Kinematics of Turbulence Convected by a Random Wave Field

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ABSTRACT

Turbulent velocity spectra measured beneath wind waves show a large enhancement about the central wave frequency. A "$-\frac{5}{3}$" frequency dependence can be seen both above and below the central peak, but with an apparent increase in spectral density at high frequencies.

We show that these features can be understood via a generalization of Taylor's hypothesis to the case in which frozen, isotropic, homogeneous turbulence is bodily convected past a fixed probe by a combination of drift and wave orbital motions. In a monochromatic wave field turbulent energy is aliased into harmonics of the wave frequency f_p. We show qualitatively how drift currents or a random wave field broaden these lines into a continuous spectrum, and present the results of direct calculations which demonstrate clearly the transition from "line-like" to a smooth "$-\frac{5}{3}$" spectrum. We calculate the leading asymptotic behavior in the limit of large and small frequencies for an arbitrary wave-height spectrum. For wave orbital velocities larger than the mean drift (in the direction of wave propagation) we find

$$\frac{S(f \gg f_p)}{S(f \ll f_p)} \approx 1.3\left(\frac{U_{\text{orbital}}}{U_{\text{drift}}}\right)^{2/3}, \quad U_{\text{orbital}} > U_{\text{drift}}$$

where U denotes an rms velocity. This result provides a possible explanation for the observed increase in spectral densities for frequencies above the peak.

1. Introduction

Field measurements of the velocity at various depths beneath wind waves on Lake Ontario have recently been completed, and the data separated into wave and turbulent components using linear statistical techniques (Kitaigorodskii *et al.*, 1983). This analysis assumes only that the wave velocity is related linearly to the wave height, and that it is independent of the turbulent velocity. The latter is then obtained by removing from the signal everything coherent with the wave height. A typical turbulent velocity spectrum is shown in Fig. 1. A prominent feature of these spectra is the striking enhancement around the central wave frequency ω_0. Sufficiently far from this peak the frequency dependence is consistent with $\omega^{-5/3}$ although the dissipation density is larger at high frequencies.

We propose that the shape (but not the magnitude) of these spectra can be understood without recourse to dynamics, but is a consequence of an essentially kinematic process in which frozen, isotropic turbulence is convected bodily by the orbital velocity field associated with the surface wave motion. In a monochromatic wave field the turbulent energy is aliased into harmonics of the wave frequency. We present arguments to show qualitatively how drift currents or a width to the wave spectrum can broaden these lines into a continuum.

The dynamical timescale for a turbulent eddy of wavenumber κ is given by $\tau_t \approx \epsilon^{-1/3}\kappa^{-2/3}$. If we estimate the rate of energy transfer to the turbulence by $\epsilon \approx u^3/l$, where u is the rms turbulent velocity and l is the scale of energy containing eddies, then $\tau_t \approx l^{1/3}\kappa^{-2/3}u^{-1}$. If U_c is the convective velocity then the time required to move an eddy past the probe is $\tau_c \approx (\kappa U_c)^{-1}$. The turbulence will appear frozen if $\tau_t \gg \tau_c$. Using the above estimates this condition becomes

$$(\kappa l)^{1/3} \gg u/U_c \qquad (1.1)$$

Typically $u/U_c < 1$ and so there is a good chance that the entire inertial range will appear frozen.

2. General model equations

Our model consists of a random field of linear gravity waves convecting a pattern of frozen, isotropic homogeneous inertial range turbulence past a fixed point. Consequently, the observed time dependence of the turbulent velocity induced by the convective motion will be given by

$$\mathbf{u}(t) \equiv \mathbf{u}[\mathbf{X}(t)] = \int_\kappa \exp[i\mathbf{k}\cdot\mathbf{X}(t)]\hat{\mathbf{u}}(\kappa), \qquad (2.1)$$

where $\mathbf{X}(t)$ denotes the displacement of a fluid element

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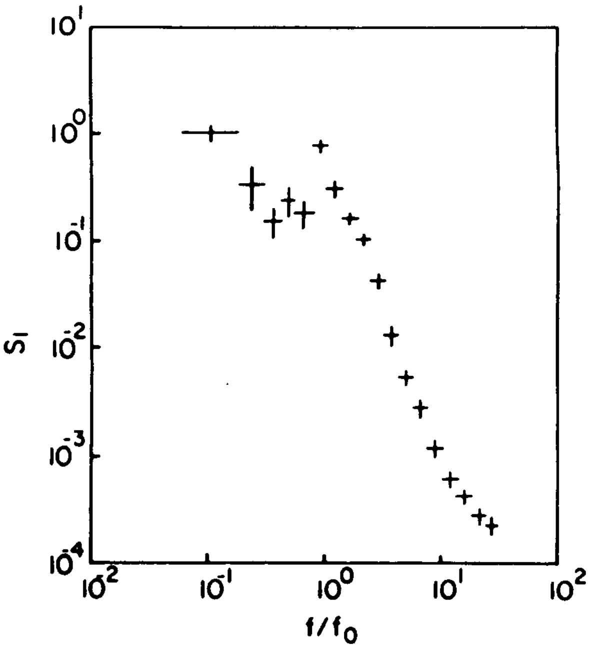

