Topographic Rectification of Tidal Currents on the Sides of Georges Bank

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ABSTRACT

The rectification of $M_2$ tidal currents on the sloping sides of Georges Bank is predicted to make an important year-round contribution to its observed mean clockwise circulation. A rectification mechanism involving continuity and Coriolis effects, but regulated by bottom friction (Huthnance, 1973), is operative. Huthnance's (1973) depth-averaged theory for the along-isobath mean Eulerian current associated with this mechanism and with a second, purely frictional, mechanism is extended to include mean current-tidal current interaction, spatially-varying bottom friction and rotary tidal currents. The ratio of cross-isobath tidal excursion $L_x$ to topographic length scale $L$ is found to be an important nondimensional parameter in determining the degree of nonlinearity of the Coriolis mechanism. A significant Stokes velocity is associated with both rectification processes, so that, for the Coriolis mechanism, the mean Lagrangian current is only about two-thirds of the mean Eulerian current.

On the sides of Georges Bank, $L_x$ and $L$ are of the same order, and the rectification is sufficiently nonlinear that interaction of the mean current with the tidal current is important. The mean Eulerian and Lagrangian currents, and the cross-isobath mean sea surface slopes, are predicted for half-sinusoidal representations of bottom topography on the northwestern, northern and open ocean sides of the Bank. The mean flow is clockwise and concentrated over the edge of the Bank, but smeared out onto the top of the Bank by the mean current-tidal current interaction. The predicted current speeds, which are greatest on the northwestern and northern sides, are of the same order as those observed.

1. Introduction

Present-day knowledge of the mean (time-averaged over tidal periods) circulation around Georges Bank goes back to Bigelow (1927). Based mainly on drifter and hydrographic observations, Bigelow suggested that the summertime nontidal circulation in the Gulf of Maine region consisted largely of two gyres—a counterclockwise eddy over the basin of the Gulf and a clockwise movement of water around Georges Bank (see Fig. 1). The results of subsequent drogue and drifter studies [for a review see Bumpus (1976)] were, in general, consistent with a clockwise circulation around Georges Bank during spring, summer and fall; however, there was some suggestion of a southerly flow across the Bank in winter.

This description of the mean circulation was appropriately based on Lagrangian measurements, and hence is indicative of the actual movement of water parcels. However, it did not resolve temporal variations in the circulation at seasonal or other subtidal frequencies, nor the details of its spatial structure. In the past few years, more detailed current measurements have been made in the Georges Bank region (Butman et al., 1977; Scarlet et al., 1979; Schlitz and Trites, 1979) using fixed current meters (thus making Eulerian measurements), as well as drogues and drifters. The more recent Lagrangian measurements (Scarlet et al., 1979; or for more detail EG&G, 1979) have been made on all sides of the Bank, and show a clockwise movement of water around the Bank throughout much of the year, with some evidence of such a circulation even in winter. There have, however, been observations of drogues and drifters moving away from the Bank especially to the southwest from its open ocean side, so that the existence of a closed gyre is uncertain.

The fixed current meter measurements have been made mainly on the northwestern, northern and southeastern sides of the Bank, and reveal mean Eulerian currents consistent with a clockwise circulation throughout most of the year. On the northwestern and northern sides, mean Eulerian currents flowing toward the northeast and east approximately parallel to isobaths with speeds in the range of $0.2-0.3$ m s$^{-1}$, and greatest near the edge of the Bank, have been observed in spring, summer and fall (Butman et al., 1977; Scarlet et al., 1979; Schlitz and Trites, 1979), giving rise to the suggestion of a current jet along the Bank's north slope. An important observation, in the context of this paper, has been that variations in the strength of this mean current at a particular depth are closely correlated with variations in the amplitude of the tidal current at that depth (Spiegel and Magnell, 1979). In addition, a relatively steady along-isobath mean current was
observed to persist into the winter of 1977–78 at a site on the 200 m isobath just off the northwestern edge of the Bank, until it was replaced in late February 1978 by a weaker cross-isobath flow (EG&G, 1979). However, current meter measurements beginning on 23 April 1978 at a site further up on the Bank on the 85 m isobath showed an along-isobath mean current to be present (EG&G, 1979), so that the persistence of the current jet throughout the winter over some portion of the north slope cannot be ruled out. On the southeastern side of the Bank, mean Eulerian currents toward the southwest have been observed to persist year-round, with speeds in the range of 0.1–0.2 m s⁻¹ in summer but generally less in winter (Butman, 1979; EG&G, 1979).

In summary, the results of recent observational studies are consistent with Bigelow's notion of a clockwise gyre around Georges Bank, subject perhaps to the qualifications that in places the clockwise flow persists throughout much of the year (at times even in winter), that there is more flow to the southwest from the open ocean side of the Bank than Bigelow perceived, and also that the circulation may be altered significantly when Gulf Stream eddies enter the region. However, there remains uncertainty about the temporal and spatial details of this circulation, as well as the associated cross-isobath flows. The relative importance of this circulation versus tidal- and wind-induced turbulent mixing and the episodic influence of Gulf Stream eddies to the exchange processes of water on the Bank has key biological and environmental implications. A satisfactory resolution of these uncertainties, as well as any predictive capability, clearly requires some understanding of the dynamics of the observed circulation.

Perhaps the first suggestion on the driving mechanism of the Georges Bank gyre was by Huntsman (for a discussion see Bigelow, 1927), who claimed that the deflection of tidal oscillations by the effect of the Earth's rotation would result in a clockwise circulation around banks in the Northern Hemisphere. In view of the conclusions to be presented in this paper, Huntsman may appear to have had significant insight with regard to the forcing of the gyre; however, the dynamics of his generation mechanism are obscure and their validity for Georges Bank very doubtful. Csanady (1974) examined the barotropic response of the Gulf of Maine region to wind stress and external pressure gradients, and found that the stress associated with northeasterly winds could produce a double-gyre circulation through the generation of vorticity over sloping bottom topography. He argued that northeasterly winds are the most effective in transferring momentum to the sea surface in spring (since air blowing off the continent is much warmer than the sea surface and produces only low stresses), and hence might cause the spin-up of the gyres. However, the recent observations of the persistence of the Georges Bank gyre throughout much of the year suggest that there are significant contributions from other forcings.

It has also been long recognized that the circulation in the Georges Bank region may be, at least partially, a geostrophic response to horizontal density gradients set up by variations in tidal mixing. The vertical mixing associated with strong tidal currents on the Bank is known to maintain vertically well-mixed conditions throughout the year (Garrett et al., 1978), whereas significant vertical stratification develops off the Bank during the heating season and exists year-round at the shelf/slope front to the south. Other work (in preparation) by the present author indicates that the flow associated with the resulting mean density structure around the Bank is an important contribution to its clockwise circulation during times of the year when the depth-averaged density of water on the Bank is sufficiently less than that of the same depth of water off the Bank. Such conditions have been observed in late summer and fall, appearing to be largely a result of seasonal heating in water columns of different depths with different vertical mixing rates.

This paper points out that a significant year-round contribution to the mean circulation around Georges Bank is expected from the rectification of tidal currents on its sloping sides. The resonance of the $M_2$ tide in the Gulf of Maine-Bay of Fundy system results in the ebb and flood of a large volume of water...
across Georges Bank with tidal current amplitudes of over 1 m s\(^{-1}\) on its central part. On its sloping sides, the depth-averaged tidal currents weaken with increasing depth giving rise to large cross-isobath gradients in the velocity field. The nonlinear advective terms in the Eulerian equations of motion are important in such regions, and, when time-averaged over tidal periods, make significant contributions to the mean momentum balances. These terms appear in the mean Eulerian equations as divergences in the tidal stress, analogous to the Reynolds stress in a turbulent flow, and can be considered to drive mean Eulerian currents or create mean pressure fields. A scale analysis of the mean Eulerian equations appropriate to the northwestern and northern sides of Georges Bank reveals these terms to be of the order of, or greater than, other forcing terms, with associated accelerations of the order of 10\(^{-5}\) m s\(^{-2}\).

