega \theta \rangle^2] + \frac{\Delta T}{L} = D_T \langle \nabla \theta^2 \rangle,$$

$$\frac{1}{D_c} [\langle (\omega c)^2 \rangle - \langle \omega c \rangle^2] + \frac{\Delta C}{L} = D_c \langle \nabla c^2 \rangle,$$

where

$$\langle A \rangle = \frac{1}{V} \int_V \bar{A} \, dV = \frac{1}{L} \int_0^L \bar{A} \, dz,$$

$$\langle \nabla A \rangle = \frac{1}{V} \int_V \nabla \bar{A} \, dV = \frac{1}{L} \int_0^L \bar{A} \, dz, \nabla$$

and

$$\Delta T = T_1 - T_2,$$

$$\Delta C = C_1 - C_2,$$

$$U_i = u_i (U_i = 0),$$

$$T = T + \theta,$$

$$C = \bar{C} + c.$$
It is useful to make the three integral equations dimensionless by introducing scales of length ($L$), temperature ($\Delta T$), concentration ($\Delta C$), and velocity ($D_T/L$). The nondimensional variables may then be defined as

$$\begin{align*}
W^* &= \frac{W}{D_T} \\
C^* &= \frac{C}{\Delta C} \\
T^* &= \frac{T}{\Delta T}
\end{align*}$$

The integral equations then become

$$\begin{align*}
\text{Ra}_T(W^*T^*) &= \frac{\text{Ra}_T}{Lc}(W^*C^*) = \langle \nabla U^* \rangle, \\
\langle W^*T^* \rangle^2 - \langle (W^*T^*)^2 \rangle &= \langle \nabla T^* \rangle^2,
\end{align*}$$

$$\text{Le}^2(W^*C^*)^2 - \langle (W^*C^*)^2 \rangle + \text{Le}(W^*C^*) = \langle \nabla C^* \rangle^2,$$

where

$$\begin{align*}
\text{Ra}_T &= \frac{g\beta_T\Delta TL^3}{\nu D_T} \quad \text{(thermal Rayleigh number)} \\
\text{Ra}_c &= \frac{g\beta_c\Delta CL^3}{\nu D_c} \quad \text{(concentration Rayleigh number)} \\
\text{Le} &= \frac{D_T}{D_c} \quad \text{(Lewis number)}
\end{align*}$$

In addition to the above second-moment integral constraints, the following first-moment relations can be obtained from (4) and (5):

$$\begin{align*}
D_T \frac{d^2 T}{dz^2} &= \frac{d\omega}{dz}, \\
D_c \frac{d^2 C}{dz^2} &= \frac{d\omega}{dz}.
\end{align*}$$

The steady-state condition requires that the mean vertical heat and mass transport rates per unit horizontal area be constant, these fluxes being designated by $F_H$ and $F_C$, respectively. Eqs. (14) and (15) can be integrated once, the constants of integration being the flux terms $F_H$ and $F_C$; thus,

$$\begin{align*}
F_H &= \rho C_p D_T \left( \frac{d^2 T}{dz^2} \right) + \rho C_p \bar{w} \theta, \\
F_C &= D_c \left( \frac{d^2 C}{dz^2} \right) + \bar{w} \bar{c}.
\end{align*}$$

Eq. (16) is integrated over the domain, and, since $F_H$ is constant, the vertical kinematic heat flux appears as

$$\frac{F_H}{\rho C_p} = \frac{D_T}{L} \frac{\nabla T}{\bar{\omega} \theta}.$$

The analogous solute flux may also be obtained from Eq. (17) as

$$\frac{F_C}{D_c} = \frac{\Delta C}{L} + \langle \bar{w} \bar{c} \rangle.$$

Using the same dimensionless variables as before, we obtain

$$\begin{align*}
\frac{F_H}{k \Delta T/L} &= \text{Nu} = 1 + (W^*T^*), \\
\frac{F_C}{D_c \Delta C/L} &= \text{Sh} = 1 + \text{Le}(W^*C^*),
\end{align*}$$

where $\text{Nu}$ and $\text{Sh}$ are the Nusselt and Sherwood numbers, respectively.

### 4. Methods of solution

From the integral constraints (11), (12) and (13), together with Eqs. (20) and (21), one can express the Nusselt and Sherwood numbers in terms of the three dissipation integrals $\langle \nabla U^* \rangle^2$, $\langle \nabla T^* \rangle^2$ and $\langle \nabla C^* \rangle^2$. These integrals are indications of the state of the flow field, but they do not provide any information on the internal flow field. Other analyses are required which do not average over this information.

