Internal Wave Spectra in the Presence of Fine-Structure

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ABSTRACT

Phillips has shown that an undulating motion of a layered medium relative to a measuring instrument will result in a σ^{-2} spectrum (frequency or wavenumber) over a bandwidth determined by the thickniss of the layers and of the sheets separating them. We show, for *any* (temperature) fine-structure statisteally stationary in depth with covariance $r_\theta(y_1-y_2)=\langle\theta(y_1)\theta(y_2)\rangle$, that the covariance of the observed time ceries can be expressed in terms of r_θ and the covariance in the vertical displacement ζ, assuming ζ to be josintly normal. An explicit expression for the spectrum is given for the case that the rms value of ζ is large compared to the vertical coherence scale of the fine-structure. We tentatively conclude that the fine-structure dominates in the upper few octaves of the internal wave spectra, and then extends the spectra beyond the cutoff frequency (wavenumber). The loss of vertical coherence due to fine-structure occurs over a distance inversely proportional to frequency, in general agreement with an empirical rule proposed by Webster.

1. Introduction

With the development of instruments capable of high resolution it has become evident that a step-like fine-structure is superimposed on the gross vertical profiles of temperature and salinity (Woods and Fosberry, 1966; Woods, 1968; Stommel and Fedorov, 1967; Cox *et al.*, 1969). Accordingly, for a smooth up-and-down motion of the water column, time series of temperature and salinity (moored or towed) display many small steps. If there is an associated fine-structure in the Väisälä frequency $n(y)$, then the vertical shear du/dy is concentrated at the steps (Phillips, 1966, p. 168) and this leads to analogous features in the time series of horizontal flow (see Section 9). Phillips (1971) has given an excellent account of the evidence in both ocean and atmosphere.

Cox (1966) has pointed out the difficulty, associated with fine-structure, in the derivation of internal wave spectra from temperature-time series. At a Royal Society meeting in November 1970, Gould (1971) discussed the case of a sinusoidal fine-structure. Phillips (1971) derives the contribution to the internal wave spectrum that is associated with the step-like fine-structure, and finds it to be proportional to σ^{-2} (frequency or wavenumber) over a certain bandwidth. The fact that internal wave spectra roughly proportional to σ^{-2} have been observed is at least a warning that fine-structure effects may be important. Reid (1971) gives an explicit solution for the special case of a transducer located at the mean boundary of two uniform layers. We find that a quite general case involving only the statistics of the layering can be treated somewhat along the lines of Reid's paper.

2. Temperature covariance

Let $T(y)$ designate the temperature profile in the absence of internal waves. If the observed temperature fluctuations at some fixed depth y_0 are entirely the result of vertical displacements $\zeta(t)$, then $T(t)=T[y_0-\zeta(t)]$. The covariance of temperature measurements separated in time by τ (or similarly in horizontal distance) can be written as

$$R_T(\tau)=\langle T(t)T(t+\tau)\rangle,$$
$$=\int\int_{-\infty}^{\infty} T(y_0-\zeta_1)T(y_0-\zeta_2)p(\zeta_1,\zeta_2;\,\tau)d\zeta_1\,d\zeta_2, \quad (1)$$

where $\langle\ \rangle$ is the time (or similarly space) average, and $p(\zeta_1,\zeta_2;\tau)$ the joint probability density of a displacement ζ_1 at some time t and ζ_2 at $t+\tau$.

Suppose the temperature profile near y_0, and referred to $T(y_0)$, is written as

$$T(y)=(y-y_0)T'+\theta(y),\quad \bar{\theta}(y)=0, \quad (2)$$

thus consisting of a mean gradient $T'=\overline{dT/dy}$ and a superimposed fine-structure $\theta(y)$. Then

$$T(y_0-\zeta_1)T(y_0-\zeta_2)$$
$$=T'^2\zeta_1\zeta_2-T'[\zeta_1\,\theta(y_0-\zeta_2)+\zeta_2\,\theta(y_0-\zeta_1)]$$
$$+\theta(y_0-\zeta_1)\theta(y_0-\zeta_2). \quad (3)$$

The first term gives the expected "gradient covariance"

$$R_{T^{gd}}(\tau)=T'^2\int\int\zeta_1\zeta_2p(\zeta_1,\zeta_2;\,\tau)d\zeta_1\,d\zeta_2=T'^2R_\zeta(\tau), \quad (4)$$

where $R_\zeta(\tau)=\langle\zeta(t)\zeta(t+\tau)\rangle$ is the covariance of the displacement ζ.