The generation of mean Eulerian currents by tidal currents flowing over variable bottom topography has been discussed by Huthnance (1973, 1980) and Zimmerman (1978). Huthnance (1973), investigating tidal current asymmetries over the Norfolk Sandbanks, suggested two mechanisms of mean current generation, and Zimmerman (1978) examined the same rectification processes from a vorticity point of view. One of the mechanisms, involving a combination of continuity and Coriolis effects but regulated by bottom friction (and which will be referred to as Coriolis rectification), should be operative on the sides of Georges Bank. Huthnance (1973) developed a depth-averaged theory for the mean Eulerian velocity resulting from both rectification mechanisms, but did not discuss the associated mean Lagrangian velocity. His analytical solution for the mean Eulerian velocity is appropriate to weak nonlinearity, but is inadequate for the sides of Georges Bank where mean current-tidal current interaction is important. In this paper, the Huthnance (1973) theory for both rectification mechanisms is extended to include mean current-tidal current interaction, spatially-varying bottom friction, and rotary tidal currents.

An important feature of the resulting circulation is that the mean Lagrangian velocity can be significantly different from the mean Eulerian velocity, due to the existence of a Stokes velocity. Longuet-Higgins (1969) has pointed out that there can be a Stokes velocity associated with time-varying ocean currents and Zimmerman (1979) has discussed its calculation for tidal currents. It arises here due to a water column oscillating horizontally in a region where there are large horizontal gradients in the velocity field. A key result of the present work is that a significant Stokes velocity is predicted to be associated with Coriolis rectification on the sides of Georges Bank, so that the mean Lagrangian current is generally less than the mean Eulerian current.

The detailed spatial structure of this velocity field is influenced by local bottom topography, the local density structure and interactions with other flows in the region, so that its prediction is clearly beyond the scope of an analytical treatment. The major emphases of this paper are an examination of the physics of Coriolis rectification on the side of a large bank, and an estimation of the associated depth-averaged circulation around Georges Bank. In Section 2 the depth-averaged Eulerian equations of motion are formulated and the general approach used in their solution is discussed. Huthnance's (1973) analytical solution for weak nonlinearity is presented in Section 3 with spatially-varying bottom friction and rotary tidal currents allowed, and the physics of Coriolis rectification is discussed in terms of the associated Stokes velocity. In Section 4 the solution of the equations for stronger nonlinearity when mean current-tidal current interaction becomes important is discussed, and some limiting cases are examined. The theory is used in Section 5 to predict mean Eulerian and mean Lagrangian velocities for parts of Georges Bank.

2. Theory

a. Formulation of Eulerian equations

The side of a large bank is considered, with straight parallel isobaths assumed along its monotonically sloping side, and zero bottom slope on its shallow top and in the adjacent deeper region. A right-handed Cartesian coordinate system \((x, y, z)\) is used (Fig. 2) with the \(x\) axis perpendicular to isobaths in the direction of decreasing depth, the \(y\) axis parallel to isobaths with the shallow region to its right, and \(z = 0\) a geopotential surface through mean sea level away from the effects of the sloping bottom. Assuming no variations in the along-isobath direction (except possibly in sea surface elevation), and neglecting horizontal friction and surface stress, the depth-averaged Eulerian equations of motion and continuity for the case of constant density can be written
\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - f v = -g \frac{\partial \zeta}{\partial x} \left( \frac{ku}{(H + \zeta)} \right) \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + f u = -g \frac{\partial \zeta}{\partial y} \left( \frac{kv}{(H + \zeta)} \right) \\
\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x} \left( (H + \zeta)u \right) + \frac{\partial \zeta}{\partial y} = 0
\]

(1)

where \(u(x,t)\) and \(v(x,t)\) are the depth-averaged horizontal components of Eulerian velocity, \(\zeta(x,y,t)\) is the position of the sea surface, \(H(x)\) the depth, \(f\) the Coriolis parameter, \(g\) the gravitational acceleration, and \(k(x)\) a linear bottom friction coefficient. (This treatment of bottom friction is discussed in Section 2b.)

The situation to be examined is an elaboration on that discussed by Huthnance (1973). In the deep region away from the influence of the bank (and indicated by subscript \(d\)), the velocity is specified to be at a single frequency \(\omega\) (the dominant tidal frequency) and can be written

\[
u_d = \frac{1}{2} u_{1d} e^{i \omega t} + \frac{1}{2} u_{1d}^* e^{-i \omega t} \\
= U_d \cos(\omega t + \phi_0)
\]

(2)

\[
v_d = \frac{1}{2} v_{1d} e^{i \omega t} + \frac{1}{2} v_{1d}^* e^{-i \omega t} \\
= R U_d \cos(\omega t + \phi_0 + \phi_1)
\]

where \(u_{1d}\) and \(v_{1d}\) are the complex amplitudes of the cross-isobath and along-isobath components, respectively, an asterisk indicates the complex conjugate, and \(U_d, R\) and \(\phi_0, \phi_1\) are real constants. The phase \(\phi_0\) of the cross-isobath tidal current at \(t = 0\) is arbitrarily taken to be zero in this paper, so that the specification of the forcing is done with \(U_d, R\) and \(\phi_1\).

The along-isobath component of velocity and the sea surface position are allowed to consist of means (obtained by time-averaging over a tidal period) and contributions at the tidal frequency \(\omega\) and its higher harmonics, while the cross-isobath component of velocity is taken to consist of a tidal contribution only (this assumption will be discussed shortly). They can be represented

\[
u(x,t) = \frac{1}{2} u_{1d} e^{i \omega t} + \frac{1}{2} u_{1d}^* e^{-i \omega t} \\
v(x,t) = \tilde{v} + \frac{1}{2} v_{1d} e^{i \omega t} + \frac{1}{2} v_{1d}^* e^{-i \omega t} \\
+ \frac{1}{2} v_{2e} e^{i 2 \omega t} + \frac{1}{2} v_{2e}^* e^{-i 2 \omega t} + \cdots
\]

(3)

\[
\zeta(x,y,t) = \tilde{\zeta} + \frac{1}{2} \zeta_{1d} e^{i \omega t} + \frac{1}{2} \zeta_{1d}^* e^{-i \omega t} \\
+ \frac{1}{2} \zeta_{2e} e^{i 2 \omega t} + \frac{1}{2} \zeta_{2e}^* e^{-i 2 \omega t} + \cdots
\]

where the overbar represents time-averaging over a tidal period and hence indicates a mean quantity, and \(u_{1d}, v_{1d}, \zeta_{1d}, \) etc., are spatially-varying complex amplitudes with the subscripted numbers referring to the harmonic being represented. Eqs. (1) can now be rewritten as a series of equations at zero frequency, and at the tidal frequency and its harmonics, by substituting (3) in (1). If the amplitude of all variations in sea surface position is assumed to be much less than the depth (i.e., \(\zeta_H \ll H\); \(\zeta_H \ll H\), etc.) and if terms arising from the third term in the continuity equation in (1) are neglected (both of which can be justified for the sides of Georges Bank), the equations become

**Mean (frequency 0)**

\[
\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} - f \tilde{v} = -g \frac{\partial \zeta_H}{\partial x}
\]

**Tidal (frequency \(\omega\))**

\[
i \omega u_1 - f v_1 = -g \frac{\partial \zeta_1}{\partial x} - \frac{ku_1}{H}
\]

\[
i \omega v_1 + u_1 \frac{\partial \tilde{v}}{\partial x} + v_2 \frac{\partial \tilde{v}}{\partial x} + \frac{\partial f_1}{\partial x} = -g \frac{\partial \zeta_1}{\partial y} - \frac{kv_1}{H}
\]

**Second Harmonic (frequency \(2\omega\))**

\[
\frac{\partial u_2}{\partial x} - f v_2 = -g \frac{\partial \zeta_2}{\partial x}
\]

Equations of similar form to (6) occur at the higher harmonic frequencies. The bracketed terms in the continuity equations in (4) and (6) are the largest terms that are being omitted from these continuity equations by the assumptions of zero cross-isobath mean current, and zero cross-isobath current at harmonic frequencies. To ensure that continuity is satisfied, these terms can be included, and the
continuity equations can be used to estimate magnitudes for \( \bar{u} \) and \( |\mu_2| \), if such is the case. For the \( M_2 \) tide on the northwestern and northern sides of Georges Bank, continuity can be satisfied by \( \bar{u} = 0.01 \text{ m s}^{-1} \), and by the cross-isobath currents at harmonic frequencies having amplitudes of a few hundredths of a meter per second. In the momentum equations in such regions, the acceleration, Coriolis and bottom friction terms involving these currents are small compared to the tidal stress terms, so that the neglect of these terms from the momentum equations in (4)–(6) is possible.