FIG. 1. Longitudinal turbulent velocity spectrum obtained from field data. The spectrum has arbitrary scale and is plotted versus frequency, normalized on the peak wave frequency. For this case: $U^{\rm rms}_{\rm orbital}/U_{\rm drift} = 1.3$.

by the combined wave and drift velocities. If the wave and turbulent motions are statistically independent, then the velocity correlation becomes

$$\rho_{ij}(\tau) \equiv \langle u_i(\tau)u_j(0)\rangle = \int_{\mathbf{k}} \Phi_{ij}(\kappa)C_{\Delta X}(\kappa, \tau), \quad (2.2)$$

where

$$C_{\Delta X}(\kappa, \tau) = \langle \exp[i\kappa \cdot \Delta \mathbf{X}(\tau)]\rangle \quad (2.3)$$

is the characteristic function for the relative displacement $\Delta\mathbf{X} = \mathbf{X}(\tau) - \mathbf{X}(0)$, and Φ_{ij} is the spectral tensor

$$\Phi_{ij}(\kappa) = \frac{E(\kappa)}{4\pi\kappa^2}[\delta_{ij} - \kappa_i\kappa_j/\kappa^2]. \quad (2.4)$$

We assume an inertial range spectrum for the turbulence,

$$E(\kappa) = \begin{cases} a\epsilon^{2/3}\kappa^{-5/3}, & \text{for} \quad \kappa_0 \leqslant \kappa \leqslant \kappa_d \\ 0, & \text{otherwise.} \end{cases} \quad (2.5)$$

Note that ϵ is a free parameter which determines the magnitude of the spectral density of turbulent fluctuations.

For infinitesimal waves, $\mathbf{X}$ will be linear in the surface displacement $\eta(\mathbf{x}, t)$, where $\mathbf{x} = (x_1, x_2)$. To good approximation this latter quantity exhibits Gaussian statistics, so that the characteristic function may be evaluated in terms of the auto-correlation, $c_{ij}(\tau) = \langle X_i(\tau)X_j(0)\rangle$, to give

$$C_{\Delta X}(\kappa, \tau) = \exp\{-\kappa_i\kappa_j[c_{ij}(0) - c_{ij}(\tau)]\} \exp[i\kappa \cdot \mathbf{U}_D\tau]. \quad (2.6)$$

Note we are allowing for a uniform drift $\mathbf{U}_D$.

Finally, we need a relation between $c_{ij}(\tau)$ and the wave-height spectrum, $\Phi_\eta(\mathbf{p}, \omega)$. To first order in the wave slope $\kappa\eta$ the equations governing the velocity potential ϕ for deep water gravity waves are (Phillips, 1977)

$$\left.\begin{aligned} (\partial_t^2 + g\partial_z)\phi|_{z=0} = 0, \quad \nabla^2\phi(\mathbf{x}, z, t) = 0, \\ \dot{\eta} = \partial_z\phi|_{z=0}, \quad \partial_z\phi|_{z\to-\infty} = 0 \end{aligned}\right\}, \quad (2.7)$$

while to the same order, the equation of motion for $\mathbf{X}$ is

$$\mathbf{X}(t) = \nabla\phi. \quad (2.8)$$

This system is linear, and is easily solved to give

$$X(t) = \int_{P,\omega} \exp[i(\mathbf{P}\cdot\mathbf{x} - \omega t)] \times \exp[Pz]\mathbf{a}(\mathbf{p})\hat{\eta}(\mathbf{P}, \omega), \quad (2.9)$$

where $\mathbf{a}(\mathbf{p}) = (i\mathbf{p}, 1)$, and $\mathbf{p}$ is a unit vector in the 1–2 (horizontal) plane. Since different wave modes are independent,

$$\langle \hat{\eta}(\mathbf{P}, \omega)\hat{\eta}^*(\mathbf{P}', \omega)\rangle = \delta(\mathbf{P} - \mathbf{P}')\delta(\omega - \omega')\Phi_\eta(\mathbf{P}, \omega). \quad (2.10)$$

Consequently

$$c_{ij}(\tau) = \int_{P,\omega} \exp[-i\omega\tau] \times \exp[2Pz]a_i(\mathbf{P})a_j^*(\mathbf{P})\Phi_\eta(\mathbf{P}, \omega). \quad (2.11)$$

To further reduce this expression, we will assume the waves are long-crested (e.g. uni-directional) and choose the 1-axis along the direction of propagation. Then the full spectrum can be written

$$\Phi_\eta(\mathbf{P}, \omega) = \frac{g}{\omega^2}\delta\left(P - \frac{\omega^2}{g}\right)\delta(\theta)\Phi_\eta(\omega). \quad (2.12)$$

The area under our frequency spectrum $\Phi_\eta(\omega)$ is normalized to the rms wave height. Now the p-integrations can be done explicitly, leading to

$$\kappa_i\kappa_jc_{ij}(\tau) = \kappa_p^2 \int_\omega \exp[-i\omega\tau] \exp\left[2\frac{\omega^2 z}{g}\right]\Phi_\eta(\omega), \quad (2.13)$$

where $\kappa_p^2 = \kappa_1^2 + \kappa_3^2$.