We could proceed in a similar deterministic manner with the remaining terms, using the measured $\theta(y)$ in (3), but the result would be a sensitive function of time, place and depth of experiment. Since the fine-structure varies slowly in time (distance), the sensible thing to do is to assume $\theta(y)$ a random variable and to use the expected values for the remaining terms. Since $\bar{\theta}=0$, the expected value of $[\zeta_1\,\theta(y_0-\zeta_2)+\zeta_2\,\theta(y_0-\zeta_1)]$ is zero, and the last term yields the "fine-structure covariance"

$$R_T{}^{fs}(\tau)=\int\int r_\theta(\zeta_1-\zeta_2)p(\zeta_1,\zeta_2;\,\tau)d\zeta_1\,d\zeta_2, \qquad (5)$$

where $r_\theta(\zeta_1-\zeta_2)=\langle\theta(y_0-\zeta_1)\theta(y_0-\zeta_2)\rangle$, with $\langle\ \rangle$ here implying an ensemble average over many realizations of the fine-structure. In practice the average $\langle\ \rangle$ is formed with respect to depth in the neighborhood of y_0.

3. Gaussian wave field

To go further we need a form for $p(\zeta_1,\zeta_2;\tau)$. We assume that the displacement at any point comes from many different, largely independent frequencies and wavenumbers. By the central limit theorem, $\zeta(t)$ is then normally distributed and $\zeta(t)$, $\zeta(t+\tau)$ are jointly normal, i.e.,

$$p(\zeta_1,\zeta_2;\,\tau)=\frac{1}{2\pi Z^2(1-\rho^2)^{\frac{1}{2}}}\exp\left[-\frac{\zeta_1{}^2-2\rho\zeta_1\zeta_2+\zeta_2{}^2}{2Z^2(1-\rho^2)}\right], \qquad (6)$$

where $\rho(\tau)=R_\zeta(\tau)/Z^2$ is the autocorrelation of the displacement, and $Z^2=\overline{\zeta^2}=R_\zeta(0)$ is the displacement variance. Eq. (5) then becomes

$$R_T{}^{fs}(\tau)=[4\pi(1-\rho)Z^2]^{-\frac{1}{2}}\times\int_{-\infty}^{\infty}r_\theta(z)\exp\left[-\frac{z^2}{4(1-\rho)Z^2}\right]dz, \qquad (7)$$

where we have changed one variable, ζ_1 say, to $z=\zeta_1-\zeta_2$ and integrated over the other, ζ_2, using

$$\int_{-\infty}^{\infty}\exp(-\lambda x^2)dx=(\pi/\lambda)^{\frac{1}{2}}.$$

Eq. (7) may be written alternatively as

$$R_T{}^{fs}(\tau)=\pi^{-\frac{1}{2}}\int_{-\infty}^{\infty}r_\theta[2(1-\rho)^{\frac{1}{2}}Zx]\exp(-x^2)dx. \qquad (8)$$

We see immediately that $R_T{}^{fs}(0)=r_\theta(0)=\overline{\theta^2}$ as required. As $\tau\rightarrow\infty$, $\rho(\tau)\rightarrow 0$, but

$$R_T{}^{fs}(\tau)\rightarrow\pi^{-\frac{1}{2}}\int_{-\infty}^{\infty}r_\theta(2Zx)\exp(-x^2)dx,$$

which is not, in general, zero. Thus, the temperature measurements remain correlated by the fine-structure even when the vertical displacements are no longer correlated. This residual correlation is removed if we allow for the fact that the fine-structure is a slowly variable function of time (distance), $\theta(y,t)$, with covariance $r_\theta(z,\tau)=\langle\theta(y_0-\zeta_1,t)\theta(y_0-\zeta_2,t+\tau)\rangle$ that tends to zero as $\tau\rightarrow\infty$. Replacing $r_\theta(z)$ by $r_\theta(z,\tau)$ in (7) then gives $R_T{}^{fs}(\infty)=0$. For the special case of no internal waves, $p(\zeta_1,\zeta_2;\tau)=\delta(\zeta_1)\delta(\zeta_2)$, and Eq. (5) becomes simply $R_T{}^{fs}(\tau)=r_\theta(0,\tau)$. We shall neglect this weak dependence on τ and write $r_\theta(z)$ for $r_\theta(z,0)$.

4. The fine-structure approximation

If $r_\theta(z)$ and $\rho(\tau)$ are known, then (8) may, in principle, be evaluated and Fourier-transformed to give the spectrum, $F_T{}^{fs}(\sigma)$, of temperature fluctuations due to the presence of layers. Further simplifications result if the vertical displacement Z of the internal waves is large compared to the scale of the layering (as is implied by the term "fine-structure"). A suitable definition of the fine-structure scale K^{-1} in terms of the fine-structure spectrum $F_\theta(k)$ is given by

$$K^2=\int k^2F_\theta(k)dk\bigg/\int F_\theta(k)dk=\frac{\overline{\theta'^2}}{\overline{\theta^2}}=\frac{-r_\theta''(0)}{r_\theta(0)}. \qquad (9)$$

[If $\theta(y)$ were normally distributed, K/π would be the average number of fine-structure zero-crossings per unit vertical distance.] Similarly

$$S^2=\int\sigma^2F_\zeta(\sigma)d\sigma\bigg/\int F_\zeta(\sigma)d\sigma=-\rho''(0). \qquad (10)$$

