On Spindown in the Ocean Interior

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ABSTRACT

In a rotating stratified fluid, with small Ekman number E and Rossby number Ro, vertical diffusion of momentum is balanced by local deceleration for large values of the Burger number S, and hence leads to an increase in S. For small values of S a feature spreads laterally and S decreases; in this case a transformation to density coordinates leads to a horizontal-diffusion equation, which can be generalized to allow for arbitrary values of S and Ro. If Ro ≪ S, as well as Ro ≪ 1, the potential-vorticity equation can be linearized and the relative effects of vertical and horizontal diffusion of either momentum or mass can be examined.

1. Introduction

In most studies of geophysical fluid flows at small Ekman number, it is assumed that the effects of friction are confined to thin boundary layers adjacent to solid boundaries. However, in some situations with fronts below free surfaces, or with the flow confined to the fluid interior, some account must be taken of frictional effects within the fluid. Very often, though, when this is done in a numerical model, any role played by friction is dominated by other inviscid effects such as frontogenesis (associated with large-scale convergence), geostrophic adjustment or baroclinic instability. Indeed, these may be the primary interest of the modeller, with viscous effects added just for completeness. Given the major uncertainty in parameterization of momentum and mass transfer by processes such as internal waves, it seems worthwhile to study the effect of dissipative processes, on lower frequency flows, in situations that are as simple as possible.

One useful concept was introduced by Csanady (1972, unpublished). He discussed the possibility of a double Ekman layer, at a tilted frontal interface between fluids of different density, arising as a consequence of interfacial friction. The lateral mass flux in these Ekman layers can be related to a lateral spread of the frontal interface (e.g., Garrett and Loder, 1981).

This account of the effect of friction is obviously inadequate if the pycnocline in a front is thicker than an Ekman layer. In that case Gill (1981) and Garrett and Loder (1981) found that, under certain conditions, structures in the flow tend to diffuse laterally, with a diffusivity of $(N^2/f^2)\nu$, as a consequence of secondary flow induced by the presence of a vertical component of viscosity, with coefficient ν. Here N and f are the Väisälä and Coriolis frequencies, respectively.

The purpose of this note is to examine further the assumptions made by Gill (1981) and Garrett and Loder (1981), and to discuss the viscous spindown of interior flow in situations where the assumptions do not hold.

2. Governing equations

We envisage a density field $\rho(x, z, t)$, with z upward, and a flow field (u, v, w), also independent of the coordinate y. We take as governing equations:

$$u_t + uu_x + wu_z - fv + \rho^{-1}p_x = (\nu u_z)_z \,, \quad (2.1)$$

$$v_t + uv_x + wv_z + fu = (\nu v_z)_z \,, \quad (2.2)$$

$$g\rho + p_z = 0, \quad (2.3)$$

$$\rho_t + u\rho_x + w\rho_z = (\kappa\rho_z)_z \,, \quad (2.4)$$

$$u_x + w_z = 0. \quad (2.5)$$

Mixing of momentum or mass either horizontally or along isopycnals has been neglected for the moment, although we do not know whether this is justified in the ocean. Here the assumption is made so as to isolate the effect of vertical mixing, which we represent via "eddy" coefficients ν, κ, although, again, the extent to which this is justified is not known. The mixing should perhaps be regarded as diapycnal, rather than vertical, but for the small slopes of isopycnal surfaces in the ocean the distinction is not important.

If $\nu = \kappa = 0$, the flow is steady and geostrophically balanced with $u = w = 0$ and, from (2.1), $v = f^{-1}\rho^{-1}p_x$. Combined with the Boussinesq approximation we have the thermal-wind equation

$$fv_z = b_x \,, \quad (2.6)$$

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where the buoyancy $b = -g\rho_0^{-1}(\rho - \rho_0)$, with ρ_0 a reference density.

a. Scales

If we now allow non-zero ν and κ, then (2.2) requires either a change in v, or a cross-frontal flow u, or both. Let H, L be the vertical and horizontal scales of the flow, with velocity scales U, V, W. The continuity equation (2.5) requires $U/L = W/H$. Two dimensionless numbers are the Ekman number $\mathrm{E} = \nu(fH^2)^{-1}$ and the Rossby number $\mathrm{Ro} = V(fL)^{-1}$. Eq. (2.2) shows that the rate of spindown is no greater than $\mathrm{E}f$ and U is no greater than $\mathrm{E}V(1 + \mathrm{Ro})^{-1}$. The ageostrophic terms in (2.1) are then negligible if $\mathrm{E}^2 \ll 1 + \mathrm{Ro}$, at least away from Ekman layers at fluid boundaries. [No-slip boundary conditions generally lead to the faster spindown rate $\mathrm{E}^{1/2}f$ (Pedlosky, 1979) associated with Ekman layer suction. The present study is thus confined to problems without solid boundaries.] We assume that the Ekman number E is small enough for the interior cross-frontal geostrophic balance, and hence the thermal wind (2.6), to be valid.

