A Numerical Study of Nonfrictional Decay of Mesoscale Eddies

D. C. SMITH IV

Mesoscale Air–Sea Interaction Group, Florida State University, Tallahassee 32306

R. O. REID

Department of Oceanography, Texas A&M University, College Station 77843

(Manuscript received 27 May 1981, in final form 15 December 1981)

ABSTRACT

The decay of mesoscale eddies can be attributed to either frictional dissipation of kinetic energy through viscous effects or through dispersive spreading of the different constituent Rossby wave components at their own characteristic wave speeds. Several previous investigations of eddy decay have examined the role of variable friction in the spindown process. In addition to frictional results, these studies have shown that nonlinear advective processes can stabilize the vortex against dispersive effects. The quantification of this relation between nonlinear stabilization and beta dispersion is the primary focus of this paper.

Results are obtained using a finite difference “equivalent barotropic” numerical model with a fixed biharmonic friction formulation. Variable parameters in the study are vortex size and strength. Initial conditions are in the form of a Gaussian height field in gradient balance. Nonfrictional vortex decay is parameterized in terms of lateral spreading. This spreading is determined by the rate of increase of the second radial moment weighted by potential energy density. Estimates are made for the time required for this length to double in magnitude. Moments based on other weightings are also investigated.

1. Introduction

The existence of intense, quasigeostrophic, mesoscale oceanic vortices which derive from major currents such as the Gulf Stream has been documented by numerous investigators (Fuglister, 1977; Lai and Richardson, 1977). The formation process as related to frontal meandering is well accepted (Hansen, 1979; Olson, 1979). Due to their long lifetimes (order of several years) and large numbers (10–14 per year for the Gulf Stream) they may be important in the overall heat and energy budgets of a given oceanic region. By virtue of their formation process, eddies represent an efficient mechanism by which heat, salt and momentum are transferred across frontal zones which otherwise act as barriers to mixing between different water masses. For these reasons it is of interest to study the motion and decay characteristics of these mesoscale features.

The motion of isolated vortices has been examined analytically (Bjerknes and Holmboe, 1944; Warren, 1967; Flierl, 1977) experimentally (Firing and Beardsley, 1976) and numerically (McWilliams and Flierl, 1979; Mied and Lindemann, 1979). These studies show a westward component of motion for eddies to be the result of the beta effect with a meridional component of motion induced by nonlinear advective effects.

The decay of isolated vortices has been examined analytically by Flierl (1977) for linear vortices, and numerically by McWilliams and Flierl (1979) and Mied and Lindemann (1979) for more intense structures. Flierl (1977) shows that the linear vortex in the absence of friction decays rapidly due to dispersive effects on a beta plane. For stronger, larger amplitude vortices, McWilliams and Flierl (1979) and Mied and Lindemann (1979) show that nonlinearity stabilizes the eddy against beta dispersion and allows the eddy to propagate as a stable unit for longer periods of time than their linear counterparts. A closer examination of this relation between nonlinear eddy strength and beta dispersion is the primary focus of this paper.

The decay or spindown of an eddy has been estimated several ways. Observationally estimates have been based on subsidence of isotherms (Vastano et al., 1980) or decay of available potential energy (APE) (Cheney and Richardson, 1976). Subsidence of isotherms as measured by decay of central peak amplitude is not the best measure of eddy spindown as this value can actually increase in association with shape changes even though the vortex is decaying energetically.

An alternative more suitable measure of vortex decay is proposed here. Lateral spreading as deter-
mined by rate of increase of a second radial moment weighted by potential energy gives a measure of non-
frictional-vortex decay. We determine this spreading rate from a series of numerical experiments using a
primitive equation "equivalent barotropic" model with variable eddy size and strength as input vari-
ables.

In Section 2 we will present the details of the nu-
merical model formulation, parameter selection, and
model verification. The initialization of the experi-
ments and the results pertaining to motion and decay of
isolated vortices are discussed in Section 3. A dis-
cussion of these results in view of previous studies
follows in Section 4 with conclusions in Section 5.

2. The model

a. Numerical technique

The equations of motion which govern a two-layer
oceanic system with the lower layer at rest are taken as
\[ \frac{\partial U}{\partial t} + \nabla \cdot \left( QU/H \right) - fV = 0, \quad (1.1) \]
\[ \frac{\partial V}{\partial t} + \nabla \cdot \left( QV/H \right) + fU = 0, \quad (1.2) \]
\[ \frac{\partial H}{\partial t} + \nabla \cdot \mathbf{Q} = 0, \quad (1.3) \]
where \( \mathbf{Q} = U \mathbf{i} + V \mathbf{j} \) is the vector transport and \( H \)
the thickness of the upper layer. The equations are
hydrostatic and Boussinesq. The coordinate system
is Cartesian with \(+x\) to the east and \(+y\) to the north.
Appendix A gives definitions of all terms and con-
stants used in this study. The model is not capable of
addressing the role of bottom topography or fric-
tion, surface wind stress, or vertical modal interac-
tions. Nor will the effects of mean flow advection,
ring–Gulf Stream or ring–ring interactions on eddy
motion or decay be examined.