For narrow spectra and for z restricted to be close to the surface, we can replace the factor $\exp[2\omega^2 z/g]$ by its value at the peak frequency ω_0. Then

$$\kappa_i\kappa_jc_{ij}(\tau) = \kappa_p^2R^2c_\eta(\tau), \quad (2.14)$$

where $R = \eta_{\rm rms}\exp[\omega_0^2 z/g]$ is the orbital radius at depth z, and $c_\eta(\tau)$ is defined by

$$c_\eta(\tau) = \frac{\int_\omega e^{-i\omega\tau}\Phi_\eta(\omega)}{\int_\omega \Phi_\eta(\omega)}. \quad (2.15)$$

Substituting these expressions back into equations (2.6) and (2.2) we find

$$\rho_{ij}(\tau) = \int_\kappa \Phi_{ij}(\kappa) \exp\{-\kappa_p{}^2R^2[1 - c_\eta(\tau)]\} \times \exp[i\kappa \cdot \mathbf{U}_D\tau] \quad (2.16)$$

which gives the spectrum

$$S_{ij}(\omega) = \int_{\tau/2\pi} \exp[-i\omega\tau]\rho_{ij}(\tau). \quad (2.17)$$

Eqs. (2.16) and (2.17) constitute our basic set, and in the next three sections we investigate the consequences of various models for $c_\eta(\tau)$.

But first we want to make a few general observations based on these equations. Since

$$\int_\omega S_{ij}(\omega) = \int_\kappa \Phi_{ij}(\kappa) = \langle u_iu_j\rangle, \quad (2.18)$$

averaged quantities are unchanged by the wave modulation. This of course follows directly from the lack of a wave-turbulence interaction in our model. The spatial distribution of turbulent fluctuations is merely redistributed temporally by the combination of orbital and drift velocities. In frequency regimes where $\mathbf{U}_D$ can be neglected we find $S_{11}(\omega) = S_{33}(\omega)$. This is a consequence of our assumption that the wave spectrum is uni-directional and is not true in general—for example, if the waves have a $\cos^2\theta$ dependence on direction then $S_{11} \neq S_{33}$ even when $\mathbf{U}_D$ is identically zero.

In the rest of this paper we restrict our numerical computations to S_{11} and further assume that any net drift is along the direction of wave propagation.

3. Monochromatic waves

In this section we assume zero drift, $U_D = 0$, and a monochromatic wave-height spectrum

$$\Phi_\eta(\omega) = \frac{1}{2}\eta_{\rm rms}^2[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]. \quad (3.1)$$

Then $c_\eta(\tau) = \cos\omega_0\tau$, so that $\rho_{11}(\tau)$ is periodic at the orbital period $2\pi/\omega_0$. Consequently we can expand the 1–1 correlation in discrete Fourier modes

$$\rho_{11}(\tau) = \sum_{n=-\infty}^{+\infty} S_n \exp[in\omega_0\tau], \quad (3.2)$$

where

$$S_n = \int_\kappa \Phi_{11}(\kappa) \exp[-\kappa_p{}^2R^2]I_n(\kappa_p{}^2R^2), \quad (3.3)$$

and I_n is the modified Bessel function of order n. Note $S_n = S_{-n}$ and for $|n| \geq 1$ the scaling form of Φ_{ij} may be extended over all κ. Then the integrals converge and are known explicitly. The result is

$$S_n = \left(\frac{7}{110}\right)2^{1/3}\Gamma\left(\frac{1}{3}\right)a\epsilon^{2/3}R^{2/3}\Gamma\left(n - \frac{1}{3}\right)/\Gamma\left(n + \frac{4}{3}\right), \quad n \geq 1. \quad (3.4)$$

Finally, taking the Fourier transform of Eq. (3.2) we find the line spectrum

$$S(\omega) = \frac{1}{\omega_0}\sum_{n=-\infty}^{+\infty} S_n\delta\left(\frac{n - \omega}{\omega_0}\right). \quad (3.5)$$

From the asymptotic form of the Gamma function for large argument it is easy to show that the line strengths S_n decrease as $n^{-5/3}$ for large harmonic numbers.

4. Uniform drift

In this section we consider the effects on the line spectrum of a uniform drift in the direction of wave propagation. Note we still take the wave-height spectrum to be monochromatic.

Because of the drift, frequency components are doppler shifted by an amount κ_1U_D. Consequently the spectrum is continuous with contributions from all frequencies ω which satisfy $\omega - \kappa_1U_D = n\omega_0$, and we expect the degree of broadening each line experiences to depend on the ratio of the orbital to drift velocity, ω_0R/U_D. For large values of this parameter the spectrum will retain considerable structure in the neighborhood of $n\omega_0$, tending ultimately to the line spectrum found earlier.