It is convenient to introduce a transformation which will normalize the integral correlations $\langle W^*T^* \rangle$ and $\langle W^*C^* \rangle$:

$$\begin{align*}
\tilde{U} &= \langle W^*T^* \rangle \langle W^*C^* \rangle \tilde{U}, \\
T^* &= \langle W^*T^* \rangle \langle W^*C^* \rangle \tilde{T}, \\
C^* &= \langle W^*T^* \rangle \langle W^*C^* \rangle \tilde{C},
\end{align*}$$

where

$$\langle \tilde{u} \tilde{T} \rangle = \langle \tilde{u} \tilde{T} \rangle = 1.$$

We see that

$$\langle \tilde{u} \tilde{C} \rangle = \langle \tilde{u} \tilde{C} \rangle = 1;$$

moreover, using (20)-(24), the integral relations (11)-(13) take the form

$$\begin{align*}
\text{Ra}_T(\text{Nu} - 1) - \frac{\text{Ra}_c}{\text{Sh} - 1} &= \frac{\text{Le}}{\text{Le}^2} \phi, \\
1 + (\text{Nu} - 1)I_\theta &= \frac{\text{Le}}{\text{Sh} - 1}, \\
1 + (\text{Sh} - 1)I_\epsilon &= \frac{1}{\text{Le}^2 (\text{Nu} - 1)^2}.
\end{align*}$$

**FIG. 1.** Predicted upper bound of the Nusselt number vs the thermal Rayleigh number with the stability number for a Lewis number of 100.
close to the upper limit that transport is greatly suppressed.

Another ratio of interest is the ratio of the Nusselt number to the Sherwood number as obtained from (42) and (43), i.e.,

\[
\frac{(Nu-1)^*}{(Sh-1)^*} = Le^{-\varepsilon/11}.
\]

This result is quite consistent with other turbulent heat and mass transfer situations; for example, in a wetted tube in turbulent flow (Kreith, 1965), we have

\[
\frac{Nu}{Sh} = Le^{-1}.
\]

In either case, the transport process is controlled by wall effects, which are dominated by molecular properties.

The present analysis indicates no discernible change in the transport characteristics at Stabilities > 1. Thus, no conclusion may be made here as to the tendency of the system to form layers. It should be noted, however, that it has been shown in the Appendix that the present power-law analysis is valid even for a layered system. It is further concluded that for Sy > Le^{5/11}, no convection is possible, as the transport coefficients become complex.

The predictions, (40) and (41), stand to be tested against data. In view of the qualitative nature of the analysis it is unrealistic to expect quantitative agreement. Even so, the correlating parameters and their arrangement are offered with some confidence for the organization of future experiments.

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**APPENDIX**

**Estimates of the Integrals I_s and I_c**

Estimations of upper bounds on I_s and I_c are made using techniques similar to the methods of Howard (loc. cit.) and Malkus (1968). An improvement on their analysis is made by using symmetry arguments; the improvement will be outlined after the basic estimation method is discussed.

The definitions of I_s and I_c are

\[
I_s = 1 - \langle \mathcal{W}_s^2 \rangle,
\]

\[
I_c = 1 - \langle \mathcal{W}_c^2 \rangle.
\]

It is convenient to express these in a slightly different manner. Using (25), it is readily verified that

\[
I_s = - \langle \mathcal{W}_s - 1 \rangle^2 \leq 0
\]

\[
I_c = - \langle \mathcal{W}_c - 1 \rangle^2 \leq 0.
\]

An upper bound on I_s or I_c is then a lower bound on the bracketed terms. Moreover, since

\[
\langle \mathcal{W}_s - 1 \rangle^2 = \int_0^1 (1 - |\mathcal{W}_s|)^2 dz + \int_0^1 (1 - |\mathcal{W}_s|) dz,
\]

an upper bound on \(|\mathcal{W}_s|\) (and \(|\mathcal{W}_c|\)) as a function of \(\varepsilon\) is thus sought.