As we have assumed $\zeta(t)$ to be normal, S/π is the average number of internal wave zero-crossings per unit time. When substituted into (8) the Taylor expansions (the first derivatives must vanish so that the mean-square gradients are finite[1])

$$r_\theta(z)=\overline{\theta^2}(1-\tfrac{1}{2}K^2z^2+\cdots), \qquad (11)$$

$$\rho(\tau)=1-\tfrac{1}{2}S^2\tau^2+\cdots, \qquad (12)$$

give the temperature covariance

$$R_T{}^{fs}(\tau)=\overline{\theta^2}(1-\tfrac{1}{2}K^2Z^2S^2\tau^2+\cdots). \qquad (13)$$

The fine-structure approximation $KZ\gg 1$ then implies that $R_T{}^{fs}(\tau)$ drops off much more sharply than $\rho(\tau)$, so that the parabolic approximation for $\rho(\tau)$ is adequate. Thus, using (12) [but not (13)], the Fourier transform

[1] The two-layer model of Reid (1971), and Phillips' (1971) first example (zero sheet thickness), have infinite mean-square gradients.

of (8) becomes

$$F_T^{fs}(\sigma)=\frac{4}{\pi^{\frac{1}{2}}}\int_0^\infty d\tau\cos\sigma\tau\int_0^\infty r_\theta(\sqrt{2}S\tau Zx)\exp(-x^2)dx,$$

$$=\frac{2}{\pi^{\frac{1}{2}}}\int_0^\infty dx\exp(-x^2)(\sqrt{2}SZx)^{-1}$$

$$\times\frac{2}{\pi}\int_0^\infty r_\theta(z)\cos\left[\frac{\sigma Z}{\sqrt{2}SZx}\right]dz,$$

$$=\frac{2}{\pi^{\frac{1}{2}}}\int_0^\infty \exp(-x^2)(\sqrt{2}SZx)^{-1}F_\theta\left(\frac{\sigma}{\sqrt{2}SZx}\right)dx, \quad (14)$$

$$=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{1}{SZ}\int_0^\infty \exp\left[-\left(\frac{\sigma}{\sqrt{2}SZk}\right)^2\right]k^{-1}F_\theta(k)dk. \quad (15)$$

The contribution of fine-structure to an internal wave spectrum thus depends on the internal waves only through their rms frequency S and rms displacement Z. The fine-structure comes in as a weighted integral of the spectrum $F_\theta(k)$.

The situation is perhaps a bit simpler in terms of the spectrum of fine-structure *gradient*, $F_{\theta'}(k)=k^2F_\theta(k)$. Changing the dummy variable to $q=x^2$, Eq. (14) can be written as

$$F_T^{fs}(\sigma)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}SZ\sigma^{-2}\int_0^\infty e^{-q}F_{\theta}'\left(\frac{\sigma}{\sqrt{2q}SZ}\right)dq. \quad (16)$$

If we further change the variable to q/σ^2, then $F_T^{fs}(\sigma)$ is related to $F_{\theta'}(k)$ by a Laplace transform.

5. Two fine-structure models

Suppose the gradient spectrum of fine-structure is

$$F_{\theta'}(k)=\overline{\theta'^2}/(k_S-k_L),\quad k_L\leqslant k\leqslant k_S, \quad (17)$$

and zero otherwise; this is roughly the situation when the gradient $\theta'(y)$ consists of a series of narrow spikes of typical thickness $2\pi k_S^{-1}$ (dimension of sheets) separated by $2\pi k_L^{-1}$ (layers). Then from (16)

$$F_T^{fs}(\sigma)=(2/\pi)^{\frac{1}{2}}SZ\sigma^{-2}(k_S-k_L)^{-1}$$

$$\times\overline{\theta'^2}[\exp(-\sigma^2\tau_S^2)-\exp(-\sigma^2\tau_L^2)], \quad (18)$$

where $\tau_{S,L}=(\sqrt{2}SZk_{S,L})^{-1}$ and $2\sqrt{2}\pi\tau_{S,L}$ is the time required for crossing a sheet (layer) at a speed SZ. The function in brackets behaves like

$$\left.\begin{array}{l}\sigma^2\tau_L^2 \text{ for } \sigma\ll\tau_L^{-1};\quad 1 \text{ for } \tau_L^{-1}\ll\sigma\ll\tau_S^{-1}\\ \exp(-\sigma^2\tau_S^2) \text{ for } \tau_S^{-1}\ll\sigma\end{array}\right\}, \quad (19)$$

so that the fine-structure contribution is relatively small at very low and very high frequencies, and proportional to σ^{-2} in the important intermediary band. This is consistent with the findings by Phillips.