Next we investigate the asymptotic behavior of the spectrum S_{11} in the limit of large and small frequencies. In the limit $\omega \to 0$ we can neglect the wave motion with respect to the drift and the problem reduces to the well known case of uniform translation. Briefly restated, the argument is that fluctuations with wavenumber κ appear at $\omega = \kappa_1U_D$. The distribution of spatial fluctuations, for a given κ_1, is described by the one-dimensional longitudinal spectrum

$$F_1(\kappa_1) = \int_{\kappa_2,\kappa_3} \Phi_{11}(\kappa). \quad (4.1)$$

$F_1(\kappa_1)$ is normalized so that its area is just $\langle u_1{}^2\rangle$. Then for frequencies satisfying $\omega \ll \omega_0$ we have

$$S_{11}(\omega) \to \frac{1}{U_D}F_1\left(\frac{\omega}{U_D}\right). \quad (4.2)$$

This result is quite general and correctly describes the small ω region for any physically reasonable wave height spectrum. The $F_1(\kappa_1)$ scales as $\kappa_1^{-5/3}$ in the inertial subrange and tends to a constant as $\kappa_1 \to 0$. Hence if $\omega > \kappa_0U_D$, where κ_0^{-1} is the length scale of the large eddies, then

$$S_{11}(\omega) \rightarrow \frac{9}{55} a\epsilon^{2/3} U_D^{2/3} \omega^{-5/3}. \quad (4.3)$$

In this range of frequencies $S_{33} = \frac{4}{3}S_{11}$.

For high frequencies the situation is more complex. The spectrum in this case is dominated by the short time behavior of the correlation function, $\rho_{11}(\tau) = \langle u_1(t+\tau)u_1(t)\rangle$. We assume the wave motion is harmonic and set $U_D = 0$. For small τ

$$\rho_{11}(\tau) \sim \int_\kappa \Phi_{11}(\kappa) \times \langle \exp[i\omega_0 R\tau(\kappa_1 \sin\omega_0 t - \kappa_3 \cos\omega_0 t)]\rangle, \quad (4.4)$$

where the brackets around the exponential represent a phase average in t over the orbital period. Transforming to rotating coordinates so that the t-dependent phase is constant and taking the Fourier transform we find for $\omega \gg \omega_0$,

$$S_{11} \rightarrow \langle\cos^2\omega_0 t\rangle F_1 + \langle\sin^2\omega_0 t\rangle F_3 = \frac{7}{6}\cdot\frac{9}{55} a\epsilon^{2/3}(\omega_0 R)^{2/3}\omega^{-5/3}, \quad (4.5)$$

since $F_3 = \frac{4}{3}F_1$ in the inertial subrange.

Finally we must average this expression over the distribution of surface maxima, which for a narrow wave height spectrum, is given by the Rayleigh form (Phillips, 1977)

$$P(R) = \left(\frac{R}{\eta_{\rm rms}^2}\right)\exp\left[\frac{R^2}{2\eta_{\rm rms}^2}\right], \quad (4.6)$$

so that $\langle R^{2/3}\rangle = (\frac{1}{3})2^{1/3}\Gamma(\frac{1}{3})\eta_{\rm rms}^{2/3}$ at the surface.

From Eqs. (4.5) and (4.6) we find

$$\langle S_{11}\rangle \rightarrow \left(\frac{7}{110}\right)2^{1/3}\Gamma\left(\frac{1}{3}\right)a\epsilon^{2/3}(\omega_0\eta_{\rm rms})^{2/3}\omega^{-5/3}. \quad (4.7)$$

Note S_{33} also tends to this limit for large ω.

Intermediate frequencies must be calculated numerically, and we proceed from the complete representation

$$S_{11}(\omega) = \int_{\tau/2\pi} \exp[-i\omega\tau]\int_\kappa \Phi_{11}(\kappa) \times \exp[-\kappa_p^2 R^2(1-\cos\omega_0\tau)]\exp[i\kappa_1 U_D\tau]. \quad (4.8)$$

Due to the drift the integrand is no longer periodic, but we can still expand the exponential cosine term into a discrete Fourier series. Hence the spectrum has the harmonic decomposition.

$$S_{11}(\omega) = \sum_{n=-\infty}^{+\infty} S_n(\omega), \quad (4.9)$$

where

$$S_n(\omega) = \int_\kappa \Phi_{11}(\kappa)\exp[-\kappa_p^2R^2]I_n(\kappa_p^2R^2) \times \delta(\omega - n\omega_0 - \kappa_1 U_D). \quad (4.10)$$

Power counting shows that these integrals remain convergent if we extend the scaling form of Φ_{11} to all $\boldsymbol{\kappa}$. So to leading order the expansion does not depend in an essential way on the size of the inertial range. With this simplification, two of the integrations in Eq. 4.10 may be performed explicitly and we find

$$S_n(\omega) = \frac{a\epsilon^{2/3}R^{5/3}}{5\sqrt{\pi}U_D}\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{5}{6}\right)}K_n(\beta_n), \quad (4.11)$$

where

$$\left.\begin{aligned} K_n(\beta_n) &= \int_{\beta_n}^\infty dx\,\frac{\exp[-x^2]I_n(x^2)}{x^{5/3}(x^2-\beta_n^2)^{1/2}} \times [1 - 8\beta_n^2/11x^2] \\ \beta_n &= \frac{\omega_0 R}{U_D}\left|\frac{\omega}{\omega_0} - n\right| \end{aligned}\right\}. \quad (4.12)$$

