On the Motion of Isolated Lenses on a Beta-Plane

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(Manuscript received 22 July 1982, in final form 22 October 1982)

ABSTRACT

This paper examines the motion and propagation of an isolated layer of anomalous water on a beta-plane, considered previously by Nof (1981). His perturbation analysis is extended, to show the following:

1) Only westward propagation can occur, induced by the beta-effect; the eddy's speed must be less than two-thirds of the long Rossby-wave speed (unless the potential vorticity of the eddy is somewhere negative, which would be unlikely).

2) The eddy must be at least $2\sqrt{2}$ deformation radii in radius.

3) The shape and velocity structure of the eddy has a simple structure, which is calculated for one range of cases.

4) The unperturbed eddy (on an f-plane) is stable to small disturbances making it likely that the eddy can propagate great distances before decaying.

1. Introduction

The role of eddies in oceanic circulation is not well understood. Even when direct or semi-direct estimates of the heat transported by eddies are made (e.g., Hall and Bryden, 1982), the results are equivocal: in the Drake Passage, eddies transport a great deal of heat; at 25°N in the North Atlantic they do not. Part of our lack of understanding presumably stems from the variety of eddies in the "eddy zoo." Some seem approximately wavelike (cf. McWilliams and Flierl, 1976) and some, like the "meddy" (McDowell and Rossby, 1978), are obviously capable of transporting a specific water mass a long way across the Atlantic with little or no apparent mixing.

Most studies of individual eddies have used quasi-geostrophic dynamics. While suitable for wavelike eddies, these dynamics must be suspect for isolated lens-shaped eddies like the "meddy" which both have a Rossby number which is not small and—because of the blob of Mediterranean water—are not a small perturbation on a uniform stratification.

Nof, in two calculations (1981, 1982), has successfully used a frictionless, reduced gravity, one-layer model to described the steady westward translation of a single isolated eddy, using an expansion based on the smallness of a parameter measuring the relative variation of the Coriolis parameter across the blob. He found an expression for the velocity of propagation of the eddy and applied this to Loop Current eddies in the Gulf of Mexico and to the meddies.

However, Nof's work left many questions unanswered, and this paper seeks to answer some of them. For example, are there any bounds on the eddy propagation velocity? Nof (1981) suggests the long baroclinic Rossby wave speed as an upper bound. Here we show that the eddy must propagate westward, and, assuming its absolute vorticity to be everywhere positive, the eddy travels slower than two-thirds of the long Rossby wave speed. (If the—physically implausible—class of eddies with negative absolute vorticity are included, there are no formal restrictions on the eddy's speed.) Further, by an adaption of Ball's (1963) analysis, the equation of motion for the blob centroid can be solved, to show that steady westward propagation (plus perhaps inertial oscillations) is the only steady motion permitted. It is also shown that all blobs must be at least $2\sqrt{2}$ deformation radii in radius.

Nof's (1981) analysis does not include any discussion of the shape of the blob or its velocity structure, save that it has certain symmetry properties and is a small perturbation on a radially symmetric circulation. It is shown here that if the radially symmetric flow is a polynomial in radius, then the perturbations are also simple polynomials in radius, but multiplied by $\sin\theta$ or $\cos\theta$, where θ is an angular coordinate measured from the eastward direction. An example is given here.

A final question to be answered concerns the stability of the eddy. Although there are many modes of instability possible, it is at least shown here that the original radially symmetric eddy is stable to small disturbances. This suggests that the eddy should have a fairly long lifetime unless other instabilities can act to destroy it.

2. Formulation

We consider an isolated homogeneous blob of fluid of typical depth $\bar{h}$, in one of the several possible con-

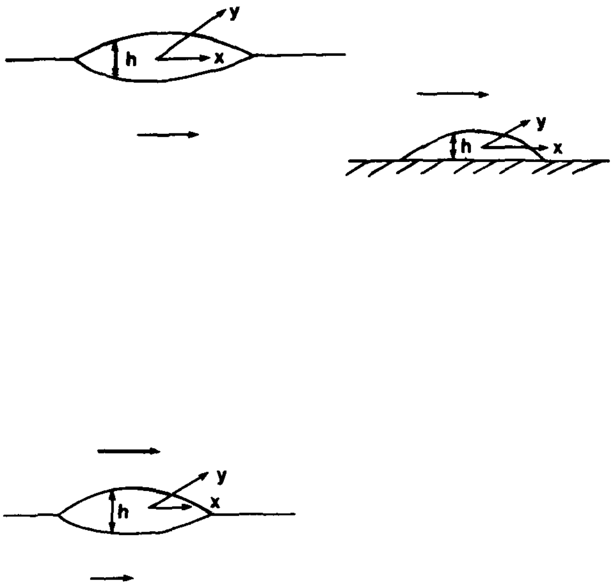

FIG. 1. The eddy under consideration. It is taken to be above, below or contained in an ocean of different density/densities.

figurations sketched in Fig. 1. (The blob can lie above an infinitely deep ocean; below such an ocean; or can exist between two layers of such an ocean.) The remaining ocean may be of several layers, each of which is assumed to possess a uniform east–west velocity (different for each layer). These velocities may be zero, of course.