Lateral friction is represented by a biharmonic
dissipation rather than the traditional Laplacian
form. The biharmonic operator, now in common
usage (Semtner and Mintz, 1977; Holland, 1978),
dissipates energy at high wavenumbers preventing
numerical instability and leaving the geostrophic
scales of motion relatively undamped. The selection of
the friction coefficient value is discussed below.

The numerical scheme employs centered differ-
ences in time (leapfrog) and space on a spatially
staggered grid, except for friction terms which are
lagged in time for numerical stability. The advection
scheme employed corresponds to scheme C of Gram-

The horizontal boundary conditions are those of
a closed system (i.e., no slip walls). The unrealistic
nature of these conditions is reduced by a coordinate
transformation which removes the boundaries to a
more remote position in the far field, away from the
central computing region. The stretching functions
used to transform to a system with variable \( \Delta x, \Delta y \)
grid spacing is
\[ \xi(x) = A_0 \tanh kx, \]
\[ \eta(y) = A_0 \tanh ky, \]
where \( x = y = 0 \) at the center of the region being
modeled. \( A_0 \) and \( k \) are defined in Appendix A. The
transformation effectively increases the domain of
integration compared to that of a uniform grid hav-
ing the same number of grid points (51 \times 51 in the
present case). As eddy size is increased in some ex-
periments the relative resolution is maintained by
proportional increases in the central grid size, \( \Delta s \),
thus allowing proportionally larger time steps. The
domain of integration is a square region 660 km
per side (centered on a latitude of 37°N) and the
simulations are 300 days for the nonlinear eddy cases
where \( \gamma \) is defined below.

b. Parameter selection

The parameters of interest are those which give a
measure of vortex size and vortex strength. As a
measure of size we take the nondimensional ratio
\( \gamma = L/R_d \) for radial length scale \( L \) and Rosby radius
of deformation \( R_d \). Vortex strength is parameterized
by \( Q^* = V_0/\beta R_d^2 \) for maximum tangential velocity
\( V_0 \). Unless otherwise noted the following discus-
sions involving vortex and strength will imply varia-
tions in \( \gamma, Q^* \). A range of \( \gamma, Q^* \) parameters is selec-
ted to represent a range of observed Gulf Stream ring
sizes and strengths. The Rossby radius of deforma-
tion is taken as fixed \( (R_d = 50 \text{ km}) \), which implies
a far-field depth \( (H_{\text{max}}) \) which is roughly repre-
sentative of the far-field depth of the 10° isotherm in
the Sargasso Sea. Fig. 1 shows graphically the dis-
tribution of the selected parameters. The nondimen-
sional \( Q^* \) values of 6.7, 13.3, 26.7 correspond to
dimensional maximum tangential velocities of 0.3,
0.6, 1.2 m s\(^{-1}\) since \( R_d \) is fixed.

Table 1 compares the eddy size and strength pa-
rameters chosen here with those of previous studies
and contrasts the various methods of investigation
used. This study includes a wider range of values
than the two former numerical studies (McWilliams
and Flierl, 1979; Mied and Lindemann, 1979) and
has a strength value \( (Q^*) \) which is not limited in
magnitude by the quasigeostrophic assumption
(McWilliams and Flierl, 1979). Where possible,
model results here will be compared with those of
these previous studies when input parameters are
similar in value. Although both of these former in-
vestigations have allowed vertical modal interac-
tions, several upper-ocean eddy cases of Mied and Lin-
demann (1979) and the single mode results of Mc-
Williams and Flierl (1979) are similar in nature to cases here.

The friction coefficient $B$ was chosen so as to maintain a cell Reynolds value less than 100 and to give equal energy decay rates for the range of eddy sizes. For Gaussian eddies with biharmonic dissipation, Smith (1980) shows (his Appendix B-4) that

$$-1/Ed(KE)dt = 12B/L^4,$$

implying that $B$ should be a function of $\Delta S^4$. This scaling however resulted in an accumulation of kinetic energy in the largest ($\gamma = 4$) case. An additional $\gamma = 4$ case was performed with a friction coefficient $B$ proportional to $\Delta S^5$. This frictional scaling is that of McWilliams and Flierl (1979). They scale time by $1/\beta L$, so that $B = B_{\text{dimensional}}/\beta L$. Although it is not the purpose of this study to examine energy transfer rates or variable friction, this scaling for the $\gamma = 4$ case gave dissipation rates (% per month) values which were more consistent with smaller eddy simulations. For $\gamma = 1$ eddies, the McWilliams and Flierl (1979) friction coefficient is nearly equivalent to the value chosen here.

**c. Model verification**

A number of tests were made to assure proper behavior of the numerical model. These tests consist of comparison of independent analytic solutions with linear numerical simulations ($Q^* \ll 1$). Any distortion effect of the stretched coordinated system on the results was found to be negligible for the time scales considered here. The conditions of volume conservation are met in all simulations using double precision computations on an Amdahl 470. The numerical scheme also conserves energy and enstrophy in the absence of friction.