3. Limiting cases

If v_t is the dominant term on the left-hand side of (2.2) the spindown time is clearly $(\mathrm{E}f)^{-1}$. On the other hand, if fu is the dominant term we use (2.6) to write $u = f^{-2}(\nu b_x)_z$. The continuity equation (2.5) gives $w = -f^{-2}(\nu b_x)_x$ and (2.4) becomes a closed-form equation for density (Gill, 1981; Garrett and Loder, 1981), i.e.,

$$b_t + f^{-2}(\nu b_x)_z b_x - f^{-2}(\nu b_x)_x b_z = (\kappa b_z)_z\,. \quad (3.1)$$

[In general w should contain an arbitrary function of x, but this is zero if the flow is confined to the fluid interior, or if w is matched to the divergence of an Ekman layer at the free surface (Garrett and Loder, 1981).]

Gill (1981) pointed out that for small departures from a state of uniform stratification $|ub_x| \ll |wb_z|$ and (3.1) becomes the diffusion equation

$$b_t = (N^2/f^2)(\nu b_x)_x + (\kappa b_z)_z\,, \quad (3.2)$$

where $N^2 = b_z$ is the square of the Väisälä frequency. Eq. (3.2) is also a special case of the linearized potential-vorticity equation of Müller (1976). Garrett and Loder (1981) showed that if the depth $z(x, b, t)$ of an isopycnal is taken as the dependent variable, rather than $b(x, z, t)$, then (3.1) becomes a diffusion equation for z,

$$z_t = [(N^2/f^2)\nu z_x]_x - (\kappa N^4 z_b)_b\,, \quad (3.3)$$

without the neglect of any term of (3.1), and hence in situations where N varies significantly in the x direction.

Both (3.2) and (3.3) show that, if fu dominates the left-hand side of (2.2), the decay time of a frontal feature (ignoring κ) is $L^2[(N^2/f^2)\nu]^{-1} = (\mathrm{E}f\mathrm{S})^{-1}$, where S is the Burger number $N^2H^2(f^2L^2)^{-1}$.

These results suggest that at low Rossby number [so that the advective terms are negligible in (2.2)], the viscous spindown occurs at a rate $f\mathrm{E}$ via local deceleration for "thick" features ($\mathrm{S} \gg 1$) and at the slower rate $f\mathrm{ES}$ via cross-frontal flow for "thin" features ($\mathrm{S} \ll 1$). The difference in decay rates is easily understood in energetic terms: for $\mathrm{S} \gg 1$ the energy of the feature is almost entirely its kinetic energy, whereas for $\mathrm{S} \ll 1$ the available potential energy is $O(\mathrm{S}^{-1})$ times the kinetic energy, and the decay rate is less by a factor S than if the kinetic energy alone mattered.

As remarked by Gill (1981) and Garrett and Loder (1981), Eqs. (3.2) and (3.3) suggest that when the features become sufficiently thin ($\mathrm{S} \approx \kappa/\nu$, assumed small) then diffusion of mass becomes as important as viscous spreading.

Eqs. (3.2) and (3.3) both require $\mathrm{Ro} \ll 1$ and $\mathrm{S} \ll 1$, with (3.2) requiring the further restriction that $|ub_x| \ll |wb_z|$, i.e, that the isopycnal slope $\ll H/L$, or equivalently that the change across the feature in isopycnal spacing and N^2 is small. Now the thermal wind equation (2.6) and the equation $N^2 = b_z$ give an isopycnal slope of order $fV(N^2H)^{-1}$ and the condition becomes $\mathrm{Ro} \ll \mathrm{S}$. In other words, (3.2) is valid in the parameter range $\mathrm{Ro} \ll \mathrm{S} \ll 1$, whereas the more general diffusion equation (3.3) is valid for $\mathrm{Ro} \ll 1$ and any $\mathrm{S} \ll 1$. However, we note that we require $\mathrm{Ro} \lesssim \mathrm{S}$ to avoid a change in isopycnal depth greater than order H.

4. A general equation in density coordinates

The discussion of Section 3 makes reasonably clear the mechanisms and rates of spindown at either small or large values of the Burger number S, at least for small values of the Rossby number Ro. It seems desirable to seek a general equation for the spindown, allowing for an arbitrary value of S and also, if possible, of Ro.