The nondimensional equations then become

$$u_t + uu_x + vu_y - (1 + \epsilon y)v + h_x = 0, \quad (2.1)$$

$$v_t + uv_x + vv_y + (1 + \epsilon y)(u - U) + h_y = 0, \quad (2.2)$$

$$h_t + (uh)_x + (vh)_y = 0, \quad (2.3)$$

relative to axes x east, y north. The velocities are (u, v) and t = time; h represents the depth. The radius of deformation $R = (g'\bar{h})^{1/2} f_0^{-1}$ has been used for a length scale where g' is a suitably reduced gravity (its precise definition will depend on the physical situation) and $f = f_0 + \beta y$ is the vertical Coriolis component. The velocities are scaled on $(g'\bar{h})^{1/2}$, time on f_0^{-1}, and depth on $\bar{h}$. The velocity U is a suitably averaged velocity of the mean flow (cf. Nof 1982, for an example).[1] The sole parameter

$$\epsilon = \frac{\beta R}{f_0}, \quad (2.4)$$

represents the β-effect. For $\beta \approx 2 \times 10^{-11}$ m^{-1} s^{-1}, $R \approx 30$ km and $f_0 \approx 10^{-4}$ s^{-1}, $\epsilon \approx 6 \times 10^{-3} \ll 1$ and will be assumed small in what follows.

We seek solutions which are steady in a frame of reference moving eastward at speed $(U + \epsilon c)$. (We shall show in Section 3 that no north–south propagation can occur.) The two terms are straightforward. U is simply the mean advection velocity, and plays no role in the dynamics (cf. Griffiths *et al.*, 1982), and ϵc, $c = O(1)$, is a small perturbation induced by the β-effect. (One can, as in the reference cited, prove U to be the leading term in an expansion of c.)

We define

$$X = x - (U + \epsilon c)t, \quad (2.5)$$

$$Y = y, \quad (2.6)$$

$$T = t, \quad (2.7)$$

and set $\partial/\partial T \equiv 0$, yielding (where now u is measured relative to U)

$$(u - \epsilon c)u_X + vu_Y - (1 + \epsilon Y)v + h_X = 0, \quad (2.8)$$

$$(u - \epsilon c)v_X + vv_Y + (1 + \epsilon Y)u + h_Y = 0, \quad (2.9)$$

$$(u - \epsilon c)h_X + vh_Y + h(u_X + v_Y) = 0. \quad (2.10)$$

Finally, using polar coordinates (r, θ) and velocities (U, V), we obtain

$$(U - \epsilon c \cos\theta)U_r + \frac{1}{r}(V + \epsilon c \sin\theta)U_\theta - \frac{V^2}{r} - \left(1 + \epsilon r \sin\theta + \frac{\epsilon c \sin\theta}{r}\right)V + h_r = 0, \quad (2.11)$$

$$(U - \epsilon c \cos\theta)V_r + \frac{1}{r}(V + \epsilon c \sin\theta)V_\theta + \frac{UV}{r} + \left(1 + \epsilon r \sin\theta + \frac{\epsilon c \sin\theta}{r}\right)U + \frac{h_\theta}{r} = 0, \quad (2.12)$$

$$(U - \epsilon c \cos\theta)h_r + \frac{1}{r}(V + \epsilon c \sin\theta)h_\theta + h\left(U_r + \frac{U}{r} + \frac{V_\theta}{r}\right) = 0. \quad (2.13)$$

We now pose an expansion in ϵ for h, U and V, of form

$$h \sim h_0 + \epsilon h_1, \quad U \sim U_0 + \epsilon U_1, \quad V \sim V_0 + \epsilon V_1, \quad (2.14)$$

so that to leading order the flow is radially symmetric and satisfies

$$U_0 \equiv 0, \quad (2.15)$$

$$h_{0r} = V_0 + \frac{V_0^2}{r}, \quad (2.16)$$

$$h_0 = h_0(r), \quad V_0 = V_0(r), \quad (2.17)$$

[1] In the third case, for example, U would be a linear combination of the velocities above and below the blob, with coefficients depending on the density differences between layers.

and we shall take as boundary conditions

$$h_0 = 1, \quad r = 0, \tag{2.18}$$

$$h_0 = 0, \quad r = a, \tag{2.19}$$

so that $\bar{h}$ represents the dimensional height of the center of the blob.

The O(ϵ) terms satisfy

$$\frac{V_0 U_{1\theta}}{r} - \left(\frac{2V_0}{r} + 1\right)V_1 + h_{1r} = r\sin\theta V_0 + \frac{c\sin\theta}{r} V_0, \tag{2.20}$$

$$\left(V_{0r} + \frac{V_0}{r} + 1\right)U_1 + \frac{V_0 V_{1\theta}}{r} + \frac{h_{1\theta}}{r} = c\cos\theta V_{0r}, \tag{2.21}$$

$$(U_1 - c\cos\theta)h_{0r} + \frac{V_0}{r} h_{1\theta} + h_0\left(U_{1r} + \frac{U_1}{r} + \frac{V_{1\theta}}{r}\right) = 0, \tag{2.22}$$