This upper bound will be determined for the case of a single predominant horizontal wavenumber. In his
\[(Sh - 1)^* = 0.14 L e^{3/8} \frac{\beta e^{5/11} - \frac{Ra_e}{Le}}{\beta T \Delta T} \frac{1}{3/8}. \] \[(41)\]

The group of dimensionless parameters at the end of each of these two expressions is of special physical importance; it may be rewritten as

\[\frac{Ra_e}{Le} = \frac{\beta \Delta T}{\beta T \Delta T} = Sy.\]

This dimensionless parameter Sy has been termed by Turner (1963) the stability number; it expresses the relative effects of stabilizing and destabilizing body forces which determine the motion, i.e.,

\[\frac{\Delta \rho \text{ stabilizing}}{\Delta \rho \text{ destabilizing}}.\]

A stability number of zero implies purely thermal convection, whereas a large stability number indicates a very dominant influence of the stabilizing solute gradient, perhaps to the point of completely inhibiting convective motion. With the stability number defined, (40) and (41) become

\[(Nu - 1)^* = 0.14 L e^{15/8} \frac{Ra_e^{1/8}}{(Le^{6/11} - Sy)^{3/8}}, \quad (42)\]

\[(Sh - 1)^* = 0.14 L e^{3/8} \frac{Ra_e^{1/8}}{(Le^{5/11} - Sy)^{3/8}}. \quad (43)\]

5. Discussion of results

The estimation procedure of the previous section provides a rigorous upper bound on the fluxes of heat and solute for thermohaline convection, resulting in predictions of the Nusselt and Sherwood numbers, (42) and (43), which may be directly employed in predicting the vertical transport of heat and solute. Upper bounds on the Nusselt number as a function of the thermal Rayleigh number are shown in Fig. 1 for various values of the stability number and for a Lewis number of 100 (appropriate for sea water).

The special case of thermal convection, that is \[Sy = 0\], creates a simplification of (42) and (43), where (42) is reduced to

\[(Nu - 1)^*_{\text{thermal}} = 0.14 Ra_e^{3/8}. \quad (44)\]

If the improved estimate on the integral \(J_\theta\) (see Appendix) is not included, this equation becomes

\[(Nu - 1)^*_{\text{thermal}} = 0.198 Ra_e^{3/8}, \]

which is the original power-law prediction of Howard [loc. cit., Eq. (1)]. Experimental data indicate that for thermal convection

\[Nu \approx 0.1 Ra_e^{1/8} \quad (45)\]

for high Rayleigh numbers.

The ratio of (42) and (44) is

\[\frac{(Nu - 1)^*}{(Nu - 1)^*_{\text{thermal}}} = Le^{-15/8} (Le^{6/11} - Sy)^{3/8}. \quad (46)\]

This expression is shown in Fig. 2 for a Lewis number of 100. This figure shows the inhibiting effect of solute gradient on heat transport. It is only for values of Sy

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Fig. 1. Predicted upper bound of the Nusselt number vs the thermal Rayleigh number with the stability number as a parameter for a Lewis number of 100.
close to the upper limit that transport is greatly suppressed.

Another ratio of interest is the ratio of the Nusselt number to the Sherwood number as obtained from (42) and (43), i.e.,

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\frac{(Nu - 1)^*}{(Sh - 1)^*} = Le^{-6/11}.
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This result is quite consistent with other turbulent heat and mass transfer situations; for example, in a wetted tube in turbulent flow (Kreith, 1965), we have

\[
\frac{Nu}{Sh} = Le^{-1}.
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In either case, the transport process is controlled by wall effects, which are dominated by molecular properties.

The present analysis indicates no discernible change in the transport characteristics at stabilities > 1. Thus, no conclusion may be made here as to the tendency of the system to form layers. It should be noted, however, that it has been shown in the Appendix that the present power-law analysis is valid even for a layered system. It is further concluded that for \( Sy > Le^{b/11} \), no convection is possible, as the transport coefficients become complex.

The predictions, (40) and (41), stand to be tested against data. In view of the qualitative nature of the analysis it is unrealistic to expect quantitative agreement. Even so, the correlating parameters and their arrangement are offered with some confidence for the organization of future experiments.

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APPENDIX

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Estimations of upper bounds on \( I_\theta \) and \( I_c \) are made using techniques similar to the methods of Howard (loc. cit.) and Malkus (1968). An improvement on their analysis is made by using symmetry arguments; the improvement will be outlined after the basic estimation method is discussed.