We start with (2.6), (2.2), (2.4) and (2.5), summarized here as

$$fv_z = b_x\,, \quad (4.1)$$

$$v_t + uv_x + wv_z + fu = (\nu v_z)_z\,, \quad (4.2)$$

$$b_t + ub_x + wb_z = 0, \quad (4.3)$$

$$u_x + w_z = 0. \quad (4.4)$$

We ignore the term $(\kappa\rho_z)_z$ in (4.3); the discussion of Section 3 makes it clear that it is only likely to be important (if $\nu/\kappa \gg 1$) for thin features with $\mathrm{S} \ll 1$ which must also have $\mathrm{Ro} \ll 1$ so that (3.2) or (3.3) are valid.

Given the utility of density coordinates in deriving the revealing equation (3.3) for $\mathrm{Ro} \ll 1$ and $\mathrm{S} \ll 1$, we carry out the same coordinate change for arbitrary values of these parameters, i.e., retaining the time-dependent and advective terms in (4.2). Formally, we define a new set of coordinates

$$\xi = x, \quad b = b(x, z, t), \quad \tau = t, \tag{4.5}$$

and note that for any variable A,

$$A_x = A_\xi + A_b b_x, \quad A_z = A_b b_z,$$
$$A_t = A_\tau + A_b b_t\,. \tag{4.6}$$

Hence [using in particular the results of (4.6) with $A = z$], Eqs. (4.1)–(4.4) become

$$f v_b = -z_\xi\,, \tag{4.7}$$

$$v_\tau + u v_\xi + f u = -f^{-1} z_b^{-1} (\nu z_\xi z_b^{-1})_b\,, \tag{4.8}$$

$$z_{b\tau} + (u z_b)_\xi = 0. \tag{4.9}$$

We note that w has been eliminated easily.

If v_τ and uv_ξ are negligible in (4.8) we may substitute u from (4.8) directly into (4.9). After one integration with respect to b, and using $N^2 = b_z = z_b^{-1}$, we obtain

$$z_\tau = [(N^2/f^2)\nu z_\xi]_\xi \tag{4.10}$$

as in (3.3) with $\kappa = 0$. In general (4.8) and (4.9) combine to give

$$z_{b\tau} = \left\{\frac{[(N^2/f^2)\nu z_\xi]_b + N^{-2} f^{-1} v_\tau}{1 + f^{-1} v_\xi}\right\}_\xi. \tag{4.11}$$

Eq. (4.10) is clearly one limit; another, corresponding to a vertical diffusive balance, $v_t = (\nu v_z)_z$ in (4.2), arises if $z_{b\tau}$ is negligible in (4.11).

The term $f^{-1}v_\xi$ in the denominator of the right-hand side of (4.11) clearly embodies the effect of finite Rossby number. We note that

$$f^{-1} v_\xi = f^{-1}(v_x - v_b b_x) = f^{-1} v_x - v_z^2/N^2, \tag{4.12}$$

so that the Rossby number $V(fL)^{-1}$, which is a measure of $f^{-1}v_x$, is also a measure of $f^{-1}v_\xi$ unless the Richardson number N^2/v_z^2 is small.

To present (4.11) in terms of a single variable we note that (4.7) permits the definition of a variable ϕ such that

$$v = -f^{-1}\phi_\xi, \quad z = \phi_b\,. \tag{4.13}$$

[In fact $\phi = -\rho_0^{-1} p + (b - g) z$, as is easily seen by evaluating the geostrophic equation $fv = \rho_0^{-1} p_x$ in (ξ, b, τ) coordinates.] Eq. (4.11) then becomes

$$\phi_{bb\tau} = \left[\frac{(N^2 f^{-2} \nu \phi_{b\xi})_b - N^{-2} f^{-2} \phi_{\xi\tau}}{1 - f^{-2}\phi_{\xi\xi}}\right]_\xi \tag{4.14}$$

or, as $N^2 = \phi_{bb}^{-1}$,

$$\phi_{bb\tau} = \left[\frac{(\nu \phi_{bb}^{-1} \phi_{b\xi})_b - \phi_{bb}\phi_{\xi\tau}}{f^2 - \phi_{\xi\xi}}\right]_\xi. \tag{4.15}$$

Neither (4.14) nor (4.15) offer any obvious insights into spindown akin to that afforded by (3.3), and scale analysis merely confirms the results of Section 3. The equations are presented for the sake of completeness and in the hope of future use. One minor extension, also apparent from the original equations, is that for large Ro the spindown is via lateral spread, rather than local deceleration, for S $\ll$ Ro and not just Ro $\ll$ 1.