and it is then easy to see that these are given by

$$h_1 = \sin\theta H(r), \tag{2.23}$$

$$V_1 = \sin\theta V(r), \tag{2.24}$$

$$U_1 = \cos\theta U(r), \tag{2.25}$$

so that the blob is distorted by the β-effect as sketched in Fig. 2.[2] The shape of the edge is given by (2.19), (2.14) as

$$r = a - \frac{\epsilon H_1(a)}{h_0'(a)} \sin\theta. \tag{2.26}$$

Substituting (2.23)–(2.25) into (2.22), the O(ϵ) terms give

$$-\frac{V_0}{r} U - \left(\frac{2V_0}{r} + 1\right)V + H_r = rV_0 + \frac{c}{r} V_0, \tag{2.27}$$

$$\left(V_{0r} + \frac{V_0}{r} + 1\right)U + \frac{V_0}{r} V + \frac{H}{r} = cV_{0r}, \tag{2.28}$$

$$(rh_0 U)_r + h_0 V + V_0 H = crh_{0r}, \tag{2.29}$$

which is an eigenvalue problem for c, to be solved subject to well-behaved solutions at the singularities $r = 0$ and $r = a$ (where h_0 vanishes). This latter condition is equivalent to requiring the edge of the blob to be a streamline of the flow.[3] The equations will be solved in Section 4 for quadratically varying depth profiles.

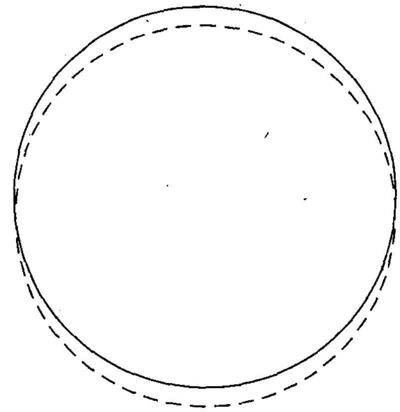

FIG. 2. The shape of the perturbed eddy, seen from above. (The curve drawn is $r = 1 + 0.1 \sin\theta$; the dashed line shows the circle $r = 1$ for comparison.)

3. Bounds on speed of propagation

Before deriving some limitations on the speed of the blob, it is worth noting a restriction on its width. Recall that the depth h_0 is scaled to 1 at the center of the blob (i.e., the deformation radius is scaled on this depth). Then it is straightforward to show that *all circular blobs must have a radius of at least $2\sqrt{2}$ deformation radii.* The proof now follows.

From (2.16), h_{0r} has a minimum value as a function of V_0, when $V_0 = -r/2$, namely

$$h_{0r} \geq -r/4. \tag{3.1}$$

Integrating from $r = 0$, and using (2.18), we have

$$h_0 \geq 1 - r^2/8, \tag{3.2}$$

which is the zero potential vorticity limit (Gill *et al.*, 1979), so that

$$h_0 \geq 0, \quad r \leq 2\sqrt{2}. \tag{3.3}$$

Hence the blob must be wider than $2\sqrt{2}$, which may help to explain why Gulf Stream rings, etc., are several deformation radii in width rather than approximately a single radius wide as might be expected on baroclinic scaling arguments. This result is in excellent agreement with Flierl's (1979) numerical results, although concealed by his nondimensionalisation.

The speed c has already been derived by Nof (1981, 1982). However, his analysis *assumes* a steady westward propagation (as we do here), whereas it can be shown by an adaptation of Ball's (1963) analysis to be the only possibility. We now use D/Dt to mean a material derivative, and then if dS denotes an element of horizontal area over which $h \geq 0$, Eq. (2.3) implies that the mass of the fluid

[2] This is in agreement with the statements of Nof (1981), although the shape of his schematic is perhaps slightly misleading. A referee has drawn my attention to the fact that G. Flierl has also found the form (2.23).

[3] That is, the material derivative of $r - a - \epsilon a_1 \sin\theta$ vanishes at the edge of the blob for some a_1, and $h_0 + \epsilon H \sin\theta$ vanishes there also. These conditions at O(ϵ) combine to give (2.29), with terms proportional to h_0 removed.

$$Q = \int h dS \tag{3.4}$$

is constant. Further, using Ball's (1963) result that for any scalar q,

$$\frac{d}{dt}\int hq dS = \int h \frac{Dq}{Dt} dS, \tag{3.5}$$

and the positions of the center of gravity $(\bar{X}, \bar{Y})$ given by

$$Q\bar{X} = \int hX dS, \quad Q\bar{Y} = \int hY dS, \tag{3.6}$$

we have

$$Q\frac{d\bar{X}}{dt} = \int h \frac{DX}{Dt} dS = \int hu dS, \tag{3.7}$$

$$Q\frac{d\bar{Y}}{dt} = \int h \frac{DY}{Dt} dS = \int hv dS. \tag{3.8}$$

Then (Ball, 1963), differentiating (3.7), (3.8), and using (2.1), (2.2), we find

$$\begin{aligned} Q\frac{d^2\bar{X}}{dt^2} &= \int h\frac{Du}{Dt} dS = \int h(1+\epsilon Y)v dS \\ &= \int hv dS + \epsilon \int hYv dS \\ &= Q\frac{d\bar{Y}}{dt} + \epsilon \int hYv dS, \end{aligned} \tag{3.9}$$

$$\begin{aligned} Q\frac{d^2\bar{Y}}{dt^2} &= \int h\frac{Dv}{Dt} dS = -\int h(u-U)(1+\epsilon Y) dS \\ &= -\int hu dS - \epsilon \int huY dS \\ &\quad + U\int h dS + \epsilon U \int hY dS. \end{aligned} \tag{3.10}$$