Each harmonic K_n has its maximum at $\omega = n\omega_0$ and is symmetric about that point. Sufficiently far from their center frequency the harmonics fall as $\omega^{-8/3}$. The height of their central maxima also decrease with the term number as $n^{-8/3}$, but the width of the peaks scale as $nU_D/\omega_0 R$. Hence for large n the nth term contributes $n^{-5/3}$ to the mean square velocity, in agreement with the line spectrum result. The behavior of the sum for small ω is dominated by the $n = 0$ harmonic, which has the scaling form $\omega^{-5/3}$ in the inertial range. The asymptotic form of the spectrum for large ω is more difficult to see from the decomposition discussed here. Roughly speaking, for large frequencies O(n) terms contribute, each with strength $n^{-8/3}$, where $n \sim \omega/\omega_0$, so that their sum produces an $\omega^{-5/3}$ behavior.

We have numerically evaluated the integrals in Eq. (4.12) for frequencies in the range $10^{-1} < \omega/\omega_0 < 10^1$, and several values of the parameter $\omega_0 R/U_D$. In each case we have computed enough terms to approximately saturate the behavior of the sum for $\Sigma K_n \sim 10$ at the largest frequencies. The results of this calculation are presented in Fig. 2.

5. Non-zero width

In this section we investigate the effect of incorporating a non-zero width in the wave height spectrum as a source of broadening. We neglect drifts and accordingly set $U_D = 0$.

Our model correlation is

$$c_\eta(\tau) = \exp[-\gamma|\tau|]\cos\omega_0\tau, \quad (5.1)$$

corresponding to the Lorentzian spectrum

$$\Phi_\eta(\omega) = \left(\frac{\eta_{\rm rms}^2}{2\pi}\right)\left[\frac{\gamma}{(\omega-\omega_0)^2+\gamma^2} + \frac{\gamma}{(\omega+\omega_0)^2+\gamma^2}\right]. \quad (5.2)$$

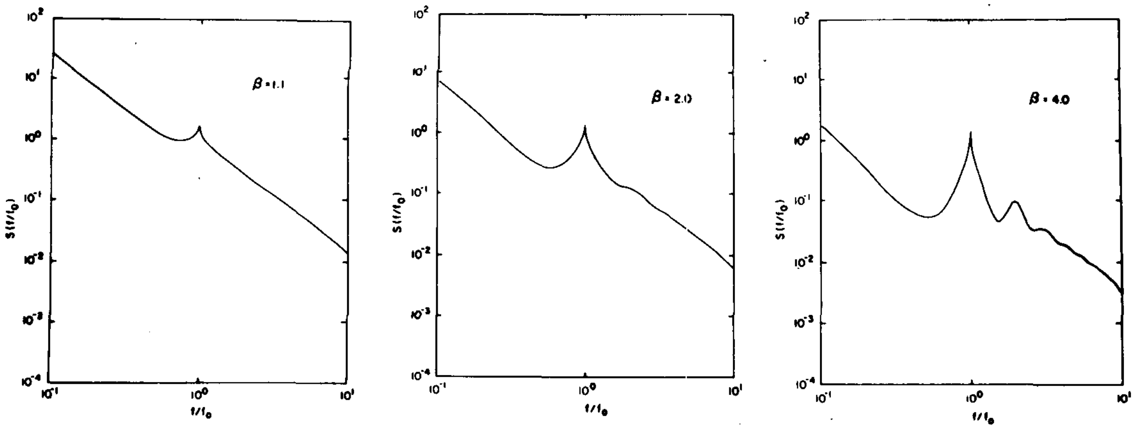

FIG. 2. Calculated longitudinal velocity spectra, using the model of Section 4, versus frequency, normalized on the peak wave frequency. The spectra have been scaled by the same (arbitrary) constant, $\beta = \omega_0 R/U$.

From Eqs. (2.16) and (2.17) we have

$$S_{11}(\omega) = \int_{\tau/2\pi} \exp[-i\omega\tau] \int_\kappa \Phi_{11}(\kappa) \times \exp\{-\kappa_p{}^2R^2[1 - \exp(-\gamma|\tau|)\cos\omega_0\tau]\}. \quad (5.3)$$

The first step is to expand the exponential as

$$\exp\left[\kappa_p{}^2R^2 \exp(-\gamma|\tau|)\cos\omega_0\tau\right] = \sum_{n=0}^{\infty} \frac{(\kappa_p R)^{2n}}{n!} \exp(-n\gamma|\tau|)\cos^n\omega_0\tau. \quad (5.4)$$

Then the spectrum can be decomposed into a sum of terms

$$S_{11}(\omega) = \sum_{n=0}^{\infty} S_n(\omega), \quad (5.5)$$

where

$$S_n(\omega) = \int_{\tau/2\pi} \exp(-i\omega\tau - n\gamma|\tau|)\cos^n\omega_0\tau \times \int_\kappa \Phi_{11}(\kappa)\frac{(\kappa_p R)^{2n}}{n!}\exp[-\kappa_p{}^2R^2]. \quad (5.6)$$