One interesting extension of previous results, however, occurs if we take Ro $\ll$ 1 but allow S to be arbitrary, subject only to the condition Ro $\ll$ S (i.e., the feature must not be too thin). In this case N^2 may be taken as independent of ξ. If we also take it to be independent of b Eq. (4.14) becomes

$$\phi_{bb\tau} + N^{-2} f^{-2} \phi_{\xi\xi\tau} = (N^2/f^2)\phi_{bb\xi\xi}\,. \tag{4.16}$$

In this situation, however, spindown is most easily understood in (x, z) rather than (ξ, b) coordinates. A discussion of this is the subject of the next section.

5. Spindown for Ro $\ll$ 1 and Ro $\ll$ S

With Ro $\ll$ 1 we may neglect the advective terms in (4.2), and if Ro $\ll$ S then $|ub_x| \ll |wb_z|$ in (4.3), and we take $N^2 = b_z$ to be constant. We also allow for density diffusion by adding the term $(\kappa b_z)_z$ to the right-hand side of (4.3). An equation for b, which is essentially the potential vorticity equation, is then (for N^2, ν, κ all constant)

$$N^2 b_{xxt} + f^2 b_{zzt} = N^2 \nu b_{xxzz} + f^2 \kappa b_{zzzz}\,. \tag{5.1}$$

This represents the extension of Gill's (1981) Eq. (3.2) to larger values of S for which both vertical exchange of momentum and lateral spread can matter. If we neglect the last term in (5.1), which only matters if S $\approx \kappa/\nu$ or less, it is clear that (5.1) reduces to the two limiting cases

$$\left.\begin{aligned} b_t &= \nu b_{zz} && \text{for} \quad S \gg 1 \\ b_t &= (N^2/f^2)\nu b_{xx} && \text{for} \quad S \ll 1 \end{aligned}\right\} \tag{5.2}$$

as expected.

The remarkable property of (5.1) (neglecting the last term) is that it shows a kind of instability at S $\approx$ 1. Features with S $\gg$ 1 diffuse vertically, and acquire even larger values of S, whereas features with S $\ll$ 1 diffuse, or rather spread, horizontally and acquire even smaller values of S.

These conclusions could change if we add to (4.2) and (4.3) terms $(\nu' v_x)_x$ and $(\kappa' b_x)_x$ to allow for lateral diffusion of momentum and mass. If the resulting extension of (5.1) is then scaled with H vertically, NH/f horizontally, and H^2/ν in time, we obtain (again for constant coefficients)

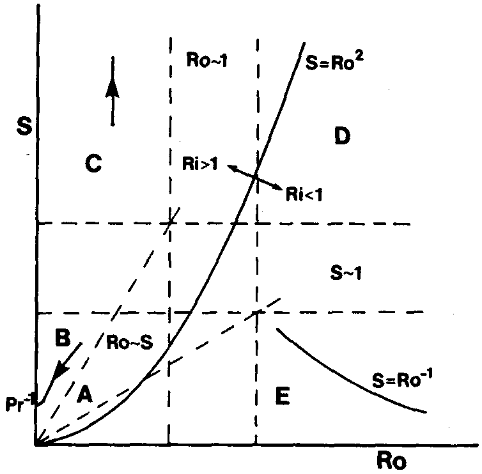

FIG. 1. Regions in [Ro = $V(fL)^{-1}$, S = $N^2H^2(f^2L^2)^{-1}$] space with different spindown characteristics. The trajectories in regions B and C are for the decay of a feature in a situation where vertical viscosity dominates.

$$(\nabla^2 b)_t = b_{xxzz} + (f^2/N^2)(\nu'/\nu)b_{xxxx} + \mathrm{Pr}^{-1}[b_{zzzz} + (f^2/N^2)(\kappa'/\kappa)b_{xxzz}], \quad (5.3)$$

in which the b_{xxzz} terms show (Müller, 1976) that vertical mixing of momentum and horizontal mixing of density act in the same way to spin down the geostrophic flow.

If we assume both $\kappa'/\kappa = N^2/f^2$ [as estimated by Young *et al.* (1982) for shear dispersion in internal waves] and $\nu'/\nu = N^2/f^2$, then (5.3) becomes (with Prandtl number $\mathrm{Pr} = \nu/\kappa$)

$$(\nabla^2 b)_t = (\nabla^2 b)_{xx} + \mathrm{Pr}^{-1}(\nabla^2 b)_{zz}\,. \quad (5.4)$$

This would lead to a lateral spread of any dynamical feature until it reached the scale ratio $\mathrm{Pr}^{-1/2}$. The spindown mechanism is primarily horizontal diffusion of momentum for thick features, and lateral spread, induced by vertical exchange of momentum, for thin features.