The proper translation of an eddy is tested by a number of linear simulations. The westward propagation speeds of the vortex center ($V_c$), computed for a range of Gaussian ring sizes, are found to be in agreement with the analytic predictions of Flierl (1977). In addition, the translational speeds of the

<table>
<thead>
<tr>
<th>Size $L/R$</th>
<th>Strength $V_0/(\beta L^3)$</th>
<th>Description of model</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>0</td>
<td>Present study primitive equation</td>
</tr>
<tr>
<td>0.50</td>
<td>0</td>
<td>&quot;reduced gravity&quot; numerical model</td>
</tr>
<tr>
<td>1</td>
<td>0, 6.67, 13.34, 26.67</td>
<td>McWilliams and Flierl (1979) quasigeostrophic two-mode numerical model</td>
</tr>
<tr>
<td>2</td>
<td>0, 1.05, 3.33</td>
<td>Mied and Lindemann (1979) primitive equation two-layer numerical model</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>Bretherton and Karweit (1975) quasigeostrophic six-layer numerical model</td>
</tr>
<tr>
<td>4</td>
<td>0, 0.41</td>
<td>Firing and Beardsley (1976) laboratory model and one-layer numerical model</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>
centroids of potential and kinetic energy are in agreement with those predicted by linear solutions for these moments. Energy centroids are defined here as the first statistical moment, weighted by either potential or kinetic energy. Fig. 2 shows the propagation speeds ($V_p$ and $V_k$) of these first moments for a range of linear Gaussian vortex sizes as derived analytically. The propagation speeds of the linear numerical simulations and the analytic results of Flierl (1977) are also shown in the figure. A mass centroid can be defined in an analogous manner. For linear Gaussian eddies the mass centroid propagates to the west at the maximum Rossby wave phase speed, $c_m (=\beta R^2)$ as derived in Appendix B. Fig. 3 shows contours of upper layer thickness for a typical linear simulation. The dispersive nature of the different Rossby wave components is evident in the east-west asymmetry which develops with time.

The proper decay of the linear Gaussian vortex is examined by comparison with analytic theory for spreading rate. The increase in ring size is determined by the increase in variance associated with the energy moments. This rate of increase is defined by $F = (\sigma^2 - \sigma_0^2)/\sigma_0^2 t^2$ where $\sigma^2$ is the variance associated with the potential or kinetic energy field at a given time, and $\sigma_0^2$ is the initial variance for that energy moment. For the linear Gaussian ring this rate of spreading is a function of vortex initial size (see Appendix B). Fig. 4 shows a comparison of numerical and analytic results. The spreading rates deduced from the linear Gaussian simulation are in reasonable agreement with the analytic results except for large vortices ($\gamma > 3$). From the figure it is evident that the spreading rate for potential energy is maximal near $\gamma = 1$, whereas the kinetic energy spreading rate has its peak shifted toward larger eddies. Small ($\gamma < 1.6$) vortices have a potential energy spreading rate which is greater than that for kinetic energy but for larger eddies the converse is true. Although discussion here has focused on spreading rates for ring energies, a spreading rate for enstrophy can also be derived (see Appendix B for that result).

3. Experiments

a. Initialization

The numerical simulations reported here are initialized with a Gaussian height field representation of a mesoscale vortex. The Gaussian initial upper layer thickness $H$ is given by

$$H(r) = H_{\text{max}} - A \exp(-r^2/2L^2),$$

for radial distance $r$, ring amplitude $A$ and radius $L$ to maximum gradient of $H$. The corresponding velocity field is taken to be in gradient balance. The properties of Gaussian vortices have been contrasted with those of observed Gulf Stream rings by Olson (1980). Previous analytic studies (Flierl, 1977) and numerical experiments (McWilliams and Flierl, 1979; Mied and Lindemann, 1979) have concentrated on Gaussian initial profiles.

b. Vortex motion

The propagational velocities for nonlinear Gaussian vortex centers observed in simulations here are in qualitative agreement with those of previous investigators (McWilliams and Flierl, 1979; Mied and Lindemann, 1979). As for the linear simulations, we report velocities for vortex mass and energy centroids as well. Unlike the linear vortex centroid velocities which exhibited a time-independent nature, centroid
FIG. 3. Contoured upper-layer thickness evolution for a linear (1, 0) Gaussian cyclonic eddy. Contour values are percentages of initial ring central amplitude. Positive (negative) values represent upward (downward) deviations of the thermocline from mean thickness.
velocities in the nonlinear simulations showed a time dependency. The general trend is for the initially different centroid velocity components to increase with time while approaching single, time-independent values. This time dependency indicates an adjustment period between initial conditions and a presumably preferable “equilibrium” state. It should be noted that this “convergence time” is a function of vortex strength and size with stronger nonlinear cases and smaller vortices equilibrating more rapidly. This time-dependent behavior is also noted by McWilliams and Flierl (1979). The tendency for the various centroid velocities to have a common value for nonlinear rings is also indicative of the resistance to dispersion that nonlinear vortices exhibit.