(We have used the fact that integrals of hh_x and hh_y vanish since h vanishes at the boundary of S.) We now substitute (3.7), (3.8), and substitute the radially symmetric solutions into integrals with ϵ coefficients [i.e., perform an O(ϵ) expansion]. Now

$$\begin{aligned} \int h_0 Y v_0 dS &= \int h_0(r) r \sin\theta V_0(r) \cos\theta r dr d\theta \\ &= 0, \end{aligned} \tag{3.11}$$

$$\begin{aligned} \int h_0 Y u_0 dS &= \int h_0(r) r \sin\theta \{-V_0(r)\sin\theta\} r dr d\theta \\ &= -\pi \int_0^a h_0 V_0 r^2 dr, \end{aligned} \tag{3.12}$$

$$\int h_0 Y dS = \int h_0 r \sin\theta r dr d\theta = 0. \tag{3.13}$$

Hence

$$Q\frac{d^2\bar{X}}{dt^2} = Q\frac{d\bar{Y}}{dt}, \tag{3.14}$$

$$Q\frac{d^2\bar{Y}}{dt^2} = -Q\frac{d\bar{X}}{dt} + \epsilon\pi \int_0^a h_0 V_0 r^2 dr + QU, \tag{3.15}$$

i.e.,

$$\bar{X}_{tt} = \bar{Y}_t, \tag{3.16}$$

$$\bar{Y}_{tt} = U - \bar{X}_t + \frac{\epsilon\pi}{Q}\int_0^a h_0 V_0 r^2 dr. \tag{3.17}$$

This shows that north–south motions of the center of gravity are purely inertial oscillations (of frequency unity), so that $\bar{Y}_t$ may be taken as zero without loss of generality. Then $\bar{X}_t$ is constant, given by

$$\bar{X}_t = U + \frac{\epsilon\pi}{Q}\int_0^a h_0 V_0 r^2 dr. \tag{3.18}$$

Ignoring the constant advection velocity U, the eddies move with a velocity

$$\bar{X}_t = \frac{\epsilon\pi}{Q}\int_0^a h_0 V_0 r^2 dr = \frac{-\epsilon \int \tfrac{1}{2} h_0 (V_0^2 + h_0) dS}{\int h_0 dS}, \tag{3.19}$$

after use of (2.16). This shows that the speed is *always directed westward* and is given by a simple ratio of blob energy to blob mass. This agrees with Stern's (1975) result that stationary eddies must have a zero ratio of energy to mass. The formula (3.19) is equivalent to Nof's (1981) Eq. (2.10), despite the fact that Nof's applies to a moving coordinate system, because the speed of movement is of order ϵ, which is small compared with the velocity V_0, of order 1.

It is of interest to seek bounds on the speed c $(=\bar{X}_t\epsilon^{-1})$. If we first impose the physically plausible restriction that the eddy has positive absolute velocity everywhere [i.e., the relative vorticity $r^{-1}(rV_0)_r$ is greater than -1], then

$$0 \geq r^{-1}(rV_0)_r \geq -1 \Rightarrow -r/2 \leq V_0 \leq 0, \tag{3.20}$$

or

$$V_0^2 \leq r^2/4. \tag{3.21}$$

Substitution into (3.19) gives

$$\begin{aligned} |c| &= \frac{\tfrac{1}{2}\int_0^a r(h_0^2 + h_0 V_0^2) dr}{\int_0^a r h_0 dr} \\ &\leq \frac{\tfrac{1}{2}\int_0^a r(h_0^2 + h_0 r^2/4) dr}{\int_0^a r h_0 dr}, \end{aligned} \tag{3.22}$$

and straightforward calculus of variations shows this to be the least when

$$h_0 = 1 - r^2/8, \qquad (3.23)$$

the zero potential vorticity value (when $V_0 \equiv r/2$, so that the value for c is attainable). Substituting (3.23) into (3.22) then yields

$$|c| \leqslant 2/3, \qquad (3.24)$$

so that with this restriction, eddies propagate at most at two-thirds of the long baroclinic Rossby wave speed. This can be compared with the numerical quasi-geostrophic calculations of McWilliams and Flierl (1979), who found a limit of precisely the long wave speed. In their calculations, however, this speed was defined by the stratification of the surrounding fluid at rest, whereas, in this study, it is the eddy itself which partially determines the wave speed through the definition of the deformation radius.

If the less plausible class of eddies which can have negative absolute vorticities is included, there is no maximum eddy propagation speed. (This assumes such eddies to be stable, which seems unlikely, although there does not seem to be an equivalent to Rayleigh's criterion for instability.) Such eddies could perhaps have come from some other latitude—by an unspecified propagation method—or be the result of an anticyclonic wind stress acting for a long time in the azimuthal direction, since the material derivative of potential vorticity depends on wind stress. To see that these eddies can travel at any speed (subject of course to stability of the eddy both as a one-layer system and to Kelvin–Helmholtz instability with the surrounding water, and also the requirement that c remain of order ϵ) we need only consider the depth profile $h_0 = 1 - \exp[(r - a)/b]$ for large a and fairly large b. There are two solutions for V_0 from (2.16). The larger in magnitude is approximately $(-a)$ everywhere, giving a large negative vorticity. The resulting value for $|c|$ is given by (3.19) as of order $b \gg 1$. We reiterate that this case is physically improbable, however, but include it for completeness.