The top-hat model (17) can be compared with the observed gradient spectrum by Cox *et al.* (1969). This consists of a decade-wide flat portion with a sharp high wavenumber cutoff at $k_S=O(1)$ rad cm^{-1}. There is no corresponding termination at low wavenumbers; in fact, the spectrum is observed to rise again with diminishing k, reaching perhaps a second plateau. This behavior is tentatively ascribed (Peter Hacker, personal communication) to a multiple thermocline structure and the resulting nonstationarity with depth. Roden (1971) has used salinity-temperature depth (STD) observations to extend the spectrum into lower wavenumbers, and finds it to rise indefinitely. The problem is further discussed in Section 11. Ultimately there must be a reduction in $F_{\theta'}(k)$ for small k at least as rapid as k in order for

$$\overline{\theta^2}=\int k^{-2}F_{\theta'}(k)dk$$

to be finite. For a model

$$F_{\theta'}(k)\propto k^2/(k_L^2+k^2),\quad 0\leqslant k\leqslant k_S, \quad (20)$$

and zero beyond, the result (after suitable normalization) is similar to (18), with the function in brackets replaced by

$$\sigma^2\tau_L^2\exp(\sigma^2\tau_L^2)E_1(\sigma^2\tau_L^2+\sigma^2\tau_S^2), \quad (21)$$

where

$$E_1(z)=\int_z^\infty t^{-1}e^{-t}dt$$

is the "exponential integral" function. The asymptotic behavior for intermediate and high wavenumbers is just the same as (19), but for $\sigma\ll\tau_L^{-1}$ we now have $\sigma^2\tau_L^2[\ln(\sigma^2\tau_L^2)-\gamma]$ in place of $\sigma^2\tau_L^2$. The difference between the two types of cutoff at low σ is not vital. The two models give the fine-structure scale [see Eq. (9)]:

$$K^2=k_Lk_S,\quad \tfrac{1}{4}\pi k_Lk_S-O(k_L^2). \quad (22)$$

6. The fine-structure "contamination"

An important consideration is the fine-structure contribution *relative* to the "gradient spectrum" [see Eq. (4)]:

$$F_T^{gd}(\sigma)=T'^2F_\zeta(\sigma). \quad (23)$$

We introduce the dimensionless numbers

$$C^2=\overline{\theta'^2}/T'^2\gg1,\quad D^2=K^{-2}/Z^2\ll1. \quad (24)$$

The "Cox number" C is the ratio of rms to mean vertical temperature gradient; the fine-structure number D is the ratio of rms fine-structure scale to rms internal wave displacement.

One can consider the additional assumption that the layers are thoroughly mixed, so that the temperature gradients in the layer vanish, i.e., $T'+\theta_L'=0$; the fine-structure was defined so that $\overline{\theta'}\propto k_L^{-1}\theta_L'+k_S^{-1}\theta_S'=0$.

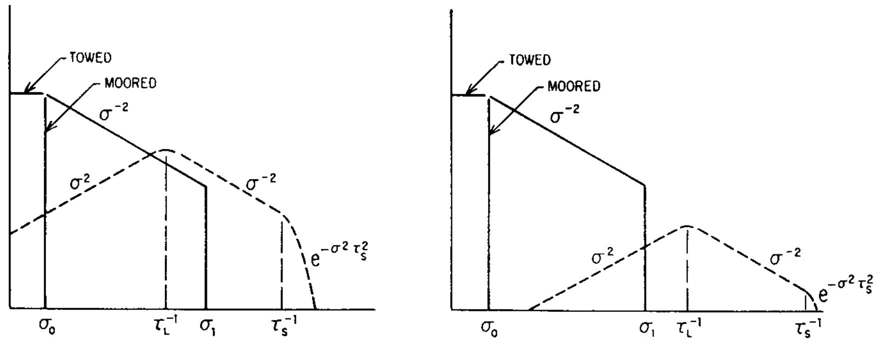

Fig. 1. Solid lines show the assumed frequency spectrum from moored sensors [Eq. (30a)] and wavenumber spectrum from towed sensors [Eq. (30b)], in a log-log presentation. For either case, the dashed curves show the fine-structure spectrum when $\gamma>1$ (left) and $\gamma<1$ (right).

It follows that

$$C^2=k_S/k_L. \tag{25}$$

Using Eq. (5), the relative contribution by the fine-structure to the temperature *variance* produced by the internal waves is

$$\frac{R_{T^{fs}}(0)}{R_{T^{gd}}(0)}=\frac{r_\theta(0)}{T'^2Z^2}=C^2D^2\approx\left(\frac{1}{k_LZ}\right)^2, \tag{26}$$

which is typically small. But the relative spectral *density* due to fine-structure may dominate at the high frequencies. From (14) and (24) we have

$$\frac{F_{T^{fs}}(\sigma)}{F_{T^{gd}}(\sigma)}=\frac{(2/\pi)^{\frac{1}{2}}C^2DS\sigma^{-2}K/(k_S-k_L)}{Z^{-2}F_\zeta(\sigma)} \times\left[\frac{k_S-k_L}{\theta'^2}\int_0^\infty e^{-q}F_{\theta'}\left(\frac{\sigma}{\sqrt{2q}SZ}\right)dq\right]. \tag{27}$$

For the top-hat model in the intermediary range, [ ]=1 and $K/(k_S-k_L)\approx(k_L/k_S)^{\frac{1}{2}}$. Using (22), (24) and (25), this ratio then becomes

$$\gamma(\sigma)=\frac{F_{T^{fs}}(\sigma)}{F_{T^{gd}}(\sigma)}=\frac{(2/\pi)^{\frac{1}{2}}}{k_L}\frac{ZS}{\sigma^2F_\zeta(\sigma)} \tag{28}$$

in the frequency interval such that

$$\tau_L^{-1}\ll\sigma\ll\tau_S^{-1}, \tag{29}$$

where $\tau_{L,S}^{-1}=\sqrt{2}k_{L,S}ZS$. We shall apply this result to observed spectra of internal waves.