The continuity equation at frequency \( \omega \) can be simplified with the assumption that \( \xi_1 / H \ll L_r / (2L_r) \), where \( L_r = 2U/\omega \) is the cross-isobath tidal excursion, \( U(x) \) the amplitude of the cross-isobath tidal current, and \( L_r \) its cross-isobath length scale. The resulting continuity equation is

\[
\frac{\partial}{\partial x} (HU_1) = 0, \tag{7a}
\]

which can also be expressed

\[
U(x) = H_d U_d / H(x), \tag{7b}
\]

where \( H_d \) is the depth in the deep water. The phase of \( u_1 \) is thus everywhere equal to zero, so that upon specification of \( U_d \) and \( H(x) \), the cross-isobath component of tidal velocity is known everywhere.

The assumption that there are no along-isobath variations (except maybe along-isobath sea surface slopes) can be used to simplify the momentum equations. If the \( x \) equation in each of (4)–(6) is differentiated with respect to \( y \), the result is

\[
\begin{align*}
\frac{\partial}{\partial y} \left( \frac{\partial \xi_i}{\partial x} \right) &= \frac{\partial}{\partial x} \left( \frac{\partial \xi_i}{\partial y} \right) = 0, \\
\frac{\partial}{\partial y} \left( \frac{\partial \xi_i}{\partial x} \right) &= \frac{\partial}{\partial x} \left( \frac{\partial \xi_i}{\partial y} \right) = 0 \quad \left( \xi_i = 1, 2, \ldots \right) \tag{8}
\end{align*}
\]

i.e., there are no cross-isobath variations in the along-isobath mean sea surface slope or in the along-isobath sea surface slopes at the tidal and harmonic frequencies. Thus, an along-isobath mean pressure gradient cannot alone balance the tidal stress terms in the mean \( y \) equation [the second of (4)] since the magnitude of the latter varies in the cross-isobath direction. Any such pressure gradient used to partially balance the tidal stress terms over the side of the bank must also exist in the deep and shallow regions (where the tidal stress terms are small), and will drive a mean current in such regions. In a confined region in the real ocean, such a return flow is probable; however, the interest here is in the dynamics of the rectification process on the side of an idealized bank and any effects on the current regime in adjacent regions will not be treated. It is thus appropriate, and also convenient, to take \( \partial \xi_i / \partial y = 0 \) and require the tidal stress terms in the mean \( y \) equation to be balanced by the bottom stress on an along-isobath mean current.

Similarly, a variation in the along-isobath sea surface slope at any of the harmonic frequencies would generate a velocity at the same frequency in the adjacent regions. In this case, the sloping bottom would be essentially radiating tides at harmonic frequencies into adjacent regions. This situation will not be considered here, where it is assumed that \( \partial \xi_i / \partial y = 0 \) for \( i = 2, 3, \ldots \), so that variations in the along-isobath sea surface slope are allowed at the tidal frequency only. (The earlier assumption that cross-isobath currents at frequencies other than \( \omega \) are unimportant to the dynamics is consistent with this treatment of the pressure gradient terms.)

The result [Eq. (8)] that there are no cross-isobath variations in the along-isobath sea surface slope at the tidal frequency can be used to simplify the along-isobath tidal equation [the second of Eqs. (5)]. In the deep region sufficiently far from the region of sloping bottom that the nonlinear terms disappear, this equation can be written

\[
i \omega r U_d + f U_d = -g \frac{\partial \xi_1}{\partial y} - \frac{k_d r U_d}{H_d}, \tag{9}
\]

where, as before, the subscript \( d \) indicates the deep region. If (9) is subtracted from the second of (5), the resulting equation [(11) below] is one in which there are no pressure gradient terms.

With these assumptions, the \( y \) equations in (4)–(6) can be rewritten as a series of equations in which, after specification of \( U(x), r, H(x) \) and \( k(x) \), the only unknowns are \( \bar{v} \) and the complex amplitudes of the other along-isobath components of velocity:

\[
\begin{align*}
\frac{\partial v_{3a}}{\partial x} + \frac{\partial v_{3b}}{\partial x} &= -\frac{k \bar{v}}{H}, \tag{10}
\end{align*}
\]

\[
\begin{align*}
i \omega (v_1 - r U_d) + U \frac{\partial \bar{v}}{\partial x} + \frac{\partial v_2}{\partial x} + f(U - U_d) &= -\left( \frac{k v_1}{H} - \frac{k_d r U_d}{H_d} \right), \tag{11}
\end{align*}
\]

\[
\begin{align*}
2i \omega v_2 + \frac{\partial v_2}{\partial x} + \frac{\partial v_3}{\partial x} &= \frac{k v_2}{H}. \tag{12}
\end{align*}
\]

The \( x \) equations in (4)–(6) remain unchanged, and the bracketed terms can be included in the continuity equations in (4) and (6), while (7b) becomes the continuity equation at tidal frequency.

**b. Solution of Eulerian equations**

Bottom friction in (10)–(12) is represented through a linear friction law. A proper representation to
include all nonlinear effects would require a quadratic bottom friction law. In this paper, the linear friction coefficient $k$ is taken to be time-independent, but spatially varying in the $x$ direction through a linear dependence on the amplitude of the cross-isobath tidal current, i.e.,

$$k(x) = C_p U(x),$$  \hspace{1cm} (13)

where $C_p$ is a drag coefficient. The right-hand side of (13) should also be multiplied by a spatially-varying factor dependent on the ellipticity of the tidal currents and between $2/\pi$ and $\frac{1}{2}$ for relatively weak mean currents. This factor is taken as 1 here, which appears to be a good first approximation for applications to Georges Bank where the tidal currents are mainly cross-isobath but become increasingly rotary in the shallow areas where bottom friction is most important.

The along-isobath components of velocity are now obtained through the solution of (10)–(12). Although nonlinear interactions are represented in the advective terms in these equations, $U(x)$ is known from (7b), effectively linearizing these terms. The presence of the complex conjugate of $\nu_1$ in (10) can be avoided by rewriting (10) as

$$\text{Re} \left\{ \frac{1}{2} U \frac{\partial \nu_1}{\partial x} \right\} = - \frac{k \tilde{\nu}}{H}. \hspace{1cm} (10a)$$

Eqs. (10a), (11), (12) and the analogous equations at higher harmonic frequencies can now be considered to be a series of $n$ linear ordinary differential equations in $n + 1$ unknown variables. Their solution requires some closure assumption.

The formulation of these equations has followed the approach of Huthnance (1973), who obtained closure by neglecting mean current-tidal current interaction [the second term in (11)] and tidal current-second harmonic current interaction [the third term in (11)]. Eq. (11) was solved for $\nu_1$ and, after differentiation with respect to $x$, the resulting expression was substituted in (10a) to give $\tilde{\nu}$. This is a lowest order closure appropriate to weak nonlinearity, and is further discussed in Section 3.

The next highest order closure is to include the second term in (11) representing mean current-tidal current interaction, but still omit interaction with the second harmonic current. The result of the substitution for $\partial \nu_1/\partial x$ in (10a) is then a second-order linear differential equation in $\tilde{\nu}$. This closure is used in Section 4, and is appropriate to stronger nonlinearity, when the mean flow generated by the lowest order nonlinearity [through (10)] is sufficiently strong that its interaction with the tidal current becomes important. The neglect of tidal current-second harmonic current interaction is justifiable if $\frac{1}{2} |\nu_2| \ll \tilde{\nu}$. Comparison of (10) and (12) suggests that this will be so provided that the acceleration term in (12) is sufficiently important, which can be shown to require that $0.7a \gg k/H$.