Fig. 5 shows “final” westward velocities as a function of vortex strength and size. Stronger vortices propagate to the west with higher velocities which are limited in magnitude by a maximum Rossby phase speed \( c_m = -3.95 \text{ km day}^{-1} \) in this study. Also for a given Rossby number, larger vortices (\( \gamma = 2 \)) propagate westward faster. “Final” meridional (northward, for the cyclones here) velocity components for the nonlinear vortices are shown in Fig. 6. Vortex size has a more apparent effect than strength, with larger rings having slower meridional velocities.

McWilliams and Flierl (1979) define a limiting northward propagation speed for cyclonic eddies (\( \frac{\nu}{\beta R_0} = 0.99 \text{ km day}^{-1} \)) here with which our results comply. The upper ocean eddy results of Mied and Lindemann (1979) (their cases 10–11) were not limited by this meridional velocity, although the reason for this discrepancy is not certain.

c. Vortex decay

As previously discussed for the linear vortices, spindown can be parameterized in terms of spreading
as determined by rate of increase of potential energy variance for nonlinear Gaussian vortices. This spreading rate for potential energy

$$\frac{\Delta \sigma^2}{\sigma_0^2} = \frac{(\sigma_{PE}^2 - \sigma_E^2)}{\sigma_0^2}$$

(2.1)

is determined from the numerical simulations to be a function of time of the general form:

$$\Delta \sigma^2/\sigma_0^2 = 3(t/t_0)^n,$$

(2.2)

where \(t_0\) is the time for \(\sigma_0\) to reach 2\(\sigma_0\).

The time histories of this spreading rate as computed in the nonlinear Gaussian simulations are shown in Fig. 7. Power laws of the form (2.2), determined by the slopes of the asymptotic portions of the curves, give spreading rates for given vortex size and strength combinations. The analytic slopes for linear rings (\(n = 2\)) are shown in the figure for comparison.

The resulting power law exponents are shown graphically in Fig. 8. These relations quantify earlier conclusions that vortex decay through dispersion is inhibited by nonlinear advective processes in the eddy. Vortex potential energy spreading rate decreases with increasing nonlinear strength (\(Q^*\)) for a given vortex size (\(\gamma\)). These results suggest that potential energy spreading rate cannot be parameterized by Rossby number \(R_o\), as different size and strength vortices with the same \(R_o\) value have different spreading rates.

From the spreading rate functions determined above it is possible to estimate an effective ring half-life (\(t_2\)), or the time required for a ring to increase its potential energy variance by a factor of two. Estimates of \(t_2\) are also shown in Fig. 8. Half-life increases with ring size for a given strength \(Q^*\), and for a given size (\(\gamma\)) increases with strength.

Since this study does not include the effects of surface wind mixing, ring-ring (or ring-Stream) interaction or bottom effects these ring lifetimes are not meant to be used for forecasting ring decay but to illustrate the role of nonlinear ring strength in the ring decay process. The combined effect of these additional factors may alter ring lifetimes significantly.

4. Discussion

The study of motion and spindown characteristics of Gulf Stream rings has been the central issue of a number of investigations using numerical, analytic, and laboratory techniques as discussed in the introduction. The comparison of results obtained in this study with these former investigations is discussed in this section.

The comparison of linear Gaussian simulations performed here with the results of analytic studies of Gaussian vortices (Flierl, 1977) gives excellent quantitative agreement. This agreement gives a measure of confidence in the numerical methods used here and insures proper model behavior.

Several numerical studies of Gulf Stream vortices using different numerical techniques provide direct comparison of numerical results. These previous studies are based on a quasigeostrophic two-mode model (McWilliams and Flierl, 1979) and a primitive equation two-layer model (Mied and Lindemann, 1979). Both of these investigations show that the lower-layer energy disperses rapidly for nearly baro-

![Fig. 7. Potential energy variance spreading rates vs. time for all \(\gamma = 1, 2, 4\) eddy simulations. The time scale is offset for each of the size classes for purposes of comparison. The parameters in parentheses are size (\(\gamma\)) and strength (\(Q^*\)) values, and for the \(\gamma = 4\) cases, the third parameter identifies the manner in which frictional dissipation is scaled (see text).](image-url)
tropic eddies leaving the upper-layer vortex to propagate independently giving some justification for a study of upper ocean eddies as this one. Several of their experiments are directly comparable with those of this study. There exists a qualitative if not quantitative agreement between these experiments. The magnitude of the westward propagation speed for ring centers of a range of vortex strengths is in good agreement with both of these studies. Whereas Mied and Lindemann (1979) used a fixed ring size ($\gamma$), agreement with McWilliams and Flierl (1979) is found in the fact that zonal velocity increases with ring size as well as being limited by a maximum Rossby phase speed.

It is generally agreed that nonlinear advective effects induce a meridional component of motion of ring translation and stabilize the ring against internal dispersion. Ring decay is thus impeded. The result here that the magnitude of the meridional component decreases with vortex size and is relatively eddy-strength independent (once $Q^*$ is sufficiently large) is in agreement with McWilliams and Flierl (1979). In addition, its magnitude is limited by a meridional Rossby phase speed ($=1/4\beta R_d^2$). They note a tendency for this component of motion to diminish for very long times ($V_e \rightarrow 0$ as $t \rightarrow \infty$) presumably as the ring decays to a linear state. That trend was not observed in this study possibly due to shorter duration of simulations here. Mied and Lindemann (1979), however, predict much larger northward velocities ($>1/4\beta R_d^2$) for upper ocean eddies which increase with increasing ring strength. Increased fractional barotropic ratio (Mied and Lindemann, 1979) or bottom topography (Smith and O'Brien, 1982) can significantly alter ring paths from the above observed trends.