7. Two internal wave models

The two models (Fig. 1)

$$F_\zeta(\sigma)=\begin{cases}0 & \text{for}\quad \sigma<\sigma_0;\quad A\sigma_0^{-r},\quad \text{for}\quad \sigma_0\leqslant\sigma\leqslant\sigma_1,\\ A\sigma_0^{-r}\end{cases} \tag{30a,b}$$

and zero beyond σ_1, represent frequency spectra from measurements at moored stations, and wavenumber spectra from rapid tows, respectively (Garrett and Munk, 1971). We obtain A from $\int F_\zeta(\sigma)d\sigma=Z^2$ and S from $\int\sigma^2F_\zeta(\sigma)d\sigma=Z^2S^2$ [Eq. (10)].

This discussion is limited to the special case $r=2$ (consistent with observational evidence); the internal wave spectrum (30) and fine-structure spectrum (18) then both have a σ^{-2} dependence in the interval defined by (29) and $\sigma_0<\sigma<\sigma_1$. The result when $\sigma_0\ll\sigma_1$ is

$$\tau_L^{-1}=\frac{2}{\sqrt{\pi}}\frac{\sigma_1}{\gamma},\quad \gamma=(1,\sqrt{2})\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{(\sigma_1/\sigma_0)^{\frac{1}{2}}}{k_LZ}, \tag{31a,b}$$

so that for $\gamma>1$ the fine-structure spectrum overlaps and exceeds the internal wave spectrum at the high frequencies (Fig. 1).[2]

For the *frequency spectra* we identify σ_1 and σ_0 with Väisälä and Coriolis frequencies n and f, respectively. In the upper oceans the ratio is $O(10^2)$. If we take 10 m for a typical layer thickness and typical vertical displacement, then $k_LZ=2\pi$, $\gamma=O(1)$ and it is difficult to say which way things will go. Since $Z\propto n^{-\frac{1}{2}}(y)$, then $\gamma\propto n(y)k_L^{-1}$, the product of the Väisälä frequency and the layer thickness.

For the *wavenumber spectra* we identify σ_0 and σ_1 with wavenumbers $\alpha_f=O(10^{-2}$ cycle km$^{-1})$ and $\alpha_n=O(10$ cycles km$^{-1})$, respectively (Garrett and Munk, 1971); α_n diminishes with depth proportional to $n(y)$. The fine-structure contamination appears to be somewhat larger, and with more overlap than in the case of the frequency spectrum. The depth dependence is $\gamma\propto n(y)k_L^{-1}$, as before.

8. The frozen fine-structure assumption

We can now verify the assumption that the fine-structure is indeed a sufficiently slowly varying function of time. Assuming the fine-structure to be time-independent, we note from (13) that $R_{T^{fs}}(\tau)$ falls off

[2] Gould (personal communication) has recently shown us a current spectrum from the Bay of Biscay which appears to be in accord with our model for $\gamma=O(1)$.

with a time scale of order

$$\frac{1}{KZS}=\left(\frac{k_L}{k_S}\right)^{\frac{1}{3}}\tau_L\approx\left(\frac{k_L}{k_S}\right)^{\frac{1}{3}}\frac{\gamma}{\sigma_1}. \quad (32)$$

For the frequency spectrum, this is about 10^{-2} times the Väisälä period, or typically 30 sec. The corresponding distance scale is 3 m. These scales are small compared to the persistence of fine-structure.

9. Currents

From internal wave theory, the relation between shear and vertical velocity (omitting a factor $e^{i\sigma t}$) is given by

$$U'(y)=-i\alpha\left[\frac{n^2(y)-\sigma^2}{\sigma^2-f^2}\right]V(y). \quad (33)$$

Horizontal motion vanishes at the Väisälä frequency n, vertical motion at the Coriolis frequency f. For a sum of contributions with the same frequency σ, but possibly different wavenumbers α, $U'=A(n^2-\sigma^2)$, where

$$A=-i\sum_\alpha \alpha(\sigma^2-f^2)^{-1}V_\alpha.$$

Suppose $U'=U_{gd}'+U_{fs}'$ and $n^2=n_{gd}^2+n_{fs}^2$, where n_{fs}^2 will alternate between high values in the sheets and approximately $-n_{gd}^2$ in the layers. Then

$$\frac{U_{fs}'}{U_{gd}'}=\frac{n_{fs}^2-\sigma^2}{n_{gd}^2-\sigma^2}\approx\frac{n_{fs}^2}{n_{gd}^2}\approx\frac{\theta'(y)}{T'}. \quad (34)$$

The first approximation applies when $\sigma\ll n_{gd}$, and the second if the ratio of fine-structure to mean gradient is the same for salinity as for temperature. This is by no means established (Pingree, 1969; Cox and Gregg, personal communication).