In strongly nonlinear situations where interactions with harmonic currents are important, the approach to closure is straightforward. In harmonic equations, the acceleration term becomes increasingly important relative to the bottom friction term, so that the amplitude of successively higher harmonic currents decreases. On this basis, the third term can be omitted from (12) or the analogous equation at higher harmonic frequency. The result is then $n$ equations in $n$ unknown variables ($\tilde{\nu}, \nu_1, \ldots, \nu_n$), but their solution becomes difficult for $n > 3$.

The $x$-momentum balances in (4)–(6) are achieved through cross-isobath sea surface slopes. After determining the along-isobath component of velocity at any frequency, the amplitude of the cross-isobath sea surface slope at that frequency can be obtained from the relevant $x$ equation.

c. Integrated mean momentum balance

Details of the spatial distribution of along-isobath mean current will clearly be dependent on the proper representation of the nonlinear interactions and on the actual bottom topography. However, some overall properties of the velocity field are dependent only on the forcing parameters, $U_d, R$ and $\phi_r$, and on the depths in the deep and shallow regions, $H_d$ and $H_s$, respectively.

In the shallow region sufficiently far from the sloping topography that there are no nonlinear interactions locally, the complex amplitude $\nu_{is}$ of the along-isobath component of tidal velocity can be obtained directly from (11) by omitting the nonlinear terms and using (7b):

$$\nu_{is} = \frac{U_d}{(F_s^2 + 1)^{1/2}} \left[ (F_d - F_s)R \sin \phi_r + (1 + F_s F_d)R \times \cos \phi_r + F_s - \frac{f}{\omega} \left( 1 - \frac{H_d}{H_s} \right) \right]. \hspace{1cm} (14)$$

Here $F_d$ and $F_s$ are the values in deep and shallow water, respectively, of $F(x)$, which is the ratio of the bottom friction term to the acceleration term in the along-isobath tidal equation, given by

$$F(x) = \frac{k(x)}{\omega H(x)}. \hspace{1cm} (15)$$

Thus, the change in along-isobath tidal current from the deep to the shallow region is independent of the
details of the interactions over the sloping topography. For this to be valid, of course, it is necessary that the cross-isobath tidal current satisfy the simplified continuity equation (7b).

Eq. (14) can be used to obtain an expression for the cross-isobath integral of mean bottom stress. Integration of (10a), after using (7b) and rearranging, yields

$$\rho \frac{H_d U_d}{2} [\text{Re} \{v_1\}]_{x_d}^x = -\rho \int_{x_d}^x k \delta x,$$  \hspace{1cm} (16)

where \(\rho\) is density, and \(x_d\) and \(x\) are positions in the deep and shallow regions sufficiently far away from the sloping topography that \(\partial v_1/\partial x = 0\). This represents a net along-isobath mean momentum balance for the problem, stating that the net cross-isobath divergence in the cross-isobath mean flux of along-isobath momentum is balanced by the net bottom stress on the resulting mean flow. Upon substitution of (14) in (16), the result is

$$\int_{x_d}^x k \delta x = -\frac{H_d U_d^2}{2(F_d - F_s)} [F_d - F_s] R(F_s \cos \phi_r + \sin \phi_r) + F_s \frac{f}{\omega} \left(1 - \frac{H_d}{H_s}\right),$$  \hspace{1cm} (17)

an integral constraint on the mean Eulerian current, useful as a check on numerical solutions for \(\bar{v}\). This is the same as Huthnance's (1973) integrated expression (2.11), except that spatially-varying bottom friction and rotary tidal currents in the deep water are allowed here.

d. Mean Lagrangian velocity

The mean Lagrangian velocity is related to the Eulerian velocity field (3) by

$$\begin{align*}
\tilde{u}_l(x_0) &= u(x_0 + \int_0^t u_1 dt', t) \\
\tilde{v}_l(x_0) &= v(x_0 + \int_0^t u_1 dt', t)
\end{align*}$$  \hspace{1cm} (18)

where \([u_1(x_0,t), v_1(x_0,t)]\) is the depth-averaged velocity at time \(t\) of a water parcel that was at position \(x_0\) at time \(t = 0\). In the present case with \(\phi_0 = 0\), the time-averaging is done from \(t = 0\) to \(t = T\), where \(T\) is the tidal period, so that \([\tilde{u}_l(x_0), \tilde{v}_l(x_0)]\) is the mean Lagrangian velocity of a water parcel whose mean position is \(x_0\).

The usual starting point in the calculation of the mean Lagrangian velocity (Longuet-Higgins, 1969) is an expansion of (18) in a Taylor series involving derivatives of the Eulerian velocity. Without the time-averaging, the result is

$$u_l(x_0, t) = u(x_0, t) + \left(\int_0^t u_1 dt'\right) \frac{\partial u}{\partial x}(x_0, t) + \frac{1}{2} \left(\int_0^t u_1 dt'\right)^2 \frac{\partial^2 u}{\partial x^2}(x_0, t) + \cdots$$  \hspace{1cm} (19)

As discussed by Zimmerman (1979), a twofold truncation is then performed, which amounts in the present situation to (i) assuming cross-isobath variations in the cross-isobath tidal current to be small over tidal excursions so that

$$\left|\int_0^t u_1 dt'\right| \approx \int_0^t u dt'$$  \hspace{1cm} (20)

and (ii) assuming the cross-isobath length scale of the total Eulerian velocity to be sufficiently greater than the cross-isobath tidal excursion that (19) can be truncated after the terms involving the first derivative. [Successively higher order terms in (19) can be seen to contain the ratio of cross-isobath tidal excursion to cross-isobath length scale of the Eulerian velocity to successively higher powers.]

With these assumptions, the mean Lagrangian velocity becomes

$$\begin{align*}
\tilde{u}_l(x_0) &= \left(\int_0^t u dt'\right) \frac{\partial u}{\partial x}(x_0, t) = 0 \\
\tilde{v}_l(x_0) &= \bar{v}(x_0) + \left(\int_0^t u dt'\right) \frac{\partial v}{\partial x}(x_0, t)
\end{align*}$$  \hspace{1cm} (21)

from which the Stokes velocity \((\tilde{u}_s, \tilde{v}_s)\), defined as the mean Lagrangian velocity minus the mean Eulerian velocity, can be obtained. For the tidal rectification processes being examined in this paper, the above assumptions are generally not valid, so that (21) can only be used to give first estimates of the mean Lagrangian velocity for weak nonlinearity. In such cases, \(\tilde{v} \approx |v_1|\), and the along-isobath component of Stokes velocity in the present notation becomes

$$\tilde{v}_s(x_0) = \frac{H_d U_d}{2\omega H(x_0)} \text{Re} \left\{ i \frac{\partial v_1}{\partial x}(x_0) \right\}.$$  \hspace{1cm} (22)

For the situations of stronger nonlinearity of primary interest here, the mean Lagrangian velocity must be determined using the more general relations (18) with the integrations done numerically. This is equivalent to defining

$$\begin{align*}
\tilde{u}_l(x_0) &= \frac{X(x_0, T)}{T}, \\
\tilde{v}_l(x_0) &= \frac{Y(x_0, T)}{T}
\end{align*}$$  \hspace{1cm} (23)
where

\[ X(x_0, t) = \int_0^t u_L(x_0, t') dt', \]

\[ Y(x_0, t) = \int_0^t v_L(x_0, t') dt', \]

and solving the differential equations

\[ \begin{align*}
\frac{\partial X}{\partial t} &= u[x_0 + X(x_0, t), t] \\
\frac{\partial Y}{\partial t} &= v[x_0 + X(x_0, t), t]
\end{align*} \]  \quad (24)

When the cross-isobath component of velocity only has contributions at the tidal frequency satisfying (7b), the result again is \( \tilde{u}_L = 0 \).