Lateral frictional dissipation in the model of Mied and Lindemann (1979) is based on the Laplacian operator as opposed to the more scale-selective biharmonic dissipation of McWilliams and Flierl (1979) and in the present study. The tendency for biharmonic dissipation to leave the large-scale geostrophic motion relatively undamped while dissipating energy at higher wavenumbers may account for this difference in observed motion, although McWilliams and Flierl (1979) find that vortex propagation is insensitive to the value of the friction coefficient chosen. Both Mied and Lindemann (1979) and McWilliams and Flierl (1979) conclude that changes in viscosity are more important in determining ring decay rates than are changes in ring strength.

Firing and Beardsley (1976) use both a laboratory and a numerical model to study a barotropic eddy on a beta plane. Their results show a northwestward motion associated with the dominant cyclonic vortex, but a pronounced southward motion of the trailing anticyclone. This southward motion was not observed in simulations here. They attribute their observed circulation to competition between opposing mechanisms: potential vorticity conservation and a vorticity segregation mechanism which gives water columns with positive relative vorticity a tendency to move northward and those with negative relative vorticity a tendency to move south. The anticyclones which develop here are apparently of insufficient nonlinear strength to have a southward component of motion.

5. Conclusions

Estimates are made of eddy dispersion rates for a range of mesoscale oceanic vortices from numerical

![Fig. 8. Potential energy variance spreading rate asymptotic power law exponents and associated eddy "half lives."](image-url)
simulations using a primitive-equation, reduced gravity model with biharmonic friction. Dispersion is estimated in terms of the change in the second moment length scales associated with potential energy. Results quantify earlier numerical findings (McWilliams and Flierl, 1979; Mied and Lindemann, 1979) that nonlinear advection effects tend to stabilize eddies against internal dispersion. The numerical results derived here seem to confirm time scales similar to those known for observed rings.

Results have implied that Gulf Stream ring lifetimes should be strongly dependent on ring size and ring strength. Another important result is that the effect of these two influencing parameters on ring spin down cannot be parameterized in a single non-dimensional Rossby number. Wilkinson (1972) examined a range of oceanic geostrophic eddies and found their Rossby number values to be surprisingly constant, even though they ranged considerably in size, from 50 to 300 km in diameter. However, despite the fact that oceanic eddies have similar Rossby numbers, this work does not imply that they have similar decay rates.

Acknowledgments. The research reported in this paper was conducted at Texas A&M University under a grant from the Office of Naval Research (Contract NOOO14-75-0537) and was written in final form at Florida State University under ONR Contract NOOO14-80-C-0076.

APPENDIX A

List of Symbols Used in Text

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U$</td>
<td>$x$-directed component of transport</td>
</tr>
<tr>
<td>$V$</td>
<td>$y$-directed component of transport</td>
</tr>
<tr>
<td>$Q$</td>
<td>vector transport in Cartesian coordinates (= U i + V j )</td>
</tr>
<tr>
<td>$H$</td>
<td>upper layer thickness</td>
</tr>
<tr>
<td>$V_0$</td>
<td>maximum tangential velocity for a given eddy</td>
</tr>
<tr>
<td>$H_{\text{max}}$</td>
<td>far-field upper layer thickness (962 m)</td>
</tr>
<tr>
<td>$g'$</td>
<td>reduced gravitational acceleration for upper and lower layer densities $\rho_1$ and $\rho_2$ respectively (0.02 m$^2$ s$^{-1}$) (= (\rho_2 - \rho_1) / \rho_1 )</td>
</tr>
<tr>
<td>$f$</td>
<td>Coriolis parameter (= f_0 ) value at ring center</td>
</tr>
<tr>
<td>$\beta$</td>
<td>variation of Coriolis parameter with latitude ((1.83 \times 10^{-11} \text{ m}^{-1} \text{ s}^{-1}))</td>
</tr>
<tr>
<td>$R_d$</td>
<td>first baroclinic Rossby radius of deformation (=[g' H/f_0 = 50 \times 10^3 \text{ m}])</td>
</tr>
<tr>
<td>$L$</td>
<td>radius from vortex center to maximum tangential velocity</td>
</tr>
<tr>
<td>$Q^*$</td>
<td>nondimensional strength parameter (= V_0 / \beta R_d^2 )</td>
</tr>
<tr>
<td>$R_o$</td>
<td>nondimensional strength parameter (= V_0 / f L )</td>
</tr>
</tbody>
</table>
| $\sigma_E$ & $\sigma_E^2$ | initial value of $\sigma_E$ same as $\sigma_E$ for potential energy maximum Rossby wave phase speed \((3.95 \text{ km day}^{-1} \text{ in this study}) \Rightarrow \beta R_d^2 \) uniform computational grid increments uniform computational grid increments nondimensional size parameter \(= L / R_d \) coordinate stretch function coefficient \((27)\) coordinate stretch function wavenumber \(= \gamma x(233 \text{ km}^{-1}) \) biharmonic friction coefficient \(= \gamma x(0.6 \times 10^{10} \text{ m}^{-2} \text{ s}^{-1}) \) westward velocity of linear ring center westward velocity of linear ring potential energy centroid westward velocity of linear ring kinetic energy centroid westward velocity of linear ring mass centroid.

