A Review of Relative Diffusion Analysis and Results

JOHN F. MIDDLETON

G. F. D. Laboratory, Department of Mathematics, Monash University, Clayton, Victoria 3168, Australia

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ABSTRACT

Some confusion in the nature and difference between absolute and relative diffusion is found in the work of Kao and Al-Gain (1968) and Kirwan et al. (1978).

It is shown that the relative diffusion analysis used by these authors is based on an assumption of independent particle motions. Further, that as this independence implies that relative diffusion statistics asymptote to those of absolute diffusion, Kao and Al-Gain (1968) and Kirwan et al. (1978) have only examined a trivial case of the relative diffusion problem.

Although Kirwan et al. (1978) neglect to validate their implicit assumption of independent particle motions, the comparison of their estimates of relative diffusion statistics with absolute diffusion theory is justified. The large-particle separation in their data indicates a near-asymptotic state.

Confidence intervals are constructed for their results, which are then shown to be only marginally significant.

1. Introduction

Mathematical descriptions of absolute and relative diffusion have been developed by Batchelor (1949, 1952). However, analysis of diffusion data by Kao and Al-Gain (1968), Kao and Powell (1969) and more recently by Kirwan et al. (1978) indicate that the differences between the two types of diffusion are not always understood.

Absolute diffusion is described by the statistics of the displacement of a particle with time, with respect to a fixed origin. If the Eulerian velocity field is stationary and homogeneous, then the displacement process is stationary. Relative diffusion is described by the statistics of the separation of two particles. Even if the Eulerian velocity field is stationary and homogeneous, the separation process is in general nonstationary.

In the abovementioned data analysis, relative diffusion statistics are estimated and compared with relative diffusion theory derived assuming a stationary separation process. This assumption is asymptotically valid for large separation, when the particle motions are independent and relative diffusion, in effect, is absolute diffusion. The various authors fail to notice that they are actually only considering this limiting "trivial" case (Batchelor, 1952) of relative diffusion, and fail to test the stationary assumption adequately (Kao et al., 1968) or even at all (Kirwan et al., 1978).

Confidence intervals calculated here for the results of Kirwan et al. (1978) show the latter to be only marginally significant.

2. Absolute diffusion as a limiting case of relative diffusion

We consider particle motions induced by a horizontal, stationary, homogeneous Eulerian velocity field. The particles are assumed to be indistinguishable from the fluid.

Denoting the absolute velocities of two particles at time $t$ by $u_i(t)$ and $u'_i(t)$, the separation velocity is given by

$$v_i(t) = u_i(t) - u'_i(t), \quad i = 1, 2. \quad (1)$$

For simplicity it is assumed that the ensemble averages $\langle v_i \rangle$, $\langle u_i \rangle$ and $\langle u'_i \rangle$ all vanish. The separation vector is then given by

$$x_i(t) = x_i(t_0) + \int_{t_0}^{t} v_i(t')dt'. \quad (2)$$

The mean-square separation tensor is

$$\langle x_i(t)x_j(t') \rangle = \langle x_i(t_0)x_j(t_0) \rangle + \int_{t_0}^{t} \int_{t_0}^{t'} \langle v_i(t')v_j(t'') \rangle dt' dt'. \quad (3)$$

Batchelor (1952) points out that in a stationary, homogeneous Eulerian velocity field, the Lagrangian absolute velocity $u_i$ is stationary but the Lagrangian separation velocity $v_i$ is not: as two particles separate, larger and more energetic eddies become responsible for their further separation. Batchelor (1952) also points out that $\langle v_i(t')v_j(t'') \rangle$ may be conditionally dependent on the initial separation $x_i(t_0)$, which determines the size of the eddies responsible for the initial increase in separation. The conditional dependence may be denoted by

$$S_{ij}(t',t'',x|x(t_0)) = \langle v_i(t')v_j(t'') \rangle, \quad (4)$$

$$\sigma_{ij}(t|x(t_0)) = \langle x_i(t)x_j(t) \rangle. \quad (5)$$

In a stationary, homogeneous Eulerian velocity field there is a finite, constant integral space scale $J$, given by
\[ J = \int_0^\infty \langle U(y) \cdot U(y + r) \rangle \langle U^2 \rangle \, dr, \]  

