On the Time-Dependent Meandering of a Thin Jet

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ABSTRACT

A thin-jet model predicts the location of the axis of a strong current such as the Gulf Stream by using the vertical and cross-stream integrated vorticity balance, under the assumption that the meandering scales are large compared to the width of the jet. We demonstrate that such an integral provides a matching condition upon the barotropic component of the wave or eddy fields which, on either side of the jet, have north–south scales on the order of the meander wavelength. For steady meanders, these exterior fields do not influence the path and our model reproduces the dynamics of Robinson and Niiler, but for the transient case, the determination of the jet axis motion and of the external field is a coupled problem.

When the disturbances in the axis position are time-dependent but are very small, the exterior wave problem can be linearized and the matching conditions can be applied at the mean position of the jet. We can therefore derive a dispersion relation for the meandering motion, allowing us to compute the phase speed and growth rates for the meanders in terms of the wavenumber and two integral properties of the stream: the mass and momentum transports. This dispersion relation predicts instability for waves shorter than a critical scale.

We also derive via standard four-dimensional instability theory a long wave approximation to the dispersion relation for perturbations of a quasi-geostrophic jet with both horizontal and vertical shears. The result is identical to that from the thin-jet theory for an interesting class of perturbations, which we therefore identify as meandering modes. Thus thin-jet theory has been calibrated by reduction to both finite amplitude steady meandering and infinitesimal instability cases. For the understanding of large amplitude, time-dependent motions of the Gulf Stream and their role in the general circulation, the thin jet theory offers a semi-analytical approach for process studies.

1. Introduction

The variability in the track of the Gulf Stream was recognized long before the realization that mesoscale motions were prevalent throughout the ocean. Maury (1855) comments on supposed seasonal variations in the Gulf Stream position and on eddy currents. In the 1930s, analysis of hydrographic data and sea level variations led Iselin (1936, 1940), Montgomery (1938) and others to recognition of the rapid changes in position of the Stream, the large fluctuations in isotherm depths near the Stream, and even the presence of Gulf Stream rings. Ship-surveying techniques became more sophisticated with the development of the BT (Iselin and Fuglister, 1948; Fuglister, 1963) and, later, the XBT; the rapidly repeated tracks of Hansen (1970) and Robinson, Luyten and Fuglister (1974) showed the downstream progression of meanders at 5 to 20 cm s^{-1} with a gradual growth in amplitude of many features. The latter work also showed the value of aircraft surveys of infrared radiation as an indicator of deep structure; more recently, satellite-based surface temperature maps have been used to trace the structure of meanders by Maul *et al.* (1978) and Halliwell and Mooers (1979). Recent studies of the Northwestern Atlantic using satellite altimetric data have increased our knowledge of the variability (e.g., Cheney and Marsh, 1981), and Robinson *et al.* (1983) have shown that the signal of the meandering and the surrounding field can be readily detected; this technique has great promise in the future for giving synoptic data on the geostrophic surface flow field. This suite of techniques has been complemented by moored instrumentation (Luyten, 1977; Watts and Johns, 1982) and drifter data (Richardson, 1981; Schmitz *et al.*, 1981). The modern observational programs have indicated the breadth and complexity of the wavenumber and frequency spectrum for Gulf Stream meandering motions; the reader is referred to review articles by Fofonoff (1981) and (specifically focused on the variability) Watts (1983) for summaries of the time-dependent fluctuations.

Theoreticians have been concerned with the source, nature and predictability of the meandering motions. In the recent literature, there have been two approaches to the problem. The first has been the application of linearized instability theory, either barotropic or baroclinic or both, in order to calculate the properties of waves upon the current; the second approach has been the derivation of equations for changes in direction of the current using the cross-stream integrated vorticity balance. Here we contribute to the second framework by finding the proper cross-stream integrated balance for a time-dependent meandering model and explain the role of the wave fields external to the jet in determining the evolution. In addition, we derive via conventional instability theory a dispersion relationship for long waves on a baroclinic jet of width comparable to the deformation radius. From this new result, we can prove that the small amplitude limit of the thin-jet model gives the same dispersion relationship as instability theory in the long wave limit.