In the latter case the following geometric argument shows that the *relative* spectral contribution by the current fine-structure, $\gamma_U(\sigma)=F_U{}^{fs}(\sigma)/F_U{}^{gd}(\sigma)$, equals the relative contribution by the temperature fine-structure, $\gamma_T(\sigma)$ [see Eq. (28)]. Consider filtered records of $U(t)$ and $T(t)$ in some narrow frequency band centered at σ. The two sensors have traversed the same layers and sheets and the associated step-like features in the record are proportional in view of (34). If the two records are plotted so they have the same scale, then they will agree in detail and the relative statistics are also the same. We should emphasize that $\gamma_U(\sigma)=\gamma_T(\sigma)$ involves Z and not the rms current U.

10. Coherence

Phillips (1971) points out that the presence of fine-structure will reduce the coherence between temperature records at two points separated in the vertical even if the vertical motions causing the temperature fluctuations are perfectly correlated. He further suggests that the coherence scale will be of the order of the thickness of a layer of the fine-structure. We reach a more specific conclusion by extending our analysis for the fine-structure spectrum.

The covariance $\langle T(y_0,t)T(y_0+Y,t+\tau)\rangle$ between temperature measurements at depths y_0, y_0+Y is again the sum of a gradient covariance $R_T{}^{gd}(\tau,Y)$ and a fine-structure covariance

$$R_T{}^{fs}(\tau,Y)=\int\int r_\theta(\zeta_1-\zeta_2-Y)p(\zeta_1,\zeta_2;\,\tau,Y)d\zeta_1\,d\zeta_2, \quad (35)$$

where $r_\theta(z)$ is as before and $p(\zeta_1,\zeta_2;\tau,Y)$ is the joint probability distribution of ζ_1 at y_0 at time t, ζ_2 at y_0+Y at time $t+\tau$. Assuming as before that ζ is the sum of many independent waves, p is jointly normal and

$$R_T{}^{fs}(\tau,Y)=[4\pi(1-\rho)Z^2]^{-\frac{1}{2}}\int_{-\infty}^{\infty} r_\theta(z-Y)\times\exp\left[-\frac{z^2}{4(1-\rho)Z^2}\right]dz, \quad (36)$$

where

$$Z^2=[\overline{\zeta^2}(y_0)\overline{\zeta^2}(y_0+Y)]^{\frac{1}{2}},$$

and $\rho(\tau,Y)=\langle\zeta(y_0,t)\zeta(y_0+Y,t+\tau)\rangle/Z^2$ is the correlation of ζ at the two points. Eq. (36) reduces to (7) when $Y=0$. The generalization of (8) is

$$R_T{}^{fs}(\tau,Y)=\pi^{-\frac{1}{2}}\int_{-\infty}^{\infty} r_\theta[2(1-\rho)^{\frac{1}{2}}Zx-Y]\times\exp(-x^2)dx. \quad (37)$$

In general $\rho(0,Y)\neq 1$ so that further progress along the lines of Section 4 is not possible. However, in the special, and interesting, case of perfect coherence between the vertical displacement at the two depths, we again use the fine-structure approximation $\rho(\tau,Y)=1-\frac{1}{2}S^2\tau^2$ [valid now if $-Z^2r_\theta''(Y)/r_\theta(Y)\gg 1$] in (37). Since $R_T{}^{fs}(-\tau,Y)=R_T{}^{fs}(\tau,Y)$, the corresponding quadrature spectrum is zero, and the co-spectrum is

$$C_T{}^{fs}(\sigma,Y)=\frac{2}{\pi}\int_0^\infty R_T{}^{fs}(\tau,Y)\cos\sigma\tau d\tau, \quad (38)$$

$$=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{1}{SZ}\int_0^\infty \exp\left[-\left(\frac{\sigma}{\sqrt{2}SZk}\right)^2\right]k^{-1}\times F_\theta(k)\cos kYdk, \quad (39)$$

$$=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}SZ\sigma^{-2}\int_0^\infty e^{-q}F_{\theta'}\left(\frac{\sigma}{\sqrt{2q}SZ}\right)\cos\left(\frac{\sigma Y}{\sqrt{2q}SZ}\right)dq, \quad (40)$$

corresponding to (15) and (16) for the spectrum. Thus, $C_T{}^{fs}$ is related to $F_{\theta'}$ as if each element of the latter were a standing wave. This is not altogether surprising, though the consequences are completely unexpected.