3. Case of weak nonlinearity

a. Analytical solution

Huthnance's (1973) solution for the along-isobath mean Eulerian current was obtained with a lowest order closure assumption, which can be considered appropriate to situations where the nonlinear interaction between components of tidal velocity is sufficiently weak that the resulting mean current is unimportant to the dynamics of the tide. For such weak nonlinearity in the present problem, the analytical solution for \( \tilde{v}_L \) is a more general form of (14) [the same as (14), but with all subscripted \( s \)'s deleted]. It can be substituted in (10a) and (22) to give lowest order estimates of the along-isobath components of mean Eulerian velocity and Stokes velocity, i.e.,

\[ \tilde{v}_L = -\frac{H_d U_d \phi dH/\phi d\phi}{\phi^2 H^2(F^2 + 1)^2} \left[ \alpha(1 + FF_d)(1 + F^2)R \sin \phi_r \\
+ \alpha(1 + F^2)(F - F_d)R \cos \phi_r \\
+ f(1 + F^2) \left( \frac{H_d}{H} - 1 \right) \right]. \]  \quad (27)

Eq. (25) is somewhat similar to Huthnance's (1973) expression (2.9) for the mean Eulerian current, but differs due to the inclusion here of spatially-varying bottom friction and the allowance for rotary tidal currents in the deep region. The along-isobath component of mean Lagrangian velocity can now be calculated, using \( \tilde{v}_L = \tilde{v} + \tilde{v}_S \), as

\[ \tilde{v}_S = \left( \frac{H_d U_d \phi dH/\phi d\phi}{\phi^2 H^2(F^2 + 1)^2} \left[ \frac{2\alpha F(1 - F^2 - 2FF_d)R \sin \phi_r}{H_d} \\
+ fF^2 \left( \frac{3H_d}{H} - 2 \right) \right] \right. \]

\[ = \frac{H_d U_d \phi dH/\phi d\phi}{\phi^2 H^2(F^2 + 1)^2} \left[ \frac{2\alpha F(1 - F^2 - 2FF_d)R \sin \phi_r}{H_d} \\
+ fF^2 \left( \frac{3H_d}{H} - 2 \right) \right]. \]  \quad (26)
confined to locations of nonzero bottom slope with greatest magnitude in the shallower areas.

The cross-isobath mean sea surface slope can be obtained from the mean \( x \) equation [the first of (4)] by using (7b) and rearranging to give

\[
\frac{\partial \xi}{\partial x} = \frac{H \phi^2 U_{z_0}^2}{2gH^3} \frac{dH}{dx} + \frac{f}{g} \tilde{v},
\]

where (25) can be substituted for \( \tilde{v} \).

b. Rectification mechanisms

As discussed by Huthnance (1973), there are two distinct mechanisms that operate to generate the along-isobath mean current—one due to continuity and Coriolis effects but regulated by bottom friction, and the other due to the combined action of bottom friction and advection. The relative importance of these mechanisms depends upon the relative magnitudes of the Coriolis and bottom friction terms in the along-isobath tidal equation [the second of (5)].

On the sloping sides of Georges Bank, the Coriolis term dominates. The generation mechanism can be understood by referring in Fig. 3a to water column \( E \), whose mean position is over sloping bottom topography. Consider the column’s hypothetical movement over a tidal period as a result of the dominant pressure gradient, Coriolis and continuity effects. During its half-period in the deeper water, column \( E \) moves through a small half-ellipse (half of ellipse D). When it moves into the shallower region, the \( x \) component of its velocity is increased by continuity [Eq. (7b)], and consequently, its acceleration in the \(-y\) direction by the Coriolis force is increased. The column thus moves through a larger half-ellipse in the shallower water, so that over a complete tidal period it drifts a net distance in the \(-y\) direction. This drift of the water column relative to the mean Eulerian flow (assumed zero so far) is a Stokes drift (a component of the Stokes velocity), somewhat analogous to the Stokes drift of surface waves, or that usually associated with tidal flow due to the terms involving \( u_1 \xi_t \) and \( v_1 \xi_t \) in the mean continuity equation. It is distinct from the latter, though, in that it arises here due to the cross-isobath oscillation of the water column in a region where there is a significant cross-isobath gradient in the amplitude of the along-isobath current. However, if all the forces on column \( E \) during its tidal excursion in Fig. 3a are considered, there is a mean bottom stress in the \(+y\) direction, as a result of the column having a larger velocity in the \(-y\) direction as well as a larger bottom friction coefficient while in the shallower region. This is an unbalanced force in the mean Lagrangian momentum balance, and, if the tidal motion is to be non-decaying, the actual movement of column \( E \) must be, not as in Fig. 3a, but such as to have zero mean bottom stress over the tidal period. This can be attained through an additional contribution in the \(+y\) direction to the column’s Lagrangian velocity, and since this additional contribution can be shown for the present representation of bottom friction to be about three times the Stokes drift, the column’s actual movement is a spiral in the \(+y\) direction. The net result (Fig. 3b) is the generation of a clockwise mean Eulerian current pattern around banks in the Northern Hemisphere, with the mean Lagrangian circulation clockwise but only about two-thirds of the mean Eulerian due to the counterclockwise Stokes drift.

The relative magnitudes of these velocities for this case of weak nonlinearity are confirmed by (25)–(27). The Coriolis mechanism gives rise to the third and fourth terms within the outer braces in (25) and (26), and the third term in (27). For \( R = 0 \) and \( F^2 \ll 1 \), which is a limit of Coriolis rectification only, Eqs. (25) and (27) become

\[
\tilde{v} = -\frac{H_d U_{z_0}^2 f dH}{2a^2 H^2} \left( \frac{3H_d}{H} - 2 \right),
\]

\[
\tilde{v}_L = -\frac{H_d U_{z_0}^2 f dH}{2a^2 H^2} \left( \frac{2H_d}{H} - 2 \right),
\]

so that \( \tilde{v} \) and \( \tilde{v}_L \) are both positive for the case of \( f > 0 \) and \( dH/dx < 0 \) being considered here, but with the mean Lagrangian current approaching two-thirds of the mean Eulerian as \( H \) becomes much less than \( H_d \). For constant \( k \) (as in Huthnance, 1973) and \( H \ll H_d \), \( \tilde{v} \) is only two-thirds of that in (29) and \( \tilde{v}_L \) only one-half of that in (30), so that a proper representation of bottom friction is clearly important. Note that in this limit of weak bottom friction, \( \tilde{v} \) and \( \tilde{v}_L \) depend not on the magnitude of \( k \), but instead on its spatial variation [for a more general discussion see Huthnance (1980)].

For stronger nonlinearity when the mean and higher harmonic Eulerian velocities become important, there are also significant contributions to the Stokes velocity due to the water column oscillating through horizontal variations in these secondary velocity fields. The explanation of the Coriolis mechanism must then be extended to include the effects of these contributions.

The bottom friction term in the along-isobath tidal equation becomes important in the shallow central region of Georges Bank, so that mean current generation by the combined action of bottom friction and advection may be occurring. This would occur through the first two terms within the outer braces in (25) and (27), which suggest the directions and relative magnitudes of the Eulerian and Lagrangian velocities to be dependent on \( \phi_0 \) and \( F \). It is interesting that the length scales and depths of the shoals on the shallow part of the Bank are similar to those of the Norfolk Sandbanks, where Huthnance (1973)
suggested that mean current generation by both the Coriolis and frictional mechanisms is occurring. However, the frictional rectification process will not be discussed further here, since the primary interest is the circulation on the sides of Georges Bank, where the depth is sufficient for Coriolis rectification to dominate.

4. Inclusion of mean current-tidal current interaction

a. Method of solution

When mean current-tidal current interaction is included in the theory, the following second-order differential equation for the mean Eulerian current is obtained from (11) and (10a)

\[ \frac{\partial^2 \bar{u}}{\partial x^2} + A_1(x) \frac{\partial \bar{u}}{\partial x} + A_2(x) \bar{u} = A_3(x), \]

where

\[ A_1(x) = \frac{(F^2 - 3)}{H(F^2 + 1)} \frac{dH}{dx} \]
\[ A_2(x) = -\frac{2\omega^2 H^3(F^2 + 1)}{H_d^2 U_d^2} \]
\[ A_3(x) = \frac{dH}{dx} \left[ 2\omega(1 - F^2) + 2\omega F_d R \sin\phi_r + 2\omega F(2 + FF_d) \right. \]
\[ + \left. F_d R \cos\phi_r + fF^2 \left( 2 - \frac{H_d}{H} \right) + f \left( 3 \frac{H_d}{H} - 2 \right) \right] \]

This equation can be rewritten as a pair of coupled first-order differential equations, which can be solved numerically if \( \bar{u} \) and \( \partial \bar{u}/\partial x \) are known at some location at which an integration scheme can be started. The solutions in Section 5 are obtained with the International Mathematical and Statistical Library subroutine DVERK (Hull et al., 1976), using Runge-Kutta formulas of orders 5 and 6.