The instability theories have followed the well-understood technique of examining linearized perturbations upon a zonal flow and attempting to predict dispersion relations including temporal or spatial growth rates. Representative analytical theories wherein applications have been made directly to the Gulf Stream include barotropic (Haurwitz and Panofsky, 1950), baroclinic (Nikitin and Tareev, 1972) and mixed (Orlanski, 1969; Flierl, 1975; Holland and Haidvogel, 1980; Talley, 1982) models. One remarkable result is that when the best attempts are made to choose model parameters (such as flow speeds in the various layers) in a consistent fashion from integrated properties of the Stream, the dispersion relations for all the models are rather similar. Instabilities occur for wave scales ($\lambda/2\pi$) less than about 150 km with the maximum growth rate occurring at short scales; generally this growth rate is more rapid than that indicated by observational data (except in the study of Nikitin and Tareev, 1972, where a large eddy viscosity is used to weaken the downstream growth). The difficulties in extending the instability approach to finite amplitude for the thin jet are significant. Analyses along the lines of Pedlosky's (1970) work on finite amplitude baroclinic waves seem implausible for this problem. The fluctuations are clearly large with respect to the mean and it does not seem likely that changes in the mean flow by Reynolds' stress or heat flux are necessarily more important than wave–wave interactions in the equilibration of an individual quasi-steady synoptic meander (if such even occurs, rather than the wave growing and breaking).

The thin-jet approach to analyzing the finite amplitude behavior of Gulf Stream meanders was suggested by Warren (1963) and Robinson and Niiler (1967). This second technique for exploring meandering uses the approximation that the Gulf Stream's width (and the deformation radius) is small compared to the length scale of the meander pattern. By integrating the vorticity balance across the jet, one determines the change of angle of path (from due east) as one follows the jet downstream. These calculations indicated that topographic and β-controlled quasi-steady meandering could not successfully reproduce both the downstream growth and average wavelengths of the features. It was also clear in the data (Robinson, 1971) that the temporal changes of vorticity were not negligible. Robinson *et al.* (1975) (hereafter referred to as RLF) recognized the importance in a theoretical model of including the time-dependent changes in Stream position and direction while preserving the apparently coherent cross-stream density structure and the lowest order geostrophic balance. This paper indicated the possible form of such a model, but development of a time-dependent thin-jet meandering model has proved quite difficult. There are many possible choices for the relationships among the various nondimensional parameters and it is difficult to think of comparisons that might be drawn between the thin-jet and other theories.

This paper begins with a discussion of the dynamics of a thin jet (Section 2), following the work of RLF but correcting certain inconsistencies in the choice of nondimensional parameter relationships. We show that the interior dynamics of the jet manifest themselves as matching conditions for the exterior fields on either side of the jet, thereby determining the evolution of the path. (An analogous problem is determining the location of the interface between two fluids in Kelvin–Helmholtz instability.) Next, we pose the linearized version (Section 3) and derive the dispersion relation which demonstrates that there is a critical scale ($= [\langle V^2\rangle/\beta\langle V\rangle]^{1/2}$) below which meandering motions become unstable. Here the bracket is a vertical and horizontal integral and V is the downstream flow rate. In the succeeding section (Section 4) we use standard instability theory to derive the dispersion relationship for long waves of a jet with both vertical and horizontal shears. From comparison of these two models, we find the criteria for the neutral or unstable modes such that they appear as a meandering motion. Finally, we summarize the results obtained from this model and suggest applications of these ideas to the Gulf Stream (Section 5).

These analytical process studies continue to serve as a valuable complement to both the large numerical general circulation models, which are now beginning to show meandering and eddy-shedding behavior (Holland, 1978), and the process model computations of Rhines (1977), Ikeda (1981) and Ikeda and Apel (1981) which followed the evolution of a periodic jet. The synthesis of all of these approaches, together with the observational data, is necessary to achieve a deeper understanding of the phenomenon of large amplitude nonlinear meandering, its relationship to the surrounding field of rings and eddies, and its crucial role in the general ocean circulation.