The definitions of \( I_\theta \) and \( I_c \) are

\[
I_\theta = 1 - \langle \bar{\omega} \Theta^2 \rangle,
\]

\[
I_c = 1 - \langle \bar{\omega} \Gamma^2 \rangle.
\]

It is convenient to express these in a slightly different manner. Using (25), it is readily verified that

\[
I_\theta = -\langle (\bar{\omega} \Theta - 1)^2 \rangle < 0
\]

\[
I_c = -\langle (\bar{\omega} \Gamma - 1)^2 \rangle < 0.
\]

An upper bound on \( I_\theta \) or \( I_c \) is then a lower bound on the bracketed terms. Moreover, since

\[
\langle (\bar{\omega} \Theta - 1)^2 \rangle = \int_0^1 (1 - \bar{\omega} \Theta)^2 dz \geq \int_0^1 (1 - |\bar{\omega} \Theta|)^2 dz,
\]

an upper bound on \( |\bar{\omega} \Theta| \) (and \( |\bar{\omega} \Gamma| \)) as a function of \( z \) is thus sought.

This upper bound will be determined for the case of a single predominant horizontal wavenumber. In his
earliest paper on variational theory, Howard (1963) formulated his extremal solutions based on a single wavenumber concept. By showing that the Euler equations had variable separable solutions, Howard assumed that the extremum was satisfied by a single separable eigenvalue or wavenumber. This separable solution may be taken to be of the form

\[ \begin{align*}
\vec{w} &= \hat{w}(z) \lambda(x,y), \\
\Theta &= \hat{\Theta}(z) \lambda(x,y), \\
\Gamma &= \hat{\Gamma}(z) \lambda(x,y),
\end{align*} \]

where

\[ \lambda = \exp[i(k_x x + k_y y)], \quad k = (k_x^2 + k_y^2)^{1/4} \]

is the horizontal wavenumber. It will be assumed that this representation is valid for the range of Rayleigh numbers for which the single wavenumber concept is valid (Busse, 1969).

The definition of the horizontal average yields

\[ |\bar{\omega}(\Theta)| = |\hat{w}(z)| |\hat{\Theta}(z)| = |\hat{w}(z)| |\hat{\Theta}(z)|. \quad (A1) \]

Bounds are then sought for \(|\hat{w}(z)|\) and \(|\hat{\Theta}(z)|\). The boundary conditions on \(\hat{w}(z)\) and \(\hat{\Theta}(z)\) are

\[ \begin{align*}
\hat{\Theta}(z) &= 0 \quad \text{at} \quad z = 0, \\
\hat{w}(z) &= \hat{w}'(z) = 0 \quad \text{at} \quad z = 0.
\end{align*} \]

The prime denotes differentiation with respect to \(z\). With these boundary conditions, \(|\hat{\Theta}(z)|\) may be estimated as

\[ |\hat{\Theta}(z)| = \left| \int_0^z \hat{\Theta}'(z_1)dz_1 \right| \leq \left[ \int_0^z \hat{\Theta}^2(z_1)dz_1 \right]^{1/2} \left[ \int_0^z dz_1 \right]^{1/2}. \]

Since

\[ \int_0^z \hat{\Theta}^2(z_1)dz_1 \leq \int_0^1 \hat{\Theta}^2(z_1)dz_1 = (\hat{\Theta}^2)_2, \quad (A2) \]

\[ |\hat{\Theta}(z)| \leq (\hat{\Theta}^2)_2^{1/2}. \]

In a similar manner,

\[ \hat{w}(z) = \int_0^z \hat{w}'(z_1)dz_1. \]

Integrating by parts,

\[ \hat{w}(z) = \hat{w}'(z)(z_1 - z) \bigg|_0^z - \int_0^z (z_1 - z)\hat{w}''(z_1)dz_1 \]

\[ = \int_0^z (z - z_1)\hat{w}''(z_1)/z_1. \]

Taking the absolute value of each side, one obtains the following inequalities:

\[ |\hat{w}(z)| = \left| \int_0^z (z - z_1)\hat{w}''(z_1)dz_1 \right| \leq \left[ \int_0^z \hat{w}''^2dz_1 \right]^{1/2} \left[ \int_0^z (z - z_1)^2dz_1 \right]^{1/2}, \]

or

\[ |\hat{w}(z)| \leq \left( \frac{z^2}{3} \right)^{1/2}. \quad (A3) \]