The boundary conditions on \( \bar{v} \) and \( \partial \bar{v}/\partial x \) are obtainable from a consideration of (31) in the deep and shallow regions, which are labeled I and III, respectively, in Fig. 4. The origin is taken to be the boundary between region I and the region of sloping bottom topography (labeled II), and \( x = x_c \) is taken as the boundary between regions II and III. In regions I and III, \( dH/dx = 0 \) and (31) reduces to

\[ \frac{\partial^2 \bar{v}}{\partial x^2} + A_4 \bar{v} = 0 \]

with, in region I,

\[ A_2 = A_{2d} = -\frac{2\omega^2(F_d^2 + 1)}{U_d^2} \]

and, in region III,

\[ A_2 = A_{2a} = -\frac{2\omega^2 H_d^2(F_d^2 + 1)}{U_d^2 H_d^2} \]

Assuming that \( \bar{v} \) vanishes as \( x \to \pm \infty \), simple exponential decay laws result for \( \bar{v} \) in these regions

\[ \bar{v}_I = B_1 e^{-\alpha_d x} \]
\[ \bar{v}_{III} = C_1 e^{-\alpha_a (x - x_c)} \]

with

\[ \alpha_d = \frac{\omega(2(F_d^2 + 1))^{1/2}}{U_d} \]
\[ \alpha_a = \frac{\omega H_d}{U_d H_d^2} (2(F_d^2 + 1))^{1/2} \]

Here \( B_1 \) and \( C_1 \) are undetermined constants. These analytical solutions can be used to obtain the following boundary conditions on \( \bar{v} \) in region II:

\[ \text{at } x = 0 \frac{\partial \bar{v}}{\partial x} \bar{v}^{-1} = \alpha_d \]
\[ \text{at } x = x_c \frac{\partial \bar{v}}{\partial x} \bar{v}^{-1} = -\alpha_a \]

However, the magnitudes of \( \bar{v} \) and \( \partial \bar{v}/\partial x \) at these positions remain unknown. The necessary approach to finding the solution of (31) that satisfies (34) is to specify, in some organized manner, values for \( \bar{v} \) at either \( x = 0 \) or \( x = x_c \), calculate \( \partial \bar{v}/\partial x \) at that position using one of (34), and repeat the integration of (31) until a solution is found that satisfies the other of (34). The integral of \( k \bar{v} \) can then be evaluated (numerically in region II, and analytically in regions I and III) and compared with that given by (17), providing a check on the solution.

Knowing the Eulerian velocity field, the mean Lagrangian velocity can be obtained from (23) after a straightforward numerical integration of (24). The mean Lagrangian velocity in Section 5 is calculated for the mean and tidal contributions to the Eulerian.
field, the contributions from higher harmonics being small.

b. Limiting cases of Coriolis rectification

The cross-isobath length scale of the mean Eulerian velocity field can be estimated on examination of (31) for two limiting cases of rectification by the Coriolis mechanism. For \( R = 0 \) and \( F^2 \ll 1 \), the coefficients in (31) become

\[
\begin{align*}
A_1(x) &= -\frac{3}{H} \frac{dH}{dx}, \\
A_2(x) &= -8L_e^2, \\
A_3(x) &= \frac{f}{H} \frac{dH}{dx} \left( 3 - 2 \frac{H}{H_d} \right)
\end{align*}
\]

(35)

where \( L_e \) is again the cross-isobath tidal excursion \((\approx 2U_d H_d^2 / (\alpha H))\). It can be seen that \( A_1 \) and \( A_3 \) contain the inverse of the topographic length scale, \( L = H dH / dx \). \( A_3 \) is the forcing term in the differential equation in (31) and varies spatially on the topographic length scale, while the relative magnitudes of the three terms on the left-hand side of the equation are determined by the relative magnitudes of \( L \) and \( L_e \). For \( L \gg L_e \), \( A_2 \) becomes large and the third term might be expected to balance the forcing term. Since these terms must then have the same spatial dependence, the cross-isobath length scale \( L_m \) of the mean velocity must be of order \( L \). The first two terms are unimportant for such a scaling, so that \( L_m = O(L) \) is a consistent scaling for Coriolis rectification in the limit \( L \gg L_e \). This limit is the case of weak nonlinearity discussed in Section 3, and the expression for \( \bar{v} \) obtained by balancing the forcing term with the third term is that given by (29).

As the magnitude of \( L_e \) increases relative to that of \( L \), the first two terms in the differential equation in (31) become increasingly important. In the limit \( L_e \gg L \), the width of region II (in which the forcing term is nonzero) becomes small, so that the problem approaches that of forcing by a delta function. This limit can be approximated by a step in topography for which an analytical solution is obtained in the next part of this section. The overall length scale of the mean flow in this limit is determined largely by its length scale in regions I and III. In these regions \( A_1 = A_3 = 0 \) so that \( L_m = O(8^{-1/2}L_e) \), which could also have been obtained from (33). The limit \( L_e \gg L \) is thus one of strong nonlinearity with \( L_m = O(8^{-1/2}(L_e + L_{eill})) \), where \( L_{eill} \) are the tidal excursions in regions I and III, respectively.

These scale arguments are supported by the analytical solution (25) and the numerical solutions of (31). The ratio \( L_e / L \) is an important nondimensional parameter in determining the length scale of the mean Eulerian velocity field due to rectification by the Coriolis mechanism. For situations when \( L_e \ll L \), the length scales of the tidal and higher harmonic velocity fields (as well as the mean velocity field) can be shown to be of order \( L \), so that \( L_e / L \) is then the important parameter in determining the validity of a truncation of (19) in the calculation of the mean Lagrangian velocity field.

c. Solution for step in topography

An analytical solution for the mean Eulerian current for an idealized step in topography can be obtained by considering region II to vanish. The problem is then simply one of obtaining the matching conditions at the step (taken at \( x = 0 \)) for the analytical solutions of regions I and III. At the present level of closure, the matching conditions on \( \partial \bar{v} / \partial x \) and \( \bar{v} \) are obtainable, respectively, from the integration of (31) and the second of (5) (with its third term omitted) from \( x = -\epsilon \) to \( x = +\epsilon \), and consideration of the limit \( \epsilon \to 0 \). The matching condition on \( \bar{v} \) is found to be

\[
\bar{v}(0^+) = \bar{v}(0^-),
\]

(36)

where \( x = 0^+ \), \( 0^- \) indicate \( x = 0 \) approached from the positive and negative \( x \) regions, respectively. Thus, \( \bar{v} \) is continuous across the step.

The integration of (31) in its general form is tedious. However, for rectification by the Coriolis mechanism, taken as when \( R = 0 \) and \( F^2 \ll 1 \), the integration is straightforward and the matching condition on \( \partial \bar{v} / \partial x \) is

\[
\frac{\partial \bar{v}}{\partial x}(0^+) = \left( \frac{H_s}{H_d} \right)^3 \frac{\partial \bar{v}}{\partial x}(0^-) - f \left( 1 - \frac{H_s}{H_d} \right) .
\]

(37)

The constants \( B_1 \) and \( C_1 \) in (33) can now be evaluated, resulting in

\[
\bar{v}(0) = -\frac{2^{-1/2} U_d}{\omega} \frac{f}{\omega} \left( 1 - \frac{H_d}{H_s} \right) \left( \frac{H_s}{H_d} \right)^g,
\]

(38)

and exponential decays away from the step.

An alternative (and simpler) approach to determining the solution for the more general theory represented in (31) is to take (36) as one matching condition and also require that the solution satisfy the integrated constraint (17). The result is

\[
\bar{v}(0) = \frac{-\omega H_d \left[ (F_d - F_s) R(F_s \cos \phi_s + \sin \phi_s) + \frac{F_s f}{\omega} \left( 1 - \frac{H_d}{H_s} \right) \right]}{C_p [2(F_s^2 + 1)]^{1/2} \left( \frac{F_s^2 + 1}{F_d^2 + 1} \right)^{1/2} + \left( \frac{H_d}{H_d} \right)^2},
\]

(39)
which, for $R = 0$ and $F^2 \ll 1$, is the same as (38). The numerical solutions of (31) for cases when $L_e \gg L$ show agreement with those obtained using the same parameters in (39), confirming that the step in topography is a valid approximation to the limit $L_e \gg L$.