2. Thin-jet model

We shall use the notation and general procedure of RLF, to which the reader is referred for a detailed presentation of the kinematics (RLF Section 3) and basic dynamics (RLF 3.1). A coordinate system is introduced based upon an unknown Stream axis position $Y(X, t)$ (Fig. 1), which is the primary dependent variable to be predicted from the theory. A cross-stream coordinate (η) is defined normal to this curve, the equations are transformed to X, η coordinates, and the velocity field is resolved into downstream (μ) and cross-stream (ν) components. The vorticity equation which can be derived from RLF's Eqs. (3.6–3.8)[1] is

$$\frac{\partial}{\partial\eta} h \frac{D\mu}{Dt} - \lambda \cos\theta \frac{\partial}{\partial X}\frac{D\nu}{Dt} + \lambda\mu\cos\theta \frac{\partial}{\partial X}\frac{D\theta}{Dt} + h\nu \frac{\partial}{\partial\eta}\frac{D\theta}{Dt} + \lambda h\tilde{\beta}[\mu \sin\theta - \nu\cos\theta] - \left[1 + \epsilon\tilde{\beta}(Y - \lambda\eta\cos\theta) + \epsilon\frac{D\theta}{Dt}\right]\frac{\lambda}{\epsilon} h \frac{\partial w}{\partial z} = 0. \quad (1)$$

Implicit are the usual Boussinesq and beta-plane assumptions; in addition, the radius of curvature of the jet must be sufficiently large compared to the width of the current so that the Jacobean of the transformation, h, does not become zero. Topography has been neglected since the existence of a bottom flow coherent with the upper level jet is dubious (Luyten, 1977); in addition, its inclusion would not fundamentally alter the theory. In Eq. (1), changes in relative vorticity (the first two terms) occur because of Coriolis acceleration when the path angle (θ) changes (the second two), advection of planetary vorticity (the third) or vortex stretching (the last term). The nondimensional parameters in this equation are ϵ, the Rossby number based on the downstream length scale and Stream velocity; λ, the ratio of the jet width to the meander length scale; $\tilde{\beta}$, the strength of the beta effect. All symbols are defined in Table 1. The substantial derivative [see also RLF Eq. (3.5)] is written in terms of the horizontal velocities (μ, ν) and the vertical velocity (w) and various functionals of Y expressing the motion of the coordinate system fixed to the Stream axis. Of particular importance are the two "axis velocities" μ_A and ν_A which a particle fixed at a particular downstream arc length on the axis would have as the axis wiggles. The parameter a measures the propagation speed of disturbances in the axis position compared to the Stream speed.

The velocity fields (downstream, cross-stream and vertical) are decomposed into the downstream velocity $[V(\eta, z), 0, 0]$; the barotropic axis velocity field $[a\mu_A(X, t), a\nu_A(X, t), 0]$ and finally the residual field $[m\mu_m(X, \eta, z, t), m\lambda\nu_m(X, \eta, z, t), m\omega w_m(X, \eta, z, t)]$ associated with the meandering motions. The nondimensional parameter m (the ratio of the meander-induced downstream velocity to the basic jet flow velocity) must be

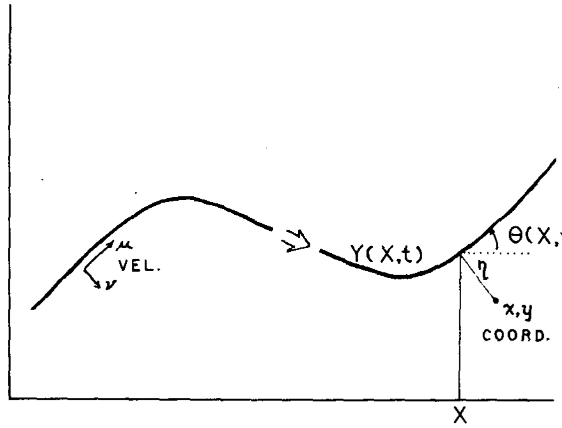

FIG. 1. The geometry of the thin-jet model. The position of the axis of the jet is labeled $Y(X, t)$. A point (x, y) off the axis can be located by coordinates (X, η). The downstream (μ) and cross-stream (ν) velocity components are also sketched.