Eq. (A1) thus becomes

\[ |\bar{\omega}(\Theta)| = |\hat{w}(z)| |\hat{\Theta}(z)| \leq \left( \frac{z}{z_*} \right)^2, \]

where

\[ z_* = 3(\hat{\Theta}^2 - 1)(\hat{w}''/z^2)^{1/2}. \quad (A4) \]

The term \(z_*\) is interpreted as a dimensionless length which is small in comparison to 1 (the depth of the fluid). The vertical average of \(|\bar{\omega}(\Theta)|\) is 1, as seen by (25). For \(z > z_*\), \(|\bar{\omega}(\Theta)| \approx 1\); thus, \(z_*\) defines a boundary layer within which \(|\bar{\omega}(\Theta)|\) deviates from its average value of 1. Both Howard (1963) and Busse (1969) have argued, from another viewpoint, that \((1 - \bar{\omega}(\Theta))\) becomes significant only in the boundary layer regions. This interpretation of \(z_*\) allows the following inequality to be written:

\[ \langle(1 - \bar{\omega}(\Theta))^2\rangle \geq \int_0^{z_*} (1 - \bar{\omega}(\Theta))^2dz + \int_{1-z_*}^1 (1 - \bar{\omega}(\Theta))^2dz \]

\[ - 2 \int_0^{z_*} (1 - |\hat{w}(z)|\hat{\Theta}(z)|)^2dz. \quad (A5) \]

Substitution of (A4) into this inequality results in the estimate

\[ \langle(1 - \bar{\omega}(\Theta))^2\rangle \geq 2 \int_0^{z_*} \left(1 - \frac{z^2}{z_*^2}\right)^2dz = \frac{16}{15}, \]

or

\[ I_s \leq \frac{16}{15} z_* \quad (A6) \]

An equivalent derivation may be applied to \(I_s\), which results in

\[ I_s \leq \frac{16}{15} z_{se}, \]

where

\[ z_{se} = 3(\hat{\Theta}^2 - 1)(\hat{w}''/z^2)^{1/2} \]

is a concentration boundary layer thickness with an interpretation similar to the one given for \(z_*\).
It remains to express \( z_s \) in terms of the dissipation integrals. Howard (1963) derived two inequalities which are based on the single wavenumber assumption. These inequalities are

\[
\phi \geq \left( k^{-2} \phi''^2 + 2 \phi'' + k^2 \phi' \right), \quad (A7)
\]
\[
\psi \geq (\phi'' + k^2 \phi^2), \quad (A8)
\]

Similarly, the following inequality may be derived:

\[
\chi \geq \left( \psi'' + k^2 \psi' \right), \quad (A9)
\]

and (A7) may be simplified to

\[
\phi \geq \left( k^{-2} \phi''^2 + k^2 \phi^2 \right). \quad (A10)
\]

Using (A8) and (A10), we find

\[
\langle \phi'' \rangle \langle \phi'' \rangle \leq k^2 \left[ \psi - k^2 \langle \phi'' \rangle \right] \left[ \phi - k^2 \langle \phi'' \rangle \right].
\]

But (25) requires that

\[
\langle \phi'' \rangle \langle \phi'' \rangle \geq (\phi')^2 = 1;
\]

therefore,

\[
\langle \phi'' \rangle \langle \phi'' \rangle \leq c b [\psi - c \phi + b \psi + c b^2] = A,
\]

where

\[
c^2 = k^4 \langle \phi'' \rangle
\]
\[
b^2 = k^2 \langle \phi'' \rangle.
\]

Since an upper bound for \( z_s \) is sought, a lower bound on \( A \) must be found. To find values of \( b \) and \( c \) which minimize \( A \), the necessary conditions on \( b \) and \( c \) are determined by

\[
\frac{\partial A}{\partial b} = 0, \quad \frac{\partial A}{\partial c} = 0,
\]

so that

\[
\begin{aligned}
\frac{c^2}{b^2} &= \psi / 3, \\
\frac{b^2}{\phi} &= \psi / 3.
\end{aligned}
\]

It follows that

\[
\langle \phi'' \rangle \langle \phi'' \rangle \leq \frac{4}{27} \phi^{4/3},
\]

and from (A4) that

\[
z_s \leq S \left( \frac{27/4}{1} \right) \psi^{-3/8} \phi^{-3/8}.
\]