(6)

where \( U(y) \) denotes the Eulerian velocity at some point \( y \). If the separation distance between two particles greatly exceeds this scale, then their motions are only weakly correlated. In the limit

\[ \langle v_i(t')v_i(t) \rangle \sim 2\langle u_i(t')u_i(t) \rangle \]  

(8a)

as

\[ (x-x_0)/J \rightarrow \infty. \]  

(8b)

The stationarity of \( u_i \) implies that the correlation \( \langle u_i u_j \rangle \) depends only on the lag \( (t' - t) \). Moreover, the particle motions eventually become independent of their initial separation. Thus we have

\[ S_{ij}(t', t', x | x(t_0)) \sim S_{ij}(t' - t'). \]  

(9)

Similarly, we have

\[ \langle x_i(t)x_i(t) \rangle \sim \langle x_i(t_0)x_i(t_0) \rangle + 2\langle X_i(t, t_0)X_i(t, t_0) \rangle, \]  

(10)

and so

\[ \sigma_{ij}(t | x(t_0)) \sim \sigma_{ij}(t - t_0), \]  

(11)

where \( X_i \) is the absolute particle displacement during the interval \( (t_0, t), \) i.e.,

\[ X_i(t, t_0) = \int_{t_0}^t u_i(t') \, dt', \]  

(12)

and \( \langle X_i(t, t_0)X_i(t, t_0) \rangle \) is the absolute displacement tensor.

3. The “relative displacement tensor” of Kao et al. (1968)

Kao et al. (1968) define a “relative displacement tensor” by

\[ \langle x_i(x(t_0)) \rangle = \langle (v_i^2) \rangle^{1/2} \int_0^\infty \int_0^\infty [R_{ij}(\tau) + R_{ji}(\tau)] \, d\tau \, d\eta + \langle x_i(0)x_i(0) \rangle, \]  

(13)

where

\[ R_{ij}(\tau) = \langle v_i(t) (v_j(t + \tau))/\langle (v_i^2) \rangle \rangle^{1/2}. \]  

(14)

They state that the separation velocity \( v_i \) must be stationary; it may be seen that (13) agrees with (10) only in that case. Kao et al. (1969) test for such stationarity by plotting \( \langle v_i^2 \rangle^{1/2} \) (\( i = 1, 2 \), against \( \langle x_i^2 \rangle^{1/2} \) (see their Fig. 5). They note that the \( \langle v_i^2 \rangle^{1/2} \) are essentially constant, indicating that \( v_i \) is stationary at the separation distances examined \( (\sim 10^8 \) km). However in neither of the papers of Kao et al. is it explicitly stated that stationarity of \( v_i \) does not obtain until the particle motions have become independent, in which case the “relative displacement tensor” is twice the absolute displacement tensor [see Eq. (10) above].\(^1\) On the other hand, Kao et al. (1968) report, without comment, a cubic growth rate in time for one component of their “relative displacement tensor,” which is in agreement with a prediction of Batchelor (1952) for dependent particle motions. It may be the case that “relative displacements” remain significantly correlated long after separation velocities are sensibly independent.

Batchelor (1949) shows that the absolute displacement tensor has the limiting behavior

\[ \langle X_i(t, t_0)X_i(t, t_0) \rangle \sim (t - t_0)^2 \langle u_i(t_0)u_i(t_0) \rangle \]  

as \( t \rightarrow t_0 \),

(15)

\[ \langle X_i(t, t_0)X_i(t, t_0) \rangle \sim (t - t_0)((\langle u_i^2 \rangle \langle u_i^2 \rangle)^{1/2} L_{ij}^{(0)} + (\langle u_i^2 \rangle \langle u_i^2 \rangle)^{1/2} L_{ij}^{(1)} \) as \( t \rightarrow t_0 \rightarrow \infty, \]  

(16)

where

\[ L_{ij}^{(0)} = \int_0^\infty [T_{ij}(\tau) + T_{ji}(\tau)] \, d\tau, \]  

(17a)

\[ L_{ij}^{(1)} = \int_0^\infty \tau [T_{ij}(\tau) + T_{ji}(\tau)] \, d\tau, \]  

(17b)

and \( T_{ij}(\tau) \) is defined analogously with (14), but for the absolute velocities \( u_i(\tau) \). Kao et al. (1968), using their assumption of stationarity for \( v_i(\tau) \), derive forms for their “relative displacement tensor” analogous with (15) and (16). However in the stationary case their derivation is really just that given by Batchelor (1949) for the absolute displacement tensor.