We have assumed that the displacements at the two depths are perfectly coherent. With the further assumption of zero phase lag, $C_T^{\varrho d}(\sigma,Y)=F_T^{\varrho d}(\sigma)$ and the quadrature spectrum is zero. Thus, the coherence is

$$\mathcal{R}_T(\sigma,Y)=\frac{1+\gamma(\sigma)\mathcal{R}_T^{fs}(\sigma,Y)}{1+\gamma(\sigma)}. \qquad (41)$$

For our first model (Section 5) of $F_{\theta'}(k)$ we have

$$\mathcal{R}_T^{fs}(\sigma,Y)=\int_{q_L}^{q_S}\exp(-q^{-2})q^{-3}\cos\lambda q dq \Big/ \int_{q_L}^{q_S}\exp(-q^{-2})q^{-3}dq, \qquad (42)$$

where

$$\lambda=\frac{\sigma Y}{\sqrt{2}SZ}, \quad q_{L,S}=(\sigma\tau_{L,S})^{-1}. \qquad (43)$$

For $\tau_L^{-1}\ll\sigma\ll\tau_S^{-1}$ the bounds may be replaced by 0 and ∞, so that

$$\mathcal{R}_T^{fs}(\sigma,Y)=2\int_0^\infty\exp(-q^{-2})q^{-3}\cos\lambda q dq=\Lambda(\lambda), \qquad (44)$$

plotted in Fig. 2. The parameter λ involves only the characteristic vertical velocity SZ of the internal waves, and so the fine-structure coherence $\mathcal{R}_T^{fs}(\sigma,Y)$ is *independent of the fine-structure* provided only $\tau_L^{-1}\ll\sigma\ll\sigma_S^{-1}$. Further, if γ is independent of frequency [as in (31)], then $\mathcal{R}_T(\sigma,Y)$ depends on frequency and separation only through the product σY. In particular, $R_T(\sigma,Y)=\frac{1}{2}$ for $\mathcal{R}_T^{fs}(\sigma,Y)=\frac{1}{2}(\gamma-1)/\gamma$; thus, for $\gamma\gg1$, $\gamma=1$, we have $\Lambda(0.67)=\frac{1}{2}$, $\Lambda(1.3)=0$, giving

$$(\sigma Y)_{\frac{1}{2}}=0.95SZ,\ 1.8SZ, \qquad (45)$$

a law which applies to measurements of current as well as temperature (see Section 9). For $S\approx(nf)^{\frac{1}{2}}\approx0.4$ cycle hr^{-1} and $Z\approx10$ m, (45) gives $(\sigma Y)_{\frac{1}{2}}\approx4$, 7 m cycle hr^{-1} for $\gamma\gg1$, $\gamma=1$, respectively. This result is suspiciously close to the empirical law $(\sigma Y)_{\frac{1}{2}}=13$ m cycle hr^{-1} found by Webster (1971) for vertical coherence of current fluctuations at Woods Hole "Site D", though Webster's result extends to frequencies much lower than those we expect to be affected by fine-structure. The actual coherence distances implied by our result range from $O(k_L^{-1})$ to $O(k_S^{-1})$ in the applicable frequency range $\tau_L^{-1}<\sigma<\tau_S^{-1}$.

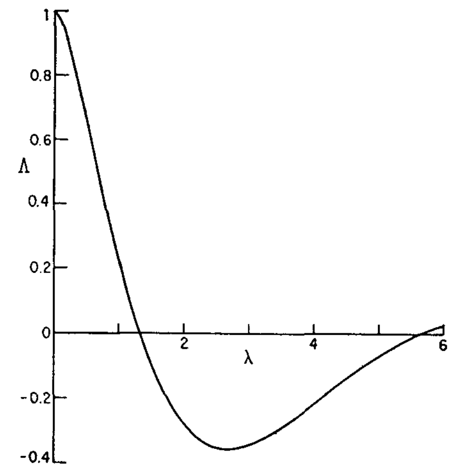

FIG. 2. The "fine-structure coherence" $\mathcal{R}_T^{fs}(\sigma,Y)=\Lambda(\lambda)$, $\lambda=\sigma Y/(\sqrt{2}SZ)$.

11. An operational interpretation of layer thickness

The principal shortcoming of our discussion has to do with the low wavenumber cutoff k_L associated with layer thickness. The relative contribution of the fine-structure varies inversely with k_L, as does the fine-structure intrusion into the band of frequencies normally reserved for internal wave activity [Eqs. (31a,b)]. We have rather glibly set $k_L=(2\pi/10)$ m, in spite of the fact that the measured spectra of microstructure show no evidence of a low wavenumber termination for $F_{\theta'}(k)$, indicating rather a hierarchy of layering, from microstructure to ministructure all the way to the main scale of oceanic stratification.[3]

The key to the problem is the separation of $T(y)$ into a deterministic mean component plus a stochastic perturbation. The separation depends on the scale of the experiment. Consider a tow from $x=0$ to $x=L$, with temperature profiles taken at such close intervals that $T(x,y)$ is adequately observed. The mean profile is operationally defined by

$$\bar{T}(y)=L^{-1}\int_0^L T(x,y)dx,$$

and clearly depends on expedition parameters. The mean field in Eq. (2) is interpreted as the first term in a Taylor expansion of $\bar{T}(y-y_0)$, and the perturbation field as $\theta(x,y)=T(x,y)-\bar{T}(y)$. Similarly

$$r_\theta(z)=L^{-1}\int_0^L r_\theta(z,x)dx.$$

The perturbation field as here defined is not the same thing as a locally high-passed $T(y)$, whether achieved by differentiation (Cox *et al.*, 1969) or by binomial filtering (Roden, 1971).