For $n = 0$ the time integration produces a delta function and we have

$$S_0 = \delta(\omega)\int \Phi_{11}(\kappa)\exp[-\kappa_p{}^2R^2]. \quad (5.7)$$

In the discussion to follow we will assume that this term has been subtracted from $S_{11}(\omega)$, and consider the sum in Eq. (5.5) from $n = 1$. For $n \geq 1$ we may extend the scaling form of Φ_{11} to all κ without affecting the convergence of the integrals, which can then be evaluated explicitly in terms of Gamma functions. Further expanding $(\cos\omega_0\tau)^n$ into a sum of Fourier modes allows the remaining time integration to be performed with the result

$$S_n(\omega') = \left(\frac{a\epsilon^{2/3}R^{2/3}}{\omega_0 R}\right)\left(\frac{7}{110}\right)\left(\frac{1}{\sqrt{\pi}}\right)\left[\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{5}{6}\right)}\right]H_n(\omega'), \quad (5.8)$$

where

$$H_n(\omega') = \left(\frac{\Gamma\left(n - \frac{1}{3}\right)}{2^n n!}\right) \times \sum_{p=0}^{n}\binom{n}{p}\frac{n\gamma'}{[\omega' - (n - 2p)]^2 + n\gamma'^2}, \quad (5.9)$$

and $\omega' = \omega/\omega_0$, $\gamma' = \gamma/\omega_0$.

Each term $H_n(\omega')$ is a symmetric function of ω' and consists of a series of peaks located at $\omega' = -n, -n + 2, \ldots, n - 2, n$. The magnitude of the $n = 1$ harmonic scales as $1/\gamma'$. All of the peaks in a given series have a width of order $n\gamma'$. It is relatively easy to show that $H_n(0)$ is bounded by $n^{-7/3}$ for large n, so that

$$H(\omega') \equiv \sum_{n=1}^{\infty} H_n(\omega') \quad (5.10)$$

tends to a constant at the origin. This is a consequence of the assumption $U_D = 0$ and is particular to this model. In addition, although this model scales for large frequencies, the exponent is not $-\frac{5}{3}$. This results from the fact that the correlation function given in Eq. (5.1) is not analytic at the origin (or equivalently, that the corresponding spectrum does not converge fast enough at infinity so that all of its moments exist). Again this feature is particular to this case and is not expected in physically realizable models. Also in this model $S_{33} = S_{11}$ for all frequencies.

Eq. (5.10) for $H(\omega')$ can be rearranged into a sum

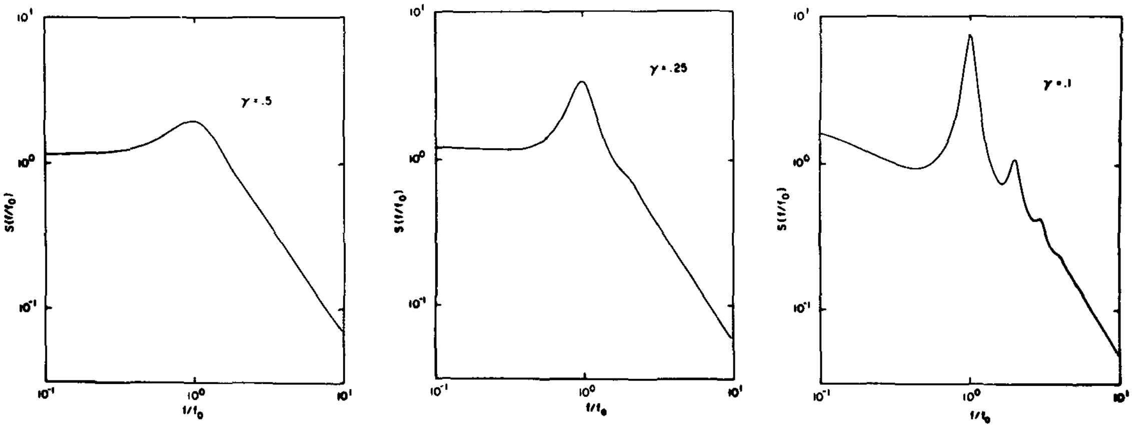

FIG. 3. As in Fig. 2 but for the model of Section (5). $\gamma = \gamma_0/f_0$ is the width of the wave-height spectrum.

of harmonics located at frequencies $\pm\,\omega' = 0, 1, 2, \ldots$. Substituting $s = n - 2p$ and inverting the order of summations gives

$$H(\omega') = \sum_{s=1}^{\infty} \{h_s(\omega') + h_{-s}(\omega')\} + h_0(\omega'), \quad (5.11)$$

where for $s \geqslant 1$

$$h(\omega') = \sum_{\{n\}_s} \frac{\Gamma\left(n - \frac{1}{3}\right)}{2^n\Gamma(1 + n/2 + s/2)\Gamma(1 + n/2 - s/2)} \times \{n\gamma'/[(\omega' - s)^2 + n^2\gamma'^2] + n\gamma'/[(\omega' + s)^2 + n^2\gamma'^2]\} \quad (5.12)$$

and $\{n\}_s$ denotes all n in the set $\{s, s + 2, s + 4, \cdots\}$. For $s = 0$,

$$h_0(\omega) = \sum_{n=2,4,6,\ldots} \frac{\Gamma\left(n - \frac{1}{3}\right)}{2^n[\Gamma(1 + n/2)]^2} \left(\frac{n\gamma'}{\omega'^2 + n^2\gamma'^2}\right) \quad (5.13)$$

We have calculated the sum $H(\omega')$ over the range $10^{-1} \leqslant \omega' \leqslant 10^1$, keeping enough terms to approximately saturate the background. The results for various values of γ' are presented in Fig. 3.