found by consistently balancing terms in the various equations of motion. We have also introduced a new parameter ω to clarify the proper scaling of the vertical velocity in the vorticity budget. The density equation

$$\frac{D}{Dt}\left(\int^{\eta} V_z + mp_{mz}\right) + N^2 \frac{\lambda}{\epsilon} m\omega w = 0,$$
$$N^2 = N^2_{\text{dimensional}} H^2/f_0{}^2 l^2,$$

will also place constraints on ω which were not considered in RLF. We will not discuss the density equation in great detail here, but our scaling ($\omega = \epsilon/m$, $m = \lambda \ll 1$) will be consistent with the deformation radius being comparable to the stream width ($N^2 \sim 1$), whereas RLF (with $\omega = \epsilon/m$, $m \sim \epsilon \ll \lambda \ll 1$) would require the fluid to be more weakly stratified, having $N^2 \sim (\epsilon/\lambda)$ or the deformation radius order $(\epsilon/\lambda)^{1/2}$ smaller than the stream width.

We now substitute the kinematic decomposition of the velocities above, defining also a decomposition of the substantial derivative,

$$\left.\begin{aligned}\frac{D_0}{Dt} &= a\frac{\partial}{\partial t} + [V + a\psi - a\lambda\eta\theta_t]\frac{\cos\theta}{h}\frac{\partial}{\partial x}\\ \frac{D_m}{Dt} &= \mu_m \frac{\cos\theta}{h}\frac{\partial}{\partial X} + \nu_m\frac{\partial}{\partial\eta} + \omega w_m \frac{\partial}{\partial z},\end{aligned}\right\}$$

and simplify part of the vorticity tendency by using

$$a\frac{\partial}{\partial\eta} h \frac{D_0}{Dt}\mu_A + ha\nu_A \frac{\partial}{\partial\eta}\frac{D_0}{Dt}\theta = \lambda\kappa a\mu_{A_t} - \lambda\kappa a\nu_A(V + a\psi - a\lambda\eta\theta_t)\frac{\cos\theta}{h}\frac{\partial\theta}{\partial X},$$

which follows from the definitions of μ_A and ν_A. The resulting vorticity equation will be written here in symbolic form

[1] RLF Eq. (3.7) has a typographical error—a factor of λ should multiply the $\partial p/\partial X$ term.

TABLE 1. Symbols and notations.

X	longitude
η	perpendicular distance from Stream axis
z	vertical distance
t	time
μ	downstream velocity
ν	cross-stream velocity
w	vertical velocity
$V(\eta, z)$	basic flow associated with stream
$Y(X, t)$	position of axis; primary dependent variable
θ	angle of axis from east ($=\tan^{-1}\partial Y/\partial X$)
κ	curvature of axis $\left(=\dfrac{\partial^2 Y}{\partial X^2}\left[1+(\partial Y/\partial X)^2\right]^{-3/2}\right)$
$\mu_A(X, t)$	downstream velocity of inextensible stream $\left(=\dfrac{\partial Y}{\partial t}-\int_0^X dq \sin\theta(q, t)\partial^2 Y/\partial X\partial t\right)$
$\nu_A(X, t)$	cross-stream velocity of axis ($=-\cos\theta\partial Y/\partial t$)
$\psi(X, t)$	correction for motion of initial point $\left(=-\int_{X_0}^X dq \sin\theta(q, t)\partial^2 Y/\partial X\partial t\right)$
h	Jacobian of transformation ($=1+\lambda\eta\kappa$)
$\dfrac{D}{Dt}=a\dfrac{\partial}{\partial t}+(\mu-a\mu_A-a\psi-a\lambda\eta\theta_t)\dfrac{\cos\theta}{h}\dfrac{\partial}{\partial X}+\dfrac{\nu-\nu_A}{\lambda}\dfrac{\partial}{\partial\eta}+w\dfrac{\partial}{\partial z}$	substantial derivative
a	ratio of axis phase speed to jet velocities
m	ratio of meander-induced velocities to jet velocities
ϵ	Rossby number based on jet velocity and downstream scale ($V_0/f_0\mathcal{L}$)
λ	ratio of cross-stream to downstream length scale ($l/\mathcal{L}$)
ω	ratio of meander-induced vertical velocities to corresponding horizontal velocities
N^2	nondimensional Brunt–Vaisala frequency squared ($N_{\text{dimensional}}{}^2 H^2/f_0{}^2 l^2$)
$\tilde{\beta}$	nondimensional beat parameter ($\beta\mathcal{L}/f_0$)

TABLE 2. Terms in vorticity equation.