A comment is appropriate on an apparent inconsistency between the conclusions of Zimmerman (1978), and the present result that the integral of $k\bar{u}$ is independent of $L$ [see (17)], so that there can be significant mean current for $L_e \gg L$. Zimmerman (1978) assumed a spectral representation of bottom topography and found that the mean square residual vorticity associated with both rectification processes was greatest when the ratio $l_1/l$ was of order one, where $l_1$ is the amplitude of the tidal displacement (=$\frac{1}{2}L_e$ in the present notation) and $l$ is the integral length scale of topography. In the limit $l_1 \gg l$, he found that the energy density of the mean field decreased to zero. The inconsistency appears to arise from Zimmerman’s assumption of sinusoidal bottom topography, implying that for every region of negative $dH/dx$ there is an adjacent region of positive $dH/dx$ in which mean current in the opposite direction is generated. The limit $l_1 \gg l$ is one in which the mean current is redistributed (or smeared out) over distances of order $l_1$ by the mean current-tidal current interaction, so that mean currents in opposite directions generated only a distance of order $l$ apart largely cancel each other. A spectral representation of bottom topography thus appears inappropriate to the monotonically sloping side of a bank in the limit $L_e \gg L$, unless the width of the bank is of order $L$.

5. Application to Georges Bank

a. Predicted circulation

The theory can now be used to obtain estimates of the mean Eulerian and Lagrangian velocities associated with tidal rectification on the sides of Georges Bank. The bathymetry of the Bank is shown in Fig. 5. The assumption of a large bank with insignificant along-isobath variations appears to be a good approximation to its northwestern (A to B in Fig. 5) and northern (B to C) sides (along which isobaths are roughly parallel and straight for distances of about 100 km), and to its open ocean side. The bottom topography will be represented by a half-sinusoid with adjacent regions of zero bottom slope such that

$$H(x) = \begin{cases} 
H_d, & x < 0 \\
\frac{1}{2}(H_d + H_s) + \frac{1}{2}(H_d - H_s) \cos \left( \frac{\pi x}{x_c} \right), & 0 \leq x \leq x_c \\
H_s, & x_c < x.
\end{cases} \quad (40)$$
The dominant tide in the region is the semidiurnal $M_2$ tide. The accuracy of any prediction of the depth-averaged flow associated with tidal rectification is obviously dependent on a knowledge of the barotropic tide, so that appropriate values of $U_d$, $R$ and $\phi_r$ can be specified, and more importantly so that the validity of (7b) can be checked. The latter can be expected to be a limiting assumption to the applicability of the theory, since a shallow bank is an obstacle to the tidal flow, so that there may be a diversion of the flow around it, thereby making (7b) invalid. The overall features of the $M_2$ tide in the Gulf of Maine-Bay of Fundy system are reproduced quite well by Greenberg’s (1979) depth-averaged numerical model. Although the model has a grid size of 22 km in the Georges Bank region so that local details of topography and hence tidal regime are not represented, it does allow along-isobath variations and hence should provide reasonably good estimates of the cross-isobath volume flux $H U$ over the Bank. Such estimates do reveal some tendency for the tidal flow to be diverted around the Bank, but the resulting reduction in volume flux is only $\sim 40\%$ of the flux in the deep water just north of the Bank. The approach here is to choose values of $U_d$ that give volume fluxes $H_d U_d$ equal to those which actually go over the Bank in Greenberg’s model. This means that the specified values of $U_d$ are smaller than those for the same location in Greenberg’s model, but it is necessary that any predictions of mean current be based on the actual cross-isobath tidal current. Fig. 6 shows the predicted along-isobath mean currents and cross-isobath mean sea surface slopes due to the rectification of $M_2$ tidal currents on the northwestern, northern, and open ocean sides of Georges Bank. The specified values of tidal and topographic parameters are given in Table 1. The choice of $R = 0$ is arbitrarily made to remove the dependence on $R$ and $\phi_r$, whose appropriate values are uncertain. However, for the regions considered, the tidal currents are mainly cross-isobath, and for values of $R$ and $\phi_r$ that are typical from Greenberg’s model, the predicted speeds in Fig. 6 are increased by less than 10%. Table 1 also gives estimates of the length scales $L$ and $L_r$ for each set of parameters (calculated at the position of minimum $L$). These length scales are of the same order, so that from the scale arguments of Section 4b, the cross-isobath length scale of the mean current is expected to be of order $L$ (or $L_r$) with some mean current-tidal current interaction occurring.

The predicted mean flow (Fig. 6) can be seen to be positive (and hence clockwise around the Bank) and concentrated in a narrow region over the Bank’s edge. The mean Eulerian current calculated numerically from (31) to include mean current-tidal current interaction is distributed over a wider area and has a lower peak magnitude than that given by the analytical solution (25) on the assumption of weak nonlinearity. However, both satisfy the integral constraint (17) so that the effect of mean current-tidal current interaction is just to redistribute the current, largely onto the top of the Bank where no mean current is predicted by the analytical solution. The mean Lagrangian current, calculated to include mean current-tidal current interaction, is significantly less than the mean Eulerian, having typically only about two-thirds of its magnitude. The strength of the mean flow is different for each of the three regions examined, with strongest flows on the northwestern (Fig. 6a) and northern (Fig. 6b) sides where mean Eulerian current speeds of up to 0.23 and 0.13 m s$^{-1}$, respectively, are predicted. On the open ocean side of the Bank, the mean flow is weaker with mean Eulerian current speeds of up to 0.06 m s$^{-1}$. The theory can also be used to predict the mean flow associated with tidal rectification on the other sides of Georges Bank, but with less confidence due to increased curvature in the isobaths and added uncertainty as to the actual cross-isobath tidal currents. The Coriolis mechanism should contribute to circulation in a clockwise direction on the Northeast Channel side of the Bank, but its importance to the circulation on the Great South Channel side is unclear.

The cross-isobath mean sea surface slopes are obtained by substitution of the numerical solutions for $\vec{v}$ in (28). It can be seen (Fig. 6) that there are alternating regions of positive and negative slope as the shallow part of the Bank is approached. An examination of (28) reveals that a positive sea surface slope is required to geostrophically balance the along-isobath mean current in regions where the tidal stress term is relatively small, while a negative slope is needed to provide a Bernoulli-type pressure gradient in regions where the tidal stress term is dominant. As might be expected from the strength of the mean flows, the largest slopes are
predicted for the northwestern side of the Bank with successively smaller slopes for the northern and open ocean sides. It is also noteworthy that the relative magnitudes of the positive and negative slopes vary for the three cases considered.

The cross-isobath pressure gradients due to these sea surface slopes are quite significant and in the presence of density structure, rotation and vertical friction can be expected to combine with barocliniticy in the tidal and along-isobath mean currents to drive depth-dependent cross-isobath mean flows. A proper assessment of the associated Eulerian and Lagrangian velocities clearly requires a depth-dependent theory taking into account this vertical structure in the currents. A first estimate of the flow for the case of constant density suggests that any net reduction in the velocity-dependent forces in the bottom boundary layer due to friction leaves
an unbalanced cross-isobath pressure gradient tending to drive a bottom flow on- or off-the-Bank depending on the sign of $\partial \zeta / \partial x$. The magnitude of such a flow can be estimated by vertically averaging the depth-dependent $x$-momentum equation over a bottom boundary layer of thickness $h_b$ giving a mean Eulerian current $\bar{u}_b$ in the layer

$$\bar{u}_b = -\frac{h_b g}{k} \frac{\partial \zeta}{\partial x},$$

(41)

where a balance between bottom stress and pressure gradient has been assumed. For the northwestern side of Georges Bank with $h_b = 5$ m, off-the-Bank Eulerian flows of up to 0.03 m s$^{-1}$ are predicted in the regions of positive sea surface slope, and on-the-Bank flows of up to 0.10 m s$^{-1}$ in the region of negative slope. For the same $h_b$ on the northern and open ocean sides, flows of order 0.01–0.02 m s$^{-1}$ are predicted. Such bottom flows in opposite directions in nearby regions should lead to adjacent zones of convergence and divergence within the bottom boundary layer, and it is tempting to speculate that the current patterns sketched in Fig. 6 might result.

b. Evidence for tidal rectification

In looking for evidence of tidal rectification in Eulerian and Lagrangian current measurements from Georges Bank, two of the following complementary approaches seem obvious:
(i) Spatial and temporal variations in the observed mean Eulerian and Lagrangian currents can be compared with those predicted by theory.