The minimum value of $|c|$ is less obvious, since there is not total freedom on depth profiles because of (3.2). However, it is straightforward to show that the minimum is zero. We examine profiles which are self-similar as a varies, i.e., $h_0 = h_0\,(\xi = ra^{-1})$. Then (2.16) gives

$$h_\xi = aV_0 + \frac{V_0^2}{\xi}, \qquad (3.25)$$

and c is given by

$$|c| = \frac{-a\displaystyle\int_0^1 \xi^2 h_0 V_0 d\xi}{\displaystyle\int_0^1 \xi h_0 d\xi}. \qquad (3.26)$$

Then

$$\frac{\partial|c|}{\partial a}\int_0^1 \xi h_0 d\xi = -2\int_0^1 \frac{\xi^2 h_0 V_0^2 d\xi}{a\xi + 2V_0}, \qquad (3.27)$$

where $\partial V_0/\partial a$ is obtained from (3.25). Solving for V from (3.25) yields

$$2V_0 + a\xi = \pm\int (a^2\xi^2 + 4\xi h_\xi). \qquad (3.28)$$

If the positive root is taken, $\partial|c|/\partial a$ is negative so that $|c|$ is least as $a \rightarrow \infty$. If the negative root is taken, $\partial|c|/\partial a$ is positive and $|c|$ is least at $a = 2\sqrt{2}$, its minimum value (which is calculable from the zero potential vorticity case).[4] The $a \rightarrow \infty$ root dominates, in fact, and (3.28) then shows that $V_0 \approx h_\xi/a$ (the geostrophic limit). Eq. (3.26) then shows that $|c|$ may be made as small as desired by choosing h to decay exponentially away from $\xi = 0$.

4. An example: Quadratic depth profiles

We consider now the specific case

$$h_0 = 1 - r^2a^{-2}, \quad a \geqslant 2\sqrt{2}, \qquad (4.1)$$

$$V_0 = \sigma r, \qquad (4.2)$$

$$\sigma + \sigma^2 = -2a^{-2}, \qquad (4.3)$$

where (4.3) is derived from (2.16).[5] (In fact any polynomial for h_0 can be treated by this method.) The solution to (2.27)–(2.29) is given by seeking polynomial expressions in r for H, U and V (here H is a cubic, U and V quadratics). The coefficients for U and V are given in terms of those for H from (2.27), (2.28); substitution into (2.29) and matching coefficients of r^n, $n = 0, 1, 2, \ldots$, then closes the problem. In this case the constant terms in (2.29) vanish identically, and so do all those proportional to the r term in H. This is because $h_1 \propto \epsilon r \sin\theta$ represents an arbitrary small displacement north or south of the quadratic profile (4.1); so without loss of generality this term can be taken as zero. Matching terms of order r^2, r^4 gives the solution as

$$c = a^2\sigma/6, \qquad (4.4)$$

$$H = \frac{3\sigma - 1}{12} r^3, \qquad (4.5)$$

$$U = -\frac{a^4\sigma^3}{12} - \frac{(\sigma - 1)r^2}{12(\sigma + 1)}, \qquad (4.6)$$

$$V = \frac{a^4\sigma^3}{12} - \frac{(\sigma + 3)r^2}{12(\sigma + 1)} \qquad (4.7)$$

[recall that $\sigma < 0$, by (4.3)].

[4] And $|c| \rightarrow \infty$ as $a \rightarrow \infty$, the unrealistic case mentioned above.
[5] Note the vorticity of the flow must lie between ± 1.

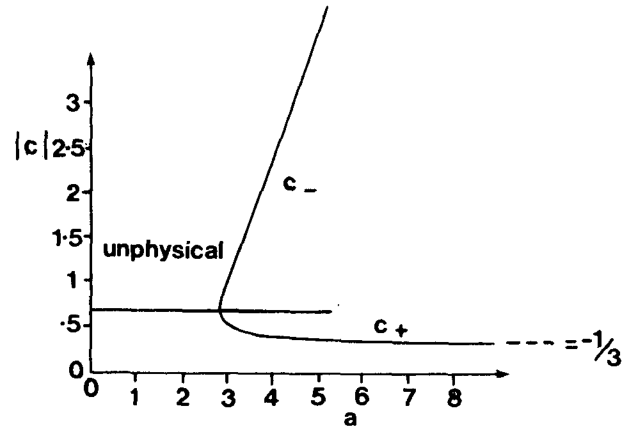

FIG. 3. The magnitude of the eddy speed $|c|$ as a function of radius a for the quadratic profile (4.1). The dimensional unit for $|c|$ is βR^2.