$$\dot{A}=\kappa\mu_{A_t}-\kappa\nu_A(V-a\psi-a\lambda\eta\theta_t)\frac{\cos\theta}{h}\theta_X-\cos\theta\frac{\partial}{\partial X}\frac{D_0\nu_A}{Dt}$$
$$+\frac{1}{a}(V+a\mu_A)\cos\theta\frac{\partial}{\partial X}\frac{D_0\theta}{Dt}$$

$$A_0=\frac{D_0\theta}{Dt}$$

$$A_1=\frac{D_1\theta}{Dt}$$

$$\dot{P}=h[(V+a\mu_A)\sin\theta-a\nu_A\cos\theta]$$

$$\dot{P}_1=\lambda h[\mu_m\sin\theta-\lambda\nu_m\cos\theta]$$

$$P_2=Y-\lambda\eta\cos\theta$$

$$S=h\frac{\partial w_m}{\partial z}$$

$$I=\frac{\partial}{\partial\eta}\left[h\frac{D_1}{Dt}(V+a\mu_A)+h\frac{D_0}{Dt}\mu_m\right]$$
$$-\lambda\cos\theta\frac{\partial}{\partial X}\left[a\frac{D_1}{Dt}\nu_A+\lambda\frac{D_0}{Dt}\nu_m\right]+\lambda\mu_m\cos\theta\frac{\partial}{\partial X}\frac{D_0\theta}{Dt}$$
$$+\lambda(V+a\mu_A)\cos\theta\frac{\partial}{\partial X}\frac{D_1\theta}{Dt}+ha\nu_A\frac{\partial}{\partial\eta}\frac{D_1\theta}{Dt}+h\lambda\nu_m\frac{D_0\theta}{Dt}$$

$$M=\frac{\partial}{\partial\eta}\left(h\frac{D_1\mu_m}{Dt}\right)-\lambda^2\cos\theta\frac{\partial}{\partial X}\frac{D_1\nu_m}{Dt}$$

$$\frac{D_0}{Dt}=a\frac{\partial}{\partial t}+(V-a\psi)\cos\theta\frac{\partial}{\partial X}$$

$$\frac{D_1}{Dt}=\mu_m\cos\theta\frac{\partial}{\partial X}+\nu_m\frac{\partial}{\partial\eta}+\omega w_m\frac{\partial}{\partial z}$$

$$a\dot{A}+\tilde{\beta}\dot{P}+\frac{m}{\lambda}(I+\tilde{\beta}\lambda\dot{P}_1)+\frac{m^2}{\lambda}M$$
$$=\frac{\omega m}{\epsilon}S(1+\epsilon\tilde{\beta}P_2+\epsilon A_0+\epsilon m A_1) \quad (2)$$

with each symbol defined in Table 2.[2] All of the terms represented by A, P, I, S, M are independent of m or ϵ and have order 1 terms and terms that are higher order in λ. The symbols represent: A, axis vorticity; P, planetary vorticity; I, interaction between axis or downstream flows and the meander field, M, meander flow vorticity; and S, vortex stretching.

The thin-jet approximation consists of assuming that the downstream scale is much larger than the cross-stream scale ($\lambda \ll 1$) and that the Rossby number based on the downstream scale is small ($\epsilon \ll 1$). However, this does not preclude the Rossby number based on the cross-stream scale (ϵ/λ) being order 1. In addition, for the velocity decomposition to be useful we expect

TABLE 2A. Linearized quasi-geostrophic terms.

$$\dot{A}=\frac{\partial}{\partial X}\left(a\frac{\partial}{\partial t}+V\frac{\partial}{\partial X}\right)\dot{Y}+\frac{1}{a}V\left(a\frac{\partial}{\partial t}+V\frac{\partial}{\partial X}\right)Y_{XX}$$

A_0: — (multiplied by ϵ)