(ii) Spatial variations in the observed tidal current regime can be compared with theoretical predictions.

However, a major problem is that the current measurements are Eulerian or Lagrangian measurements at only a few depths in the water column so that, if there is baroclinicity in the currents, estimates of the depth-averaged currents are lacking. Since the present theory is depth-averaged, the above approaches are most useful for current measurements in homogeneous water.

A detailed comparison with observations will not be attempted here, since the recent observations made by other investigators have not yet appeared in the literature. Nevertheless, as noted in Section 1, the observations are consistent with a mean clockwise circulation around the Bank, with particularly persistent along-isobath flows on its northwestern and northern sides, and some suggestion of strongest flow over sloping bottom topography. A key observation on the northwestern side (Spiegel and Magnell, 1979) has been that the magnitude of the mean Eulerian velocity at a particular depth is dependent on the tidal current amplitude at that depth, as would be predicted by a depth-dependent tidal refraction theory. The observation that variations in both the mean and tidal Eulerian currents at this location occur on non-tidal time scales of about five days (Spiegel and Magnell, 1979) may well be due to wind or other forcing altering the density structure and hence the baroclinic component of the tidal currents.

An alternative approach in assessing the relevance of the present theory to Georges Bank is to examine whether the major assumption that along-isobath variations in the tidal regime can be neglected is consistent with Greenberg’s (1979) linear numerical model, in which along-isobath variations are allowed. The cross-isobath component of tidal velocity in the model has already shown that there is some reduction in the volume flux over the Bank (which will result in some along-isobath variations), but the predicted flows in Fig. 6 are based on the reduced flux. The along-isobath component of velocity, together with the cross-isobath component, can be used to calculate cross-isobath changes in both the tidal current ellipses and the tidal stress for comparison with those predicted by the present refraction theory. The premise is then that, although Greenberg’s model does not include the nonlinear advective terms and therefore cannot reproduce the details of the mean and tidal velocity fields on the side of the Bank (which is typically within one or two grid squares), it should reproduce the overall change in tidal regime between the deep and shallow regions.

Fig. 7 shows the predicted tidal ellipses (from the present theory) for the deep and shallow regions on the northwestern, northern and open ocean sides of the Bank. The deep-water ellipses are obtained directly from \( U_d \), \( R \) and \( \phi \), while the shallow-water ellipses are obtained from (7b) and (14). The dominant tidal effects associated with Coriolis rectification are apparent. The cross-isobath and along-isobath excursions are much greater in the shallow region due to continuity and Coriolis acceleration, respectively. Typical deep- and shallow-water ellipses on the northwestern side of the Bank in Greenberg’s model are shown in Fig. 8. These ellipses exhibit a similar along-isobath amplification between the deep and shallow regions as do the predicted ellipses of Fig. 7a, and the same is true for ellipses on the northern and open ocean sides of the Bank. The observed tidal ellipses on Georges Bank (see, e.g., Bumpus, 1976) are very similar to the shallow-water ellipses of Figs. 7 and 8, confirming that the continuity and Coriolis effects involved in Coriolis rectification do actually occur around the Bank.

The tidal currents in Greenberg’s model can be used to calculate local values of the depth-averaged tidal stress tensor, from which divergences of tidal stress across each grid square can be estimated. The along-isobath mean momentum equation (in the present notation) in which these divergences appear is

\[
\frac{1}{4} \frac{\partial}{\partial x} (H u_* v^* ) + \frac{1}{4} \frac{\partial}{\partial x} (H u_* v_1^-) + \frac{1}{2} \frac{\partial}{\partial y} (H u_* v_1^-) = -k \tilde{v}, \quad (42)
\]
where the cross-isobath and along-isobath divergences of tidal stress are assumed to drive an along-isobath mean current. An integrated form of this equation,

\[-\frac{1}{2} \left[ \frac{\partial}{\partial t} \right] \left[ \mathbf{H} \mathbf{u}_1 \right] + \frac{1}{2} \left[ \mathbf{H} \mathbf{u}_1 \right] \left[ \partial \right] \mathbf{v}_1 = \int_{x_1}^{x_2} k \mathbf{d}x, \tag{43} \]

can be used to calculate the excess divergence of tidal stress across each grid square (the left-hand side), which then provides an estimate of the integral of bottom stress for comparison with that given by (17) for the predicted flows of Fig. 6. Such a comparison for the northwestern side of the Bank reveals integral values over one or two grid squares to be in the range of \((2-5) \times 10^6 \text{ cm}^3 \text{ s}^{-2}\), which compares quite favorably with the value of \(3.98 \times 10^6 \text{ cm}^3 \text{ s}^{-2}\) for the predicted flow of Fig. 6a.

The conclusion is then that the tidal dynamics involved in the Coriolis mechanism are occurring in Greenberg’s linear numerical model, in which along-isobath variations are allowed, and also appear to be occurring in the observed tidal regime on Georges Bank. A first examination of the resulting divergences of tidal stress in Greenberg’s model reveals them to be of the same order as those required to drive the mean flows predicted in this paper. However, a prediction of the spatial details of the mean circulation clearly requires a nonlinear numerical model with smaller grid size.

6. Conclusions

For a long time it has been realized that the strong tidal currents on Georges Bank may contribute to its mean circulation and hydrography, largely through tidal mixing and the resulting horizontal density gradients. However, the forcing mechanisms of the persistent clockwise circulation around the Bank have been somewhat uncertain. It is found in this paper that there is a firm theoretical basis (and also some observational support) for expecting that a major year-round contribution to this observed circulation comes from the rectification of the M₂ tidal currents on the Bank’s sloping sides.

A rectification process involving a combination of continuity and Coriolis effects, but regulated by bottom friction (Huthnance, 1973), is expected to occur, so that the nonlinear advective terms in the Eulerian equations of motion are important. The degree of nonlinearity is dependent on the ratio of cross-isobath tidal excursion to topographic length scale. On the sides of Georges Bank, these length scales are of the same order and the rectification is sufficiently nonlinear that interaction between the tidal current and the resulting mean current is important. The result is an along-isobath mean Eulerian current over the edge of the Bank, such that the net bottom stress on the mean current balances the net divergence of tidal stress in the along-isobath momentum balance. As a result of the cross-isobath gradients in both the mean and tidal velocities, a significant Stokes velocity is associated with the rectification process, so that the mean Lagrangian current speed is only about two-thirds of the mean Eulerian current speed. The cross-isobath mean momentum balance requires a significant cross-isobath mean sea surface slope.

The associated depth-averaged flows around Georges Bank are predicted to be strongest on its northwestern and northern sides where mean Eulerian current speeds of up to 0.23 and 0.13 m s⁻¹, respectively, are expected, with weaker flows expected on its other sides. However, a precise prediction of the magnitude of these flows requires more accurate representations of bottom topography and bottom friction, allowance for along-isobath variations, and inclusion of interactions with flows of other origin. In the presence of density structure, significant baroclinicity may be present and a depth-dependent theory is required before comparison with current measurements at a specific depth is appropriate. Depth-dependent effects are expected to be especially important to the cross-isobath flows so that the significance of tidal rectification to the exchange of water on Georges Bank remains uncertain.

However, the importance of the strong tidal currents is unmistakable. Not only can strong mean flows of purely tidal origin be expected, but the excursion of water parcels due to the tidal currents may enhance interactions with flows of other origin, as well as result in Stokes velocities. Tidal rectification should also be contributing to the mean circulation elsewhere in the Gulf of Maine-Bay of Fundy system, as well as in other shallow seas where there is strong tidal flow across steeply sloping bottom topography.

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