6. Conclusion

We have seen in the preceding sections that it is possible to have considerable structure in the observed frequency spectrum for turbulent motions without the necessity of a dynamical coupling between waves and turbulence. Although our models have been qualitative we believe they correctly illustrate the general features shown by the data. A more realistic calculation is probably not feasible, but the asymptotic frequency dependence for a given case can be calculated to leading order, and we briefly indicate the results here (see Appendix A).

As noted earlier the low-frequency limit given in Eq. (4.2) is generally valid whenever there is a drift in the direction of wave propagation. Similarly our estimate in Eq. (4.7) of the high frequency limit is also generally applicable to models with both drift and width provided we replace $\omega_0\eta_{\rm rms}$ by $U(z)$, the observed rms orbital wave velocity at depth z. Comparing the low and high frequency estimates given by Eqs. (4.3) and (4.7) we find for $U > U_D$,

$$\frac{S_{11}(\omega \gg \omega_0)}{S_{11}(\omega \ll \omega_0)} = \frac{7}{18}\, 2^{1/3}\Gamma(1/3)\left(\frac{U}{U_D}\right)^{2/3} \approx 1.31\left(\frac{U(z)}{U_D(z)}\right)^{2/3} > 1. \quad (6.1)$$

We have explicitly indicated the depth dependence of the velocity ratio to emphasize that it depends on the *observed* velocities appropriate to a given depth. Hence for $U > U_D$ we find $S_{11}(\omega \gg \omega_0) > S_{11}(\omega \ll \omega_0)$ and this result provides a possible explanation for the observed offset in the "$-\frac{5}{3}$ scaling form" of the spectrum between high and low frequencies. It is important to emphasize that this model does not address the question of the overall magnitude of the spectrum, which is parameterized by the dissipation rate ϵ.

Since this work was completed we have become aware that similar models have been used to understand the dynamic loading on wind turbine propellers (Kristensen, 1982). Turbulent frequency spectra, measured at the blade tip, display sharp peaks at harmonics of the rotor frequency and provide an interesting example of the monochromatic wave model discussed in Section (4).

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APPENDIX A

Asymptotic Form of the Turbulent Velocity Spectrum

In this appendix we investigate the asymptotic form of the turbulent velocity spectrum for high and low frequencies. In order to more clearly show the effects of the simplifications introduced in Section 2 we will initially drop the restriction to narrow and uni-directional wave spectra. Note however that the waves are still assumed linear. Then the wave modulation factor

$$\exp[-\kappa_p{}^2R^2(1 - c_\eta(\tau))] \quad \text{(A1)}$$

in Eq. (2.16) is modified by the replacement

$$\kappa_p{}^2R^2c_\eta(\tau) \rightarrow c(\boldsymbol{\kappa}, z, \tau), \quad \text{(A2)}$$

where

$$c(\boldsymbol{\kappa}, z, \tau) = \int_{\mathbf{P},\omega} e^{-i\omega\tau}e^{2\omega^2 z/g}[\kappa_3{}^2 + (\boldsymbol{\kappa}_p\cdot\hat{\mathbf{p}})^2]\Phi_\eta(\mathbf{P}, \omega). \quad \text{(A3)}$$

Here $\boldsymbol{\kappa}_P = (\kappa_1, \kappa_2)$ and we have used the linear wave dispersion relation $\omega^2 = gP$. With this replacement the spectrum can be written as

$$S_{ij}(\omega) = \int_{\tau/2\pi} e^{-i\omega\tau}\int_{\boldsymbol{\kappa}} \Phi_{ij}(\boldsymbol{\kappa})E(\boldsymbol{\kappa}, z, \tau)e^{i\boldsymbol{\kappa}\cdot\mathbf{U}_D\tau}, \quad \text{(A4)}$$

where

$$E(\boldsymbol{\kappa}, z, \tau) = \exp[c(\boldsymbol{\kappa}, z, \tau) - c(\boldsymbol{\kappa}, z, 0)]. \quad \text{(A5)}$$

LIMIT $\omega \rightarrow 0$.

