Comments on "The Level of No Motion in an Ideal Fluid"

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The problem of the ideal fluid thermocline equations has a long history, and it seems remarkable that we are hardly any nearer solving it today than twenty years ago. Olbers and Willebrand (hereafter OW) have written an interesting paper, although I remain unconvinced by their arguments (which is not to say I believe them to be false). There seem to me to be two ways out of the dilemma of whether LANMs exist. One is to use Shepherd's (1983) approach in Ocean Modelling 50 and to argue that real fluids are not ideal, and so our arguments are academic. I feel, however, that it is worth following the second route, to examine the mathematical implications of our assumed dynamics and see if we can accept what they yield. Hence the work of OW and Killworth (1983) (the latter paper does not use LANMs, but still finds many in the data).

To show why I am unconvinced, I present here a single argument based only on the density field rather than the functions in OW. Define $q = f\rho_z$, $r = f^2\rho_{zz}$. Then these satisfy

$$u\rho_x + v\rho_y + w\rho_z = 0, \qquad (1)$$

$$uq_x + vq_y + wq_z = 0, \qquad (2)$$

$$ur_x + vr_y + wr_z = \frac{g}{\rho_0} J_{xy} \qquad (3)$$

where

$$J_{xy} = \rho_x q_y - \rho_y q_x. \qquad (4)$$

Eqs. (1), (2) may be manipulated to give

$$uJ_{xy} = wJ_{yz}, \quad uJ_{xz} = -vJ_{yz}, \quad vJ_{xy} = -wJ_{xz}. \qquad (5)$$

The first point to make is that (1)–(3) imply that ρ satisfies a rather difficult equation (implicit in many of the references cited by OW). Either we write

$$K_{xy} = \rho_x r_y - \rho_y r_x, \text{ etc.,} \qquad (6)$$

so that

$$uD = \frac{-g\rho_z}{\rho_0} J_{yz}J_{xy}, \quad vD = \frac{g\rho_z}{\rho_0} J_{xz}J_{xy}, \qquad (7)$$

where

$$D = K_{xz}J_{yz} - J_{xz}K_{yz}. \qquad (8)$$

If we use the geostrophic relation

$$(fu)_x + (fv)_y = 0, \qquad (9)$$

(7) will yield an appalling third-order equation in z and second-order in x and y. We could also differentiate (3) again with respect to z, eliminate w and substitute (7) to give an equally complicated equation for ρ which is fourth-order in z, but first in x and y. Either way, ρ satisfies an equation. This will be important later.

The second point is to examine what happens when $w = 0$. Eq. (5) shows that at such a point either

$$u = v = 0 \qquad (10)$$

or

$$J_{xy} = 0. \qquad (11)$$

If (10) is true we have a LANM, so consider instead (11). If J_{xy} vanishes, then at that point (1)–(3) are three homogeneous equations for two unknowns (u and v). Unless (10) holds, the ratio $r_x r_y^{-1}$ is determined (i.e., $K_{xy} = 0$). This is a relation in ρ_{zz}, a *second* derivative in z, and so is not apparently predicted in advance by either of the above equations for ρ, which are of higher order.

So it looks likely that J_{xy} vanishing does not necessarily imply K_{xy} vanishing. If not, either the LANM occurs or $r_z \rightarrow \infty$ ($\rho_{zz} \rightarrow \infty$) which is unpleasant also, but might be acceptable.

For me, the problem remains open. Even if $u = v = 0$, (3) implies J_{xy} vanishes anyway. What happens on a flat bottom, where w is forced to vanish for all x, y? Perhaps we must await numerical solutions of (1)–(3) before the solution to the problem becomes obvious.

REFERENCES

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