The variances associated with the spatial distributions of energy are defined by the second statistical moment

$$\sigma^2 = \int \int (x - \bar{x})^2 + (y - \bar{y})^2 W \, dx \, dy / \int \int W \, dx \, dy$$

where $W$ is the weighting factor, which is $h^2$ for potential energy or $(U^2 + V^2) / H$ for kinetic energy

\[ \text{or } (u^2 + v^2) / H \text{ in terms of velocities}. \]

APPENDIX B

Horizontal Moments for Linear Rings

The anomaly of the upper layer thickness \(h = H_m - H\), under the constraint of linear, frictionless, quasi-geostrophic flow is governed by the following simple vorticity equation

$$\frac{\partial}{\partial t} (\nabla^2 h - \gamma^2 h) + \frac{\partial h}{\partial x} = 0,$$  \(\text{(B1)}\)

where $x, y$ are scaled by the ring scale $L$, $t$ is scaled by $(\beta L)^{-1}$, and $\gamma = L / R_d$. We assume that $h$ is initially Gaussian

$$h = A \exp\left[-\frac{1}{2}(x^2 + y^2)\right] \text{ at } t = 0.$$  \(\text{(B2)}\)

The solution of (B1) satisfying (B2), with bounded $h$ in the far field, is readily shown to be given by

$$h(x, y, t) = \frac{A}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2} (\mu^2 + \nu^2) \right] + i(\mu x + \nu y + \omega t) \, d\mu \, d\nu,$$  \(\text{(B3)}\)

where $\omega$ is a non-dimensional frequency (scaled by $\beta L$) for which

$$\omega = \frac{\mu}{\gamma^2 + \mu^2 + \nu^2}.$$  \(\text{(B4)}\)

The latter is a non-dimensional form of the Rossby
dispersion relation, in which \( \mu, \nu \) are \( x, y \) components of wavenumber scaled by \( L^{-1} \).

The evolving field of \( h \) can be characterized by its geometrical moments or by those of some transform of \( h, w = w(h) \). We define a moment generating function for \( w(h) \) by

\[
M(\mu, \nu, t) = N^{-1} \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w \exp[-i(\mu x + \nu y)]dx dy, \quad (B5)
\]

where \( N \) is the norm

\[
N = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w^2 dx dy,
\]

such that \( M(0, 0, t) = 1 \). The function \( M \) has the property that

\[
\bar{x}^m y^n = N^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^m y^n w^2 dx dy
\]

\[
= \int_{-\infty}^{\infty} \frac{\partial^{m+n} M}{\partial \mu^n \partial \nu^m} \bigg|_{\mu=\nu=0} \quad (B6)
\]

Of particular interest are the first moments \( \bar{x}, \bar{y} \), which characterize the (time dependent) centroid of \( w(h) \), and the radial second moment

\[
\sigma^2 = \bar{x}^2 - (\bar{x})^2 + \bar{y}^2 - (\bar{y})^2, \quad (B7)
\]

which characterizes the horizontal dispersion. Logical candidates for \( w(h) \) are

\[
\begin{align*}
w_0 &= h \\
w_1 &= h^2 \\
w_2 &= \nabla h^2 \\
w_3 &= (\nabla^2 h - \gamma^2 h)^2
\end{align*}
\]

which lead to moments based on the fields of mass \( (m) \), potential energy \( (p) \), kinetic energy \( (k) \) and enstrophy \( (\varepsilon) \), respectively. The simplest of these to evaluate is that for mass; however, those based on energy or enstrophy are more meaningful since their \( w \) functions are positive definite.

**a. Mass moments**

We consider this case mainly for its unique properties. The moment generating function for \( w = h \), which we denote \( M_m \), is readily shown from (B3) and (B5) to be simply

\[
M_m = \exp[-\frac{1}{2}(\mu^2 + \nu^2) + i \omega(\mu, \nu)t]. \quad (B9)
\]

Using (B4) and (B6) we find

\[
\bar{x}_m = -t/\gamma^2, \quad \bar{y}_m = 0, \quad \sigma_m^2 = 2, \quad (B10)
\]

which indicates a constant westerly drift rate of

\[
\gamma^{-2} \text{ in dimensionless form (or } \beta R_d^2 \text{ in dimensional form) for the center of mass. This corresponds to the maximum possible Rossby speed } C_m^r.
\]

While the variance of the mass field is constant \( (2L^2 \text{ in dimensional form}) \), the third zonal moment about the mean is found to be

\[
\bar{\sigma}_m^3 = -6t/\gamma^4, \quad (B11)
\]

which indicates an increasing westward skewness with time.