A_1: — (multiplied by ϵ)

$$\dot{P}=V\frac{\partial Y}{\partial X}+a\dot{Y}$$

$$\dot{P}_1=-\lambda^2\nu_m$$

P_2: — (multiplied by ϵ)

$$S=\frac{\partial w_m}{\partial z}$$

$$I=\frac{\partial}{\partial\eta}\left[\nu_m\frac{\partial V}{\partial\eta}+\left(a\frac{\partial}{\partial t}+V\frac{\partial}{\partial X}\right)\mu_m-\lambda^2\left(a\frac{\partial}{\partial t}+V\frac{\partial}{\partial X}\right)\frac{\partial\nu_m}{\partial X}\right]$$

$$M=0$$

[2] One difficulty with the thin-jet formation is the algebraic complexity. For readability, we have adopted an abbreviated symbolism for the equations in the text, with the terms written out fully in tables.

that m will also be small; its actual value will be determined by making consistent balances in the various governing equations. As yet, then, we assume that the sizes of m/λ or $\omega m/\epsilon$ are less than or of order 1. Under these assumptions the terms involving M, P_1, P_2, A_0 and A_1 can be dropped while the others are simplified as in Table 3 to reduce Eq. (2) to the form

$$a\dot{A}' + \tilde{\beta}\dot{P} + \frac{m}{\lambda} I' = \frac{\omega m}{\epsilon} S'. \quad (3)$$

Eq. (3) can also be rewritten in a form that shows explicitly the dependence of various terms upon the cross-stream and downstream coordinates

$$F_0(X, t) + V(\eta, z)F_1(X, t) + V^2(\eta, z)F_2(X, t) - \frac{\omega m}{\epsilon}\frac{\partial w_m}{\partial z} + \frac{m}{\lambda}\frac{\partial}{\partial \eta}\left[\frac{D'_m}{Dt}(V + a\mu_A) + \frac{D'_0}{Dt}\mu_m + a\nu_A \frac{D'_1}{Dt}\theta\right] = 0, \quad (4)$$

where the F_i are somewhat complicated functionals of the angle $\theta(X, t)$ defining the axis (Table 4).

At this point, we diverge from RLF in making choices of the relationships among the various parameters. The forms of Eqs. (3) and (4) suggest that the proper scaling choices are $\omega = \epsilon/m$ and $m = \lambda$ if a and $\tilde{\beta}$ are order 1. This choice of ω is also consistent with the density equation. However, RLF made the choice $m \sim \epsilon \ll \lambda$ and $\omega = \epsilon/m$, so that integration both across the stream and vertically gave an evolution equation for the path

$$F_0(X, t)d + F_1(X, t)\int_{-d/2}^{d/2} d\eta \int_{-1}^{0} dzV + F_2(X, t)\int_{-d/2}^{d/2} d\eta \int_{-1}^{0} dzV^2 = 0. \quad (5)$$

The width of the current is represented by d; it will be assumed that the downstream flow $V(\eta, z)$ vanishes

TABLE 3. Simplified vorticity terms.

$$\dot{A}' = \kappa\mu_{A_t} + \kappa\nu_A(V - a\psi)\cos\theta\theta_X - \cos\theta\frac{\partial}{\partial X}\frac{D'_0\nu_A}{Dt} + \frac{1}{a}(V + a\mu_A)\cos\theta\frac{\partial}{\partial X}\frac{D'_0}{Dt}\theta$$

$$I' = \frac{\partial}{\partial \eta}\left[\frac{D'_1}{Dt}(V + a\mu_A) + \frac{D'_0}{Dt}\mu_m + a\nu_A\frac{D'_1}{Dt}\theta\right]$$

$$S' = \frac{\partial w_m}{\partial z}$$

$$\frac{D'_0}{Dt} = a\frac{\partial}{\partial t} + (V - a\psi)\cos\theta\frac{\partial}{\partial X}$$

$$\frac{D'_1}{Dt} = \mu_m\cos\theta\frac{\partial}{\partial X} + \nu_m\frac{\partial}{\partial \eta} + \omega w_m\frac{\partial}{\partial z}$$

TABLE 4. Dependence of grouped terms upon X and t.