Finally from (A6), we have

\[
I_6 \leq \frac{16}{5 \sqrt{2}} \psi^{-3/8} \phi^{-3/8}. \quad (A11)
\]

This is the form obtained by Howard (loc. cit.). Arguments similar to those leading to (A11) may be applied to \( I_6 \), resulting in the inequality

\[
I_c \leq \frac{16}{5 \sqrt{2}} \psi^{-3/8} \phi^{-3/8}. \quad (A12)
\]

An improvement of these estimates may be made by noting the assumed even property of \( \bar{\omega}(z) \) and \( \bar{\theta}(z) \) about \( z = \frac{1}{2} \). In particular, the assumption involves the symmetry

\[
\bar{\theta}^2(z) = \bar{\theta}^2(1 - z), \quad 0 \leq z \leq 1
\]
\[
\bar{\omega}''(z) = \bar{\omega}''(1 - z), \quad 0 \leq z \leq 1.
\]

Accordingly,

\[
\int_0^1 \bar{\theta}^2(z) dz = 2 \int_0^1 \bar{\theta}^2(z) dz
\]
\[
\int_0^1 \bar{\omega}''(z) dz = 2 \int_0^1 \bar{\omega}''(z) dz.
\]

Eqs. (A2) and (A3) thus become

\[
| \bar{\omega}(z) | \leq \frac{1}{\sqrt{2}} | \bar{\theta}(z) | \psi^{1/4} \]

\[
| \bar{\omega}(z) | \leq \frac{1}{\sqrt{2}} | \bar{\theta}(z) | \psi^{1/4}
\]

and

\[
| \bar{\theta}(z) | \leq | \bar{\theta}(z) | \psi^{1/4}
\]

\[
| \bar{\omega}(z) | \leq \frac{1}{\sqrt{2}} | \bar{\theta}(z) | \psi^{1/4}, \quad \frac{1}{2} \leq z \leq 1.
\]

Thus,

\[
| \bar{\omega}(z) | \bar{\theta}(z) \leq \frac{1}{2} \frac{z}{z_s}, \quad 0 \leq z \leq \frac{1}{2}
\]

\[
| \bar{\omega}(z) | \bar{\theta}(z) \leq \left( \frac{z}{z_s} \right)^2, \quad \frac{1}{2} \leq z \leq 1.
\]

The assumption of symmetry about \( z = \frac{1}{2} \) enables the second inequality to be omitted in favor of the first, which is stronger. The integral estimate (A6) may then be improved as

\[
\langle (1 - \bar{\omega} \bar{\theta})^2 \rangle \leq 2 \int_0^{z_s} \left( 1 - \frac{z}{2} \right)^2 dz = \frac{16 \sqrt{2}}{15},
\]

or

\[
I_6 \leq \frac{16 \sqrt{2}}{15}, \quad (A13)
\]

Similarly,

\[
I_c \leq \frac{16 \sqrt{2}}{15}. \quad (A14)
\]

Since the development beyond (A6) remains identical, we have

\[
I_6 \leq \frac{16}{5} \psi^{-3/8} \phi^{-3/8}, \quad (A15)
\]

\[
I_c \leq \frac{16}{5} \psi^{-3/8} \phi^{-3/8} \quad (A16)
\]
These improved estimates are employed to get (29) and (30).

The presence of internal layers has not been taken into account in the preceding estimations. Prior to applying the present estimates to a system where layers may occur, one must examine their effect on the inequalities. Internal layers will influence the inequalities of (A5), where the effect of $1 - \bar{w}/\Theta$ is assumed to be limited to regions near the bounding surfaces. Should layers occur, $1 - \bar{w}/\Theta$ will become significant in the region near each internal layer interface where convective fluctuations are diminished. The inequality may thus be written as

\[
(1 - \bar{w}/\Theta)^2 \geq \int_{0}^{z_0} (1 - \bar{w}/\Theta)^2 \, dz + \sum_{i=1}^{n-1} \int_{z_i}^{z_{i+1}} (1 - \bar{w}/\Theta)^2 \, dz
\]

\[
+ \int_{z_i}^{z_{0}} (1 - \bar{w}/\Theta)^2 \, dz,
\]

where $n$ is the number of layers, and $z_i$ the vertical location of each layer interface. This inequality is stronger than (A5) when layers occur, but (A5) is still valid though not as strong. Should layers occur, the bounding process is seen not to be as close to reality as when layers are absent.

REFERENCES