**b. Potential energy moments**

Here we take \( w = h^2 \) and from (B3) it follows that

\[
h^2(x, y, t) = \frac{A^2}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} exp\left[-\frac{1}{2}(k_1^2 + k_2^2) + \frac{i}{2}(\mu_1 + \mu_2)x + (\nu + \nu_2)y\right.\]

\[
\left. + (\omega_1 + \omega_2)t\right] d\mu_1 d\nu_1 d\mu_2 d\nu_2, \quad (B12)
\]

where

\[
k_j^2 = \mu_j^2 + \nu_j^2 \quad \omega_j = \frac{\mu_j}{\gamma^2 + k_j^2}
\]

\[
j = 1, 2. \quad (B13)
\]

From (B5) it follows that the associated moment function \( N_P \) is given by

\[
N_P M_P(\mu, \nu, t) = \frac{A^2}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial}{\partial \mu_1} \frac{\partial}{\partial \nu_1} \delta(\mu_1 + \mu_2 - \mu)
\]

\[
\times \delta(\nu_1 + \nu_2 - \nu) d\mu_1 d\nu_1 d\mu_2 d\nu_2, \quad (B14)
\]

where \( \delta \) is the Dirac delta function. The norm \( N_P \) is found from the requirement that \( M_P = 1 \) for \( \mu = \nu = 0 \), for which (B14) reduces to

\[
N_P = \frac{A^2}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} exp(-k^2) d\mu_1 d\nu_1 = \pi A^2.
\]

To reduce the integral in (B14) for general \( \mu, \nu \), we introduce the transformations

\[
\begin{align*}
\mu_1 + \mu_2 &= \mu', & \quad \nu_1 + \nu_2 &= \nu' \\
\mu_1 - \mu_2 &= \xi, & \quad \nu_1 - \nu_2 &= \eta
\end{align*}
\]

for which the Jacobian is

\[
\frac{\partial(\mu_1, \mu_2)}{\partial(\mu', \xi)} \frac{\partial(\nu_1, \nu_2)}{\partial(\nu', \eta)} = \frac{1}{4}.
\]

The resulting relation for \( M_P \) is then

\[
M_P(\mu, \nu, t) = \frac{1}{4\pi} \exp\left[-\frac{1}{4}(\mu^2 + \nu^2) + \frac{i}{2}(\mu_1 + \mu_2)x + (\nu_1 + \nu_2)y\right]
\]

\[
\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} exp\left[-\frac{1}{2}(\xi^2 + \eta^2) + i \omega t\right] d\xi d\eta, \quad (B16)
\]

Unauthenticated | Downloaded 05/02/24 03:32 PM UTC
where
\[ \Omega = (\mu + \xi)(4\gamma^2 + (\mu + \xi)^2 + (\nu + \eta)^2) \div (\mu - \xi)(4\gamma^2 + (\mu - \xi)^2 + (\nu - \eta)^2) \div \] (B17)
The latter function vanishes for \( \mu = \nu = 0 \).
While it is not feasible to evaluate \( M_\rho(\mu, \nu, t) \) in closed form, there is no need to if we are concerned only with the lower-order moments. All that is needed is a power series expansion of \( M_\rho \) in \( \mu \) and \( \nu \) about \( \mu = \nu = 0 \), and this only needs to be carried to second degree to determine the first and second moments. The expansion of \( \exp(i\Omega t) \), accurate to second degree in \( \mu \) and \( \nu \) is
\[
\exp(i\Omega t) = 1 + i4t[\mu Q^{-1} - 2(\mu \xi^2 + \nu \xi \eta)Q^{-2}] - 8t^2[\mu^2(1 - 2\xi^2 Q^{-1})Q^{-2} - 4\mu \nu \xi \eta \xi^2 Q^{-4}] \\
\times (1 - 2\xi^2 Q^{-1})Q^{-3} + \frac{4t^3 \xi^2 \eta Q^{-4}}{3} + \text{(higher order in } \mu, \nu),
\] (B18)
where
\[ Q = 4\gamma^2 + \xi^2 + \eta^2. \]
Now letting \( \xi = \xi \cos \theta, \eta = \xi \sin \theta \) and transforming the integral in (B16) to polar \((\xi, \theta)\) form, the subsequent integration over \( \theta \) gives
\[
M_\rho = \frac{1}{2} \exp[-\frac{1}{4}(\mu^2 + \nu^2)] \int_0^\infty \exp(-\xi^2/4) \times \left(1 + i4t\mu Q^{-1} - \xi^2 Q^{-2} \right) \\
\times (1 - 2\xi^2 Q^{-1} + \frac{1}{3} \xi^4 Q^{-2})Q^{-2} \\
= \frac{1}{2} (\mu^2 + \nu^2) \exp(-\xi^2) + \text{(higher order in } \mu, \nu) \right) \right] \left( \int d\xi, \right)
\] (B18)
where \( Q = 4\gamma^2 + \xi^2 \). The remaining integration is facilitated by converting to an integration on \( Q \). The result is
\[
M_\rho = \exp[-\frac{1}{4}(\mu^2 + \nu^2)] \left[ 1 + t \frac{\mu}{\gamma^2} B_2(\gamma^2) \\
- \frac{t^2}{4\gamma^2} \mu[B_3(\gamma^2) - 2B_3(\gamma^2) + 3B_3(\gamma^2)] \\
- \frac{t^2}{4\gamma^2} \nu[B_3(\gamma^2) - 2B_3(\gamma^2) + 3B_3(\gamma^2)] \\
\text{(higher order in } \mu, \nu) \right], \quad \text{(B19)}
\] where
\[
B_n(z) = z^nE_n(z), \quad \text{(B20)}
\]
\( E_n(z) \) being the exponential integral. The function \( B_n \) has the recursion relation
\[
B_{n+1}(z) = \frac{z}{n}[1 - B_n(z)]
\]
and the asymptotic property
\[
B_n(z) \to 1 - \frac{n}{z} \quad \text{as} \quad z \to \infty.
\]
From (B19) we get finally
\[
\bar{x}_p = -\frac{t}{\gamma^2} B_2(\gamma^2), \quad \bar{y}_p = 0
\]
\[
\sigma_{p}^2 = 1 + \frac{t^2}{\gamma^2} \left[ B_3(\gamma^2) - 2B_3(\gamma^2) + 2B_4(\gamma^2) - 6B_4(\gamma^2) \right]
\] (B21)
for the non-dimensional first and second potential energy moments.
The westward translational speed of the centroid of potential energy for the linear, initially Gaussian ring is
\[
V_\rho = C_m B_2(\gamma^2),
\] (B22)
in dimensional form, where \( C_m = \beta R^2 \). This may be compared with the initial translational speed \( V_0 \) for the point of maximum \( h \), which Flierl (1977) shows is given by
\[
V_0 = C_n B_2(\gamma^2/2).
\] (B23)
For any finite \( \gamma, 0 < V_0 < V_\rho < C_m \).