$$F_0(X, t) = \kappa a\mu_{A_t} + \kappa a^2\nu_A\psi\cos\theta\theta_X - a^2\cos\theta\frac{\partial}{\partial X}\left(\frac{\partial}{\partial t} - \psi\cos\theta\frac{\partial}{\partial X}\right)\nu_A + a^2\mu_A\cos\theta\frac{\partial}{\partial X}\left(\frac{\partial}{\partial t} - \psi\cos\theta\frac{\partial}{\partial X}\right)\theta + a\tilde{\beta}(\mu_A\sin\theta - \nu_A\cos\theta)$$

$$F_1(X, t) = -\kappa a\nu_A\cos\theta\theta_X - a\cos\theta\frac{\partial}{\partial X}\left(\cos\theta\frac{\partial}{\partial X}\nu_A\right) + a\mu_A\cos\theta\frac{\partial}{\partial X}\left(\cos\theta\frac{\partial\theta}{\partial X}\right) + a\cos\theta\frac{\partial}{\partial X} \times \left(\frac{\partial}{\partial t} - \psi\cos\frac{\partial}{\partial X}\right)\theta + \tilde{\beta}\sin\theta$$

$$F_2(X, t) = \cos\theta\frac{\partial}{\partial X}\left(\cos\theta\frac{\partial\theta}{\partial X}\right)$$

for $|\eta| > \frac{1}{2}d$. This is exactly the flat-bottom form of RLF's Eq. (3.22). The simplicity of this equation for the path stems from the neglect of the last term in (4). However, this scale assumption is not consistent as we now show by integrating (4) (with the last term neglected) in the vertical only. In that case

$$F_0(X, t) + F_1(X, t)\int_{-1}^{0} dzV(\eta, z) + F_2(X, t)\int_{-1}^{0} dzV^2(\eta, z) = 0$$

(the flat bottom, $\lambda \ll 1$ limit of RLF's (3.20)) is obtained. This must hold for all values of η; but that can occur only if F_0, F_1 and F_2 are separately zero, which leads to inconsistent specifications for $\theta(X, t)$. This problem clearly still exists in RLF's (3.20) where the $\lambda \ll 1$ approximation has not been made.

This inconsistency implies that the amplitude of the meander field cannot be chosen to be much smaller than λ; rather the correct choice is $m = \lambda$. In this case Eq. (4) can still be integrated but boundary contributions (at $\eta = \pm d/2$) will not disappear:

$$F_0(X, t)d + F_1(X, t)\int_{-d/2}^{d/2} d\eta \int_{-1}^{0} dzV + F_2(X, t)\int_{-d/2}^{d/2} d\eta \int_{-1}^{0} dzV^2 = -a\left[\frac{\partial}{\partial t} - \psi\cos\theta\frac{\partial}{\partial X}\right]\int_{-1}^{0} dz\mu_m\Big|_{\eta=-d/2}^{\eta=d/2}. \quad (6)$$

This equation is our main result. It allows us to relate the discontinuities in tangential velocity of the exterior fields to the motions of the axis and the momentum and mass flux within the current. It is valid for motions with downstream length scales large compared to the jet width (and the deformation radius), and Rossby numbers based on the cross-stream width not larger than 1. The amplitude of the meander can be large—on the order of the downstream scale, but the path clearly cannot cross upon itself without violating the conditions implied in the coordinate transformation. In principle, this equation, together with a condition matching the normal flow to the motion of the axis, could be combined with solutions to the exterior nonlinear Rossby wave fields to yield an evolution equation for the axis position. The problem would be quite similar to that for nonlinear Kelvin–Helmholtz instabilities of a vortex sheet. This procedure has been formulated for the f-plane case by Campbell (1980); here we choose to explore the beta-induced effects.

Equation (6) also answers one of the puzzles in RLF—why the integrated equation for the path (5) seems to be sensitive to how far across one integrates. In the correct form, (6), the d dependence of the first term can be balanced with that of the right-hand side, in the region where the solutions are matched from the interior to the exterior regions. Eq. (6) shows the necessity for solving the external wave field and matching these waves to the interior fields, thereby determining μ_m at the edges of the jet.

Robinson and Niiler (1967) derived a path equation for a steady meander of the Gulf Stream using essentially the same scaling assumptions. In this special case, we can see from Eq. (6) that the exterior field is not important in determining the path. When the flow is time-independent, $F_0 = 0$ since it depends only upon $\partial\theta/\partial t$ (or ψ, μ_A and ν_A which all arise from $\partial\theta/\partial t$ terms) and the boundary term is likewise zero. The remaining terms give

$$\tilde{\beta}\sin\theta \int d\eta \int dzV + \cos\theta \frac{\partial}{\partial X}\left(\cos\theta \frac{\partial\theta}{\partial X}\right) \int d\eta \int dzV^2 = 0,$$

which is just Robinson and Niiler's (1967) result. Notice that the exterior field enters in this theory in exactly the same way it enters in Robinson and Niiler's (1967) work but that it is not required for finding the path. The path equation can be solved not only for small amplitude but also for finite amplitude standing Rossby wave patterns, as sketched in Robinson and Niiler (1967, Figs. 3, 4).