As a varies, there are two roots for σ given by (4.3) as

$$\sigma_{\pm} = -\tfrac{1}{2} \pm \tfrac{1}{2}(1 - 8a^{-2})^{1/2}. \qquad (4.8)$$

Here σ_+ corresponds to a slow swirl velocity and σ_- to an unrealistic rapid swirl velocity [when $-1 \leqslant \sigma \leqslant -\tfrac{1}{2}$, the relative vorticity $r^{-1}(rV_0)_r$ is less than -1; this would have to have been created by some anticyclonic wind system or topographic interaction, for example]. For large a, $\sigma_+ \approx -2a^{-2}$, $\sigma_- \approx -1$. A graph of the values of c is shown in Fig. 3, as a function of a. The $\sigma = 2\sqrt{2}$ root gives $c = -\tfrac{2}{3}$, as found previously. The c_- root increases without limit as a increases; the c_+ root asymptotes to $\tfrac{1}{3}$. The lower curve is identical with that found by Nof (1981)[6]; he has successfully compared the order of magnitude of the predicted c_+ with various observed eddies.

The corresponding velocity field for two values of a is shown in Fig. 4. U is everywhere positive, V and H everywhere negative. The limiting values are

$$a = 2\sqrt{2}: \quad \left.\begin{aligned} U &= r^2/4 + \tfrac{2}{3} \\ V &= -\tfrac{5}{12}r^2 - \tfrac{2}{3} \\ H &= -\tfrac{5}{24}r^2 \end{aligned}\right\}. \qquad (4.9)$$

$$a \to \infty, \sigma_+: \quad \left.\begin{aligned} U &= r^2/12 \\ V &= -r^2/4 \\ H &= -r^3/12 \end{aligned}\right\}. \qquad (4.10)$$

$$a \to \infty, \sigma_-: \quad \left.\begin{aligned} U &= \frac{a^2}{12}(a^2 + r^2) \\ V &= \frac{a^2}{12}(r^2 - a^2) \\ H &= -r^3/3 \end{aligned}\right\}. \qquad (4.11)$$

These yield a flow as in Fig. 5 (for the case $a = 2\sqrt{2}$, but all cases are qualitatively similar). The value of ϵ is deliberately exaggerated to show the resulting asymmetry of the flow, with a bias towards stronger motions poleward of the center of the eddy.

5. Discussion

In this paper we have shown that isolated eddies on a β-plane can have a wide range of velocities, all westward relative to any surrounding mean flow. The eddy speed is limited to two-thirds of the long Rossby wave speeds except for unrealistically energetic eddies. All eddies must be at least 2.84 deformation radii in diameter. The shape of the eddy is of the form $r = a + \epsilon A \sin\theta$ for some A; the perturbed radial velocity varies like $\cos\theta$. Mean depths which are polynomial in r have perturbations which are also polynomials. We have solved the quadratic case; the quartic case considered by Nof (1981) is straightforward but algebraically tedious, and is not presented here.

The success or failure of a perturbation solution depends on many things. Clearly the speed c cannot be too large, as the perturbation would break down. This suggests that rapidly rotating "pancakes" may have an interesting structure, which this analysis cannot reproduce. Again, it is important that the resulting steady flow be *stable*. Although the stability of the combined $O(1) + O(\epsilon)$ flow is unknown, it is straightforward but lengthy to show that the quadratic mean profile (4.1) on an f-plane is stable to all small perturbations, and this calculation is performed in the Appendix. This result is surprising, since Griffiths *et al.* (1982) have shown that the two-dimensional equivalent of the radial flow [$u = u(y)$, $v = 0$] is unstable for all velocity and height distributions.

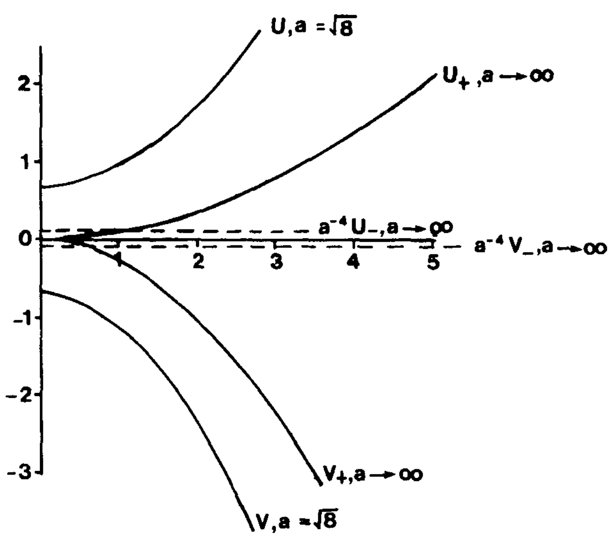

FIG. 4. U, V as functions of r for the quadratic depth profile (4.1) in the limiting cases $a = 2\sqrt{2}$, and $a \to \infty$ (in the latter case the solutions for σ_+, σ_- are both shown).

[6] Nof rejected the upper curve by requiring cyclonic potential vorticity.