The departure of \( \sigma_{p}^2 \) from its initial value is seen to increase as \( t^2 \) for a linear ring. In dimensional form
\[
\frac{(\sigma_{p}^2 - L^2)}{(C_mL^2)^2} = F_1(\gamma), \quad \text{(B24)}
\]
where \( F_1(\gamma) \) is the quantity in brackets in (B21). The maximum \( F_1(\gamma) \), and hence maximum dispersion rate for potential energy, is 0.113 and occurs for \( \gamma = 0.94 \) (i.e., for \( L = 0.94R_d \)).

c. Kinetic energy moments

In this case, \( w = |\nabla h|^2 \) and from (B3),
\[
|\nabla h|^2 = \frac{A^2}{(2\pi)^2} \int \int \int [u_1 u_2 + v_1 v_2] \\
\times \exp\{\frac{1}{2}(k_1^2 + k_2^2) + i\left[(\mu_1 + \mu_2)x \\
+ (v_1 + v_2)y + (\omega_1 + \omega_2)t \right] \} \, dx \, dy \, dz.
\] (B25)
Following the procedure employed for the potential energy moments we find the counterpart of (B16) is
\[
M_k(\mu, \nu, t) = \frac{1}{16\pi} \exp[-\frac{1}{4}(\mu^2 + \nu^2)] \int_0^\infty \int_0^\infty \left( \xi^2 + \eta^2 \right) \\
- \mu^2 - \nu^2) \times \exp[-\frac{1}{4}(\xi^2 + \eta^2 + i\Omega t)] \, d\xi \, d\eta.
\] (B26)
As before, we employ the expansion of \( \Omega \) and evaluate the integral term-by-term. The resulting first and second moments in non-dimensional form are
\[
\bar{x}_k = -t[B_1(\gamma^2) - B_2(\gamma^2)], \quad \bar{y}_k = 0 \quad \text{(B27)}
\]
and
\[
\sigma_k^2 = 2 + \frac{t^2}{\gamma^2} \left[ B_3(\gamma^2) - 2B_3(\gamma^2) + 6B_4(\gamma^2) \\
- 2B_4(\gamma^2) - \gamma^2[B_1(\gamma^2) - B_2(\gamma^2)] \right].
\] (B28)
In dimensional form the westward translational speed for kinetic energy is

\[ V_k = C_m \gamma^2 [B_1(\gamma^2) - B_2(\gamma^2)] \] (B29)

and the dispersion rate is

\[ \left( \sigma_k^2 - 2L^2 \right)/(C_m t)^2 = F_2(\gamma), \] (B30)

where \( F_2(\gamma) \) is \( \gamma^2 \) times the quantity in braces in (B28). Relative to the other translational speeds

\[ V_k < V_0 < V_p, \]

which implies that the kinetic energy centroid moves east relative to the maximum of \( h \) while the potential energy centroid moves west. The dispersion rate \( F_2(\gamma) \) has a maximum value of 0.088 which occurs for \( \gamma = 1.48 \).

\[ \begin{align*}
\text{REFERENCES} \\
Bjerknes, J., and J. Holmboe, 1944: On the theory of cyclones. \\
J. Meteor., 1, 1–22. \\
\end{align*} \]