The consequences of (5.3) are easily determined for other values of ν'/ν. In particular, we note that thick features ($\mathrm{S} \gg 1$) get thicker (i.e., S increases) by vertical viscosity if $\nu' \ll (N^2/f^2)\nu\mathrm{S}^{-1}$, but thinner by horizontal viscosity if $\nu' \gg (N^2/f^2)\nu\mathrm{S}^{-1}$.

6. Conclusions and discussion

The results of the preceding sections are best summarized in a diagram. Fig. 1 shows the (Rossby number, Burger number) parameter space which categorizes an oceanic feature. Note that the Richardson number is given locally by $\mathrm{Ri} = N^2/v_z^2$, which is given by SRo^{-2} in order of magnitude.

In regions A and B of Fig. 1 fu is the dominant term on the left-hand side of (2.2) and the feature decays by spreading laterally until it reaches a Burger number of approximately Pr^{-1}, at which vertical diffusion of density becomes as important as the lateral spread. During this phase the decay rate is $(N^2/f^2)\nu L^{-2}$, and the Rossby number is given by

$$\mathrm{Ro} = v/(fL) \approx BHL/(f^2L^3), \quad (6.1)$$

where B is a measure of the buoyancy anomaly, and BHL the integrated anomaly, which is conserved. Hence, as the vertical scale does not change, Ro $\propto \mathrm{S}^{3/2}$. When $\mathrm{S} \approx \mathrm{Pr}^{-1}$ is reached, the Rossby number decreases at the same value of S. This trajectory in (Ro, S) space is shown in Fig. 1. The details of the decay are described in density coordinates by (3.3) in either region A or B; the simpler equation (3.2) applies only in region B.

In region C the decay is by local deceleration which increases S while keeping Ro constant. A trajectory is shown. However, the more general equations (5.1) or (5.4) apply to regions B and C and show that allowing for horizontal mixing of momentum may reverse the trajectory of a feature that starts in region C, and connect it to the type of trajectory shown in region B.

In region D the advective and acceleration terms are important in (2.2); in region E the advective and Coriolis terms dominate. A prediction of spindown would depend on a solution of the basic equations or of the general equation in density coordinates [(4.15)]. The decay is likely to be rapid, as the small values of Ri in these regions will lead to either Kelvin-Helmholtz instability or the viscous overturning described by McIntyre (1970) for $\mathrm{Ri} \leq \frac{1}{4}\mathrm{Pr} \times (1 + f^{-1}v_x)^{-1}$. We also bear in mind the likely geometrical constraint $\mathrm{Ro} \leq \mathrm{S}$, which would render most of regions D and E inaccessible.

Baroclinic stability is a process that may dominate the viscous spindown discussed in this note. In particular, one wonders whether an axisymmetric lens of rotating stratified fluid will undergo a process of fission as it spreads to have a radius greater than a Rossby radius, i.e., as it enters the parameter range $\mathrm{S} < 1$. (The equations of this paper have been for a straight front, but the essential results carry over to the axisymmetric case.)

In many oceanic applications an important spindown mechanism may be associated with double-diffusive intrusions in the frontal region. These clearly lead to a negative value of κ (e.g., Garrett, 1982), but also presumably exchange momentum with a positive ν, and hence cause spindown. Actual values of ν and κ are not well-established, and clearly depend on the evolving T–S fields rather than just the density (and hence velocity) field.

Values of the viscosity and diffusion coefficients associated with internal wave processes are also far from established. Müller's (1976) arguments based on wave-wave interactions suggested a dominance of vertical viscosity over other mixing mechanisms, with a coefficient of up to 0.4 $m^2 s^{-1}$, but oceanic data (Ruddick and Joyce, 1979) limit $|\nu|$ to less than 0.02 $m^2 s^{-1}$. On the other hand, Brown and Owens (1982) claim a value of order 100 $m^2 s^{-1}$ for the horizontal eddy viscosity associated with internal waves, though they admit the incompatibility of this with the observed persistence of small-scale energetic eddies.

It is hoped that studies of the observed decay of isolated features in the ocean interior, combined with an understanding of the mechanisms of spindown explored in this paper, may lead to a deduction of the viscosity and diffusion coefficients responsible.

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