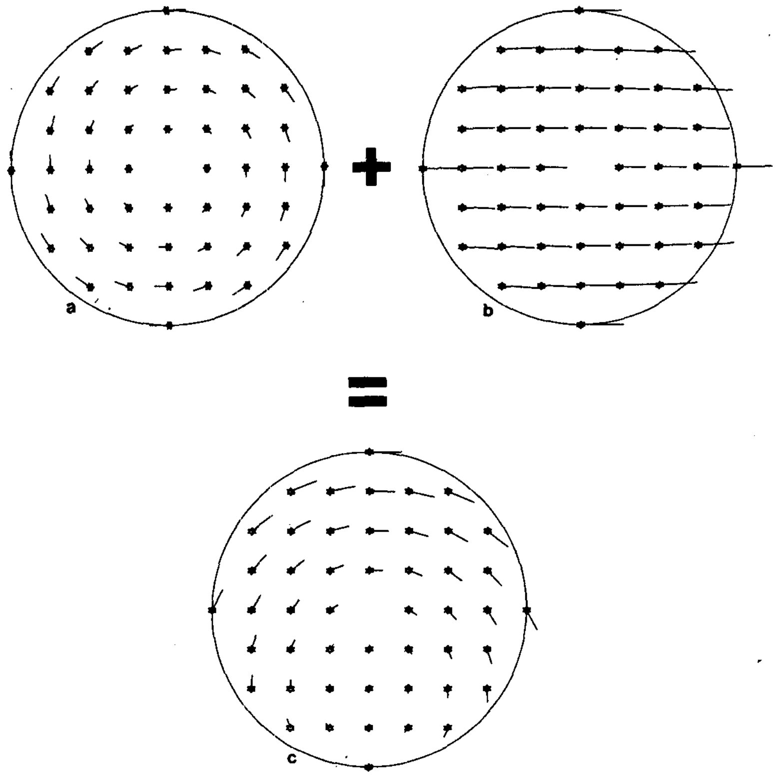

FIG. 5. A schematic of the flow in the eddy. (a) shows the undisturbed flow, (b) the perturbation. These are combined, with $\epsilon = 0.3$ to exaggerate the perturbation, in (c).

Nonetheless, the quadratic profile *is* a special case, and stability of the combined flow is *not* assured, although the growth rate can at best be $O(\epsilon)$, implying that the eddy could at worst propagate a few diameters before (possibly) breaking up.

Other forms of instability must inevitably affect the eddy. Nof (1981) notes that Kelvin–Helmholtz instability with the surrounding fluid will modify the edges of the "quadratic" eddy. Baroclinic instability, or a mixed barotropic-baroclinic instability, is certainly present in laboratory eddies even for approximately zero potential vorticity (Griffiths and Linden, 1981). Thus the interaction of the eddy with the surrounding water masses cannot be neglected. At the very least this must be manifested as some frictional term which will gradually dissipate the eddy. The very existence of "meddies" suggest that their survival depends on a net westward translation (including mean advection, for example, which cannot be too small, else diffusion would have destroyed the meddy. Nof's (1981, 1982) calculations show that speeds of about $0.3\beta R^2$ give reasonable agreement with observations.

Thus, to improve models of isolated eddies, interaction of the eddy with the rest of the ocean (i.e., baroclinic effects) must be included: a two-layer model similar to this would not be too difficult to create. Notably lacking, however, would be time variation of the eddy and of the main ocean flow. A successful model, even for a single layer, which includes regions of vanishing depth, has not to my knowledge been presented except for very restricted oceanic regions such as oceanic fronts. Such a model would be a very useful tool for the heirarchy of ocean circulation problems.

APPENDIX

Stability of the Quadratic Depth Distribution

Small perturbations to $\exp[il(\theta - \alpha t)]$ are sought to the mean flow (4.1), (4.2), where l is a wavenumber in the θ direction (an integer). Instability will occur only if $l\,\mathrm{Im}(\alpha) > 0$. Then the polar coordinate version of (2.1)–(2.3), with $\epsilon = 0$, gives

$$il\left(\frac{V_0}{r} - \alpha\right)u - \left(\frac{2V_0}{r} + 1\right)v + h_r = 0, \quad \text{(A1)}$$

$$il\left(\frac{V_0}{r} - \alpha\right)v + \left(V_{0r} + \frac{V_0}{r} + 1\right)u + \frac{ilh}{r} = 0, \quad \text{(A2)}$$

$$il\left(\frac{V_0}{r} - \alpha\right)h + \frac{1}{r}(ruh_0)_r + \frac{ilh_0}{r}v = 0, \quad \text{(A3)}$$

subject to the requirements that u, v, h be well-behaved at $r = 0, a$.[7] The coefficients of u and v in (A1) and (A2) are constant, so that (A3) becomes

$$\left\{r\left(1-\frac{r^2}{a^2}\right)h_r\right\}_r + h\left\{\left(\frac{2\gamma}{a^2\phi} - \gamma^2 + l^2\phi^2 + l^2/a^2\right)r - \frac{l^2}{r}\right\} = 0, \quad \text{(A4)}$$

where

$$\alpha - \sigma = \phi, \quad 2\sigma + 1 = \gamma, \quad \text{(A5)}$$

are conveniently defined. In particular, complex α implies complex ϕ; we shall show that $\mathrm{Im}(\phi) = 0$. A series expansion about $r = 0$ shows that $h \propto r^l$ near $r = 0$ (indeed, the solutions all turn out to be polynomials in r). So we may write

$$h = r^l M(r), \quad \text{(A6)}$$

giving

$$\left\{\left(r^{2l+1} - \frac{r^{2l+3}}{a^2}\right)M_r\right\}_r = r^{2l+1}MQ, \quad \text{(A7)}$$

where

$$Q = \gamma^2 + \frac{2l}{a^2} - l^2\phi^2 - \frac{2\gamma}{a^2\phi} \quad \text{(A8)}$$

is a (possibly complex) constant.