Using obvious notation we write $E(\boldsymbol{\kappa}, z, \tau)$ as

$$E(\boldsymbol{\kappa}, z, \tau) = \underset{[1]}{1} - \underset{[2]}{[e^{-c(0)}(1 - e^{c(\tau)})]} - \underset{[3]}{[(1 - e^{-c(0)})]}. \quad \text{(A6)}$$

Term [2] is finite for $\omega = 0$ and so contributes to the non-leading constant. It is also relatively easy to show (by rescaling $\boldsymbol{\kappa} \rightarrow \omega\boldsymbol{\kappa}$) that term [3] is $O(\omega^{1/3})$ in the inertial range and so can be neglected. Hence to leading order we may replace $E(\boldsymbol{\kappa}, z, \tau)$ by 1. The τ-integration then gives a delta function and we find

$$S_{ij}(\omega) \rightarrow \int_{\boldsymbol{\kappa}} \Phi_{ij}(\boldsymbol{\kappa})\delta(\omega - \boldsymbol{\kappa}\cdot\mathbf{U}_D). \quad \text{(A7)}$$

If the 1-axis lies along $\mathbf{U}_D$ then

$$S_{ii}(\omega) = \frac{1}{U_D}F_i\left(\frac{\omega}{U_D}\right); \quad i = 1, 2, 3, \quad \text{(A8)}$$

where the one-dimensional spectra are given by

$$F_i(\kappa_1) = \int \Phi_{ii}(\boldsymbol{\kappa})d\kappa_2 d\kappa_3. \quad \text{(A9)}$$

Because of isotropy there is of course no contribution to the off-diagonal elements of S_{ij} from the inertial range. For a general alignment of the coordinate axes the components of S_{ij} will be obtained by applying the appropriate orthogonal transformation to the diagonal matrix in equation (A8).

LIMIT $\omega \rightarrow \infty$.

The dominant contribution for large frequencies comes from the region of small τ. Expanding about $\tau = 0$ we have

$$c(\boldsymbol{\kappa}, z, 0) - c(\boldsymbol{\kappa}, z, \tau) = \frac{1}{2}\tau^2\kappa_i\kappa_j\langle U_iU_j\rangle + O(\tau^4). \quad \text{(A10)}$$

Here $\langle U_i^2\rangle$ is the mean square of the ith component of the wave velocity. We have chosen the 1-axis along the peak direction of wave propagation and assume that the wave spectrum is symmetric about this line so that $\langle U_iU_j\rangle$ is diagonal. Then the leading asymptotic behavior of S_{ij} is given by

$$S_{ij}(\omega) \rightarrow \int_{\tau/2\pi} e^{-i\omega\tau}\int_{\boldsymbol{\kappa}} \Phi_{ij}(\boldsymbol{\kappa})e^{i\boldsymbol{\kappa}\cdot\mathbf{U}_D\tau}e^{-(1/2)\kappa_i\kappa_j\langle U_iU_j\rangle\tau^2} \quad \text{(A11)}$$

This expression can be used to develop a systematic expansion in $\langle U_2{}^2\rangle/U^2$, where $U^2 = \langle U_3{}^2\rangle = \langle U_1{}^2 + U_2{}^2\rangle$, but we will restrict the discussion here to the case $U_2 = 0$ and further assume the drift velocity is aligned with the direction of wave propagation. Then Eq. (A11) reduces to ($i = 1, 3$)

$$S_{ii} = U^{2/3}\omega^{-5/3}\int_{\tau/2\pi} e^{-i\tau}\int_{\boldsymbol{\kappa}} \Phi_{ij}(\boldsymbol{\kappa})e^{-(1/2)\kappa_p{}^2\tau^2}e^{i\kappa_p\tau U_D/U}. \quad \text{(A12)}$$

When $U > U_D$ we can expand in powers of $(U_D/U)^2$. The leading term for $U_D = 0$ is easily computed to be

$$S_{11} = S_{33} = \frac{7}{110}2^{1/3}\Gamma(1/3)a\epsilon^{2/3}U^{2/3}\omega^{-5/3}, \quad \text{(A13)}$$

and we recover Eq. (4.7) with the replacement $\omega_0\eta_{\rm rms} \rightarrow U(z)$.

Non-leading terms are obtained by multiplying this result by the function

$$G_i(U_D/U) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n-\frac{1}{3}\right)}{\Gamma\left(-\frac{1}{3}\right)\Gamma(n+1)} \frac{1}{n!}\left[\frac{1+nc_i}{1+n}\right]\left[-\frac{1}{2}\left(\frac{U_D}{U}\right)^2\right]^n \quad \text{(A14)}$$

where $c_{1,3} = (\frac{3}{7}, \frac{11}{7})$.

Using Eq. (A8) and (A13) we find an offset between the high and low frequency spectral densities of

$$\frac{S_{11}(\omega \gg \omega_0)}{s_{11}(\omega \ll \omega_0)} \approx 1.31\left(\frac{U}{U_D}\right)^{2/3}, \quad U \gg U_D. \quad \text{(A15)}$$

Here $G_i(U_D/U)$ has a simple representation in terms of confluent hypergeometric (Kummer) functions (Abramowitz and Stegun, 1964)

$$G_i(z) = c_{i\,1}F_1\left(-\frac{1}{3}, 1; -z^2/2\right) - (c_i - 1)\,{}_1F_1\left(-\frac{1}{3}, 2; -z^2/2\right), \quad \text{(A16)}$$

where $z = U_D/U$. A complete asymptotic expansion for $U_D \gg U$ may be obtained from the analytic continuation of Eq. (A16) to large z. The leading term recovers the result in equation (A1) found previously for small frequencies. Hence in the limit of pure drift the spectrum scales as a single "$-\frac{5}{3}$" line throughout the entire "inertial" range.

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