4. The work of Kirwan et al. (1978)

Kirwan et al. (1978) claim to base their analysis of ocean drifting buoy data upon that of Kao et al. (1968), who examine simulated atmospheric free balloon data. However, the “relative displacement tensor” defined by the former [their Eq. (1)] does not agree with that of the latter [their Eq. (5)], which is the definition that will be used here.

A more serious error is that Kirwan et al. (1978) fail to state or test the assumption of stationarity. Thus, their Eq. (1) may not be applicable and their results non-interpretable; initial dependences and non-stationarity may be important.

However, examination of their Fig. 1, a plot of buoy trajectories, reveals that most buoys were deployed at least 200 km apart. If the integral space scale (6) is taken to be \( \sim 100 \) km (a typical mesoscale eddy diameter), then the buoy separations may well be large enough to ensure a near-independence of buoy motions.

Kirwan et al. (1978) do not calculate confidence intervals for their estimates of the relative displacement tensor. The intervals may be readily estimated. Batchelor (1949) and Csanady (1973) note that the probability density function (pdf) for components of absolute displacement may be approximated by independent normal distributions. It may then be shown

\(^1\) The first term on the right-hand side of (10) may be neglected.
that if the particle motions are independent, then the pdf’s for the components of the separation tensor will also be normal, with zero mean and variance twice the absolute displacement tensor.

Kirwan et al. (1978) estimate the trace $\langle D^2 \rangle$ of their relative displacement tensor; which shall be here denoted by $S^2$. For $N$ buoys, there are $N - 1$ independent samples of $\langle D^2 \rangle$ in the $\chi^2$ possible samples used in the estimate of Kirwan et al. (1978). Each sample is the sum of the squares of two independent normal Cartesian components, so $S^2$ is proportional to $\chi^2_{2N-2}$. It follows that the 95% confidence intervals are

(i) for the subtropical gyre

$$N = 6, \text{ Fig. 7a}$$
$$0.5S^2 \leq \langle D^2 \rangle \leq 3.0S^2$$

(ii) for the subarctic gyre

$$N = 5, \text{ Fig. 7b}$$
$$0.45S^2 \leq \langle D^2 \rangle \leq 3.65S^2$$

(iii) for both gyres

$$N = 9, \text{ Fig. 7c}$$
$$0.55S^2 \leq \langle D^2 \rangle \leq 2.5S^2.$$

In the subtropical gyre (Fig. 7a of Kirwan et al.) there is a drop in the mean-square displacement at $t \approx 150$ days which the authors attribute to large-scale convergence. The drop is statistically insignificant, and a continuation of the linear trend through $t \approx 150$ days falls within the confidence interval.

In the subarctic gyre (Fig. 7b) the break at $t = 30$ days in the quadratic trend may not be significant. The predicted quadratic and linear trends in their Figs. 7a–7c lie within the confidence intervals, but with considerable margin for error. Fig. 7c shows the most reliable result.

5. Summary

In their purported analyses of relative diffusion, Kao et al. (1968) fail to make clear that their basic assumption of a stationary separation process is valid only if the particle motions are independent. In that limit relative diffusion, in effect, is absolute diffusion and the analysis of Kao et al. (1968) is inappropriately titled. Kirwan et al. (1978) add to the confusion. They do not state that a stationary separation process is needed if their analysis is to hold, nor do they test for stationarity. There is evidence, in the initial buoy separation, that such an assumption is reasonable. Confidence intervals for their estimates of the “relative dispersion tensor” have been calculated here, using this assumption.

An initial quadratic and final linear growth in time expected of the “relative displacement tensor” is supported within these intervals, although there is quite a margin for error.

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REFERENCES