The proof now proceeds in stages. We first show that Q is real and negative for instability, and express it in terms of $\mathrm{Re}(\phi)$ only. We then minimize it as a function of $\mathrm{Re}(\phi)$ and again as a function of γ. We show this minimum to be positive, thus proving stability.

1. *Q* is real and negative for instability

We multiply (A7) by M^*, the complex conjugate of M, and integrate from $r = 0$ to a, where M vanishes. This gives

$$0 = \int_0^a \left\{r^{2l+1}\left(1 - \frac{r^2}{a^2}\right)|M_r|^2 + r^{2l+1}|M|^2 Q\right\}dr, \quad \text{(A9)}$$

after integration by parts. The imaginary part shows that $\mathrm{Im}(Q)$ is zero, while the real part shows $\mathrm{Re}(Q) < 0$.

Now if

$$\phi = \psi + i\chi, \quad \text{(A10)}$$

$\mathrm{Im}(Q) = 0$ implies

$$\chi\left(-2l^2\psi + \frac{2\gamma}{a^2|\phi|^2}\right) = 0, \quad \text{(A11)}$$

so that if χ is not zero (which would imply stability), the term in parentheses must be, or

$$\chi^2 = \frac{\gamma}{a^2l^2\psi} - \psi^2. \quad \text{(A12)}$$

[7] That these *are* the correct boundary conditions can be seen from an expansion of the full nonlinear problem.

Substitution into Q and requiring $Q < 0$ then gives

$$Q = \gamma^2 + \frac{2l^2}{a^2} - 4l^2\psi^2 + \frac{\gamma}{a^2\psi} < 0, \quad \text{(A13)}$$

with $\chi^2 \geq 0$; or from (A12)

$$\psi^2 < \frac{\gamma}{a^2l^2\psi}. \quad \text{(A14)}$$

2. Minimization of *Q*

We note that as $-1 \leq \sigma \leq 0$, $-1 \leq \gamma \leq 1$. Also, since $\gamma \to -\gamma$, $\psi \to -\psi$ leaves (A13), (A14) unaltered, we can require $0 \leq \gamma \leq 1$ in what follows. Then (A14) first implies $\psi \geq 0$; and then

$$0 \leq \psi \leq \left(\frac{\gamma}{a^2l^2}\right)^{1/3}. \quad \text{(A15)}$$

Now

$$\frac{\partial Q}{\partial\psi} = -8l^2\psi - \frac{\gamma}{a^2\psi^2} < 0, \quad \text{(A16)}$$

so Q is least when ψ is largest, i.e., $(\gamma/a^2l^2)^{1/3}$. Hence, on substitution, we see that

$$Q_{\min} = \gamma^2 + \frac{2l^2}{a^2} - \frac{3l^{2/3}\gamma^{2/3}}{a^{4/3}} \quad \text{(A17)}$$

is a function of $\lambda = \gamma^{2/3}$; $0 \leq \lambda \leq 1$. But

$$\frac{\partial Q_{\min}}{\partial\lambda} = 3\lambda^2 - \frac{3l^{2/3}}{a^{4/3}}. \quad \text{(A18)}$$

So there are two possibilities:

$$\left.\begin{array}{l} 1\colon \dfrac{l^{1/3}}{a^{2/3}} > 1, \quad \dfrac{\partial Q_{\min}}{\partial\lambda} < 0 \\ \quad Q_{\min} \text{ least at } \lambda = 1 \\ 2\colon \dfrac{l^{1/3}}{a^{2/3}} < 1, \quad \dfrac{\partial Q_{\min}}{\partial\lambda} = 0 \quad \text{at} \quad \lambda = \dfrac{l^{1/3}}{a^{2/3}} \\ \quad Q_{\min} \text{ least at } \lambda = \dfrac{l^{1/3}}{a^{2/3}} \end{array}\right\}. \quad \text{(A19)}$$

Possibility 2 gives the smallest possible value of $Q_{\min}$ as

$$Q_{\min} = \frac{2l}{a^2}(l-1) \geq 0,$$

so that this case is stable automatically (the case $l = 0$ is trivial to show stable). Possibility 1 means that

$$Q_{\min} = 1 + \frac{2l^2}{a^2} - \frac{3l^{2/3}}{a^{4/3}},$$

and that $Q_{\min}$ is an increasing function of a if $l^{1/3} \times a^{-2/3}$ is held constant. Thus $Q_{\min}$ is least when a is least, at $a = 2\sqrt{2}$ and is

$$Q_{\min} = 1 + \frac{l^2}{4} - \frac{3}{4} l^{2/3} > 0, \qquad \text{(A20)}$$

because $l_{\min}^{1/3} > a_{\min}^{2/3}$ or $l > 8$ by 1 above.

So in both cases $Q > 0$ and there is therefore stability, which ends the proof.

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