Any horizontal scales in the temperature structure much larger than L are part of the mean field, those much shorter than L part of the perturbation field. The work of Stommel and Fedorov (1967) suggests that layers of 2–40 m thickness have horizontal scales of

[3] For Garrett and Munk's model of the main stratification, the density beneath the thermocline increases with depth y as $\rho_0+\Delta\rho(1-e^{2y/b})$, with $b=O(1\text{ km})$. Harmonic analysis over any depth range h gives a power proportional to $[1+(j\pi b/h)^2]^{-1}$ for harmonic j, or roughly $F_\theta(k)\propto k^{-2}$. Roden's (1971) $k^{-8/3}$ spectra at the low wavenumber limit can presumably be attributed to the main thermal structure. In all events the spectra monotonically increase with decreasing k. The non-integrability at $k=0$ is avoided by the discretization of the spectrum imposed by finite ocean depth.

2–20 km; Cox *et al.* estimate 0.05–0.5 km for 0.1–1 m layers. Suppose the "aspect ratio" is $10^3:1$; then the perturbation field is limited to layers $<10^{-3}L$, and $k_L \approx 2\pi/(10^{-3}L)$. A typical tow is over 10 km, giving $k_L = (2\pi/10)$ m. Vector displacement diagrams for a typical moored record indicate 50-km displacements over a month (Webster, 1970), thus suggesting $k_L = (2\pi/50)$ m. But perhaps the moored spectra are more subject to *time* changes in the thermal structure (rather than spatial changes that are swept by the mooring). In that event we expect to relate the record length to the lifetime of a layer of thickness $2\pi k_L^{-1}$.

It is a consequence of this point of view that longer records (space or time) assign a larger part of the temperature structure to the perturbation field, with the result that the relative contribution of the fine-structure is likewise enhanced. If this dependence of the fine-structure theory on expedition parameters is deemed undesirable, so is the dependence (in our view) of the spectra we are trying to interpret. There is perhaps some hope that the horizontal extent of layering has an upper limit in the oceans; in that event we would find that $r_\theta(z)$ as here operationally defined would approach some asymptotic form for very large L.

Russ Davis (personal communication) has suggested that k_L may also have a lower bound determined by the range of validity of the joint normal distribution (6). If, for example, there is zero probability of $\zeta > \zeta_{\max}$, one effect will be to limit the range of integration of (7) to $|z| < 2\zeta_{\max}$. Alternatively, $r_\theta(z)$ might be taken as zero for $|z| > 2\zeta_{\max}$, producing a low wavenumber cutoff for $F_{\theta'}(k)$ at $k = O((2\zeta_{\max})^{-1})$, i.e., $2\pi k^{-1} = O(4\pi\zeta_{\max})$. Taking $\zeta_{\max} = 3Z$ gives $2\pi k^{-1} = O(12\pi Z) = O(300 \text{ m})$, which is probably much larger than the value of $2\pi k_L^{-1}$ obtained from other considerations (as above).

12. Summary

We expect spectra of internal waves to exhibit a sharp cutoff above σ_1. For "moored spectra" $\sigma_1 = n(y)$ (the Väisälä frequency); for "towed spectra" $\sigma_1 = \alpha_n(y)$, a wavenumber proportional to $n(y)$ and of order 10 cycles km^{-1} at a few thermocline depths. Spectra of currents and temperatures at fixed depths (moored or towed) are observed to have a σ^{-2} dependence with no discernible break at n or α_n {Fofonoff, 1969; Webster, 1970 [see especially Bermuda spectrum 1612(6)]; Ewart, personal communication}. Fine-structure theory predicts a σ^{-2} dependence above a frequency σ_1/γ, at a level γ times the underlying internal wave spectrum. The observed statistical properties of the fine-structure are not inconsistent with $\gamma > 1$, leading to a dominance of fine-structure at the higher frequencies and wavenumbers. It should be noted that spectra of vertical velocity (presumably not subject to fine-structure contamination) measured by Voorhis (1968) do, in fact, cut off sharply at $n(y)$.

Fine-structure theory predicts a vertical coherence distance $Y \propto \sigma^{-1}$ which is in rough accord with an empirical rule proposed by Webster. The trouble here is that Webster's rule appears to hold over a broader range of frequencies than can be accounted for by the fine-structure theory.

In our view the moored and towed spectra and the moored vertical coherence indicate a dominant role for the fine-structure at the upper frequencies and wavenumbers. Conflicting evidence consists of the successful n-normalization at different depths, and of the successful comparison of different observables on the assumption of internal wave theory (Fofonoff, 1969). But this goes beyond the scope of the present paper.

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