3. Linearized time-dependent form

The comparison above shows that the time-dependent formulation (6) will reproduce the proper steady flow patterns on the beta plane. Solving the fully nonlinear time-dependent theory appears to be quite difficult. It is of use, therefore, to explore the small-amplitude case, not only for validation of the model by comparison to more standard methods but also to make more explicit the matching process necessary to predict the motion. A further benefit of this comparison is a new insight upon the perturbation structures that will be manifested as meandering modes.

When linearizing the thin jet model, we drop all terms quadratic or higher order in θ, μ_m, ν_m and w_m. For simplicity we will take the limit corresponding to a quasi-geostrophic model by taking $\omega = \epsilon/\lambda \ll 1$ so that vertical advection becomes unimportant and the meander fields become geostrophic to lowest order. In this approximation, the various terms in the vorticity Eq. (2) simplify as shown in Table 2A. The resulting equation

$$\left[a\frac{\partial}{\partial t} + V\frac{\partial}{\partial X}\right]\left(\frac{\partial^2}{\partial\eta^2} + \lambda^2\frac{\partial^2}{\partial X^2}\right)p_m - V_{\eta\eta}\frac{\partial}{\partial X}p_m + \tilde{\beta}\lambda^2\frac{\partial}{\partial X}p_m - \frac{\partial}{\partial z}w_m + a^2Y_{Xtt} + 2aVY_{XXt} + V^2Y_{XXX} + \tilde{\beta}\left[a\frac{\partial}{\partial t} + V\frac{\partial}{\partial X}\right]Y = 0 \quad (7)$$

balances changes in meander vorticity, meander advection of stream vorticity, meander-induced advection of planetary vorticity, vortex stretching, vorticity changes due to turning of the axis, changes due to flow along a curved axis, and finally planetary vorticity changes from north or south flow of the Stream itself.

The equation analogous to (6) is formed by dropping the λ^2 terms from (7) and integrating with respect to η and z:

$$a^2dY_{Xtt} + 2a\langle V\rangle Y_{XXt} + \langle V^2\rangle Y_{XXX} + \tilde{\beta}\langle V\rangle Y_X + a^2\tilde{\beta}dY_t = -a\frac{\partial}{\partial t}\int_{-1}^{0} dzp_{m_\eta}\Big|_{\eta=-d/2}^{\eta=d/2},$$

which we can rewrite taking

$$Y = Y_0\exp[ik(X - ct)],$$

$$\int_{-1}^{0} dzp_m = p_0(\eta)\exp[ik(X - ct)]$$

as

$$-ikca\left[p_{0_\eta}\left(\frac{d}{2}\right) - p_{0_\eta}\left(-\frac{d}{2}\right)\right] + ikY_0\{-a^2dk^2c + 2a\langle V\rangle k^2c - \langle V^2\rangle k^2 + \tilde{\beta}\langle V\rangle - a\tilde{\beta}dc\} = 0. \quad (8)$$

In the exterior region, we need to calculate only the barotropic component of the flow, since the matching condition involves the depth-integrated tangential velocity. (The linearity is also important in that it decouples the modes.) However, we cannot use Eq. (7) in the exterior region because this is only valid for $\eta < \frac{1}{2}d$. Rather we can just take the linear equation of motion for the barotropic total pressure

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial \eta^2} + \lambda^2 \frac{\partial^2}{\partial X^2}\right)p + \lambda^2 \tilde{\beta} \frac{\partial}{\partial X} p = 0,$$

which has solutions

$$p = \hat{p}_0 e^{ik(X-ct)} e^{-\lambda[\tilde{\beta}(ac)^{-1}+k^2|\eta|]^{1/2}}. \quad (9)$$

Here we use the X and η coordinates basically as a Cartesian system; this is consistent with the radius of the curvature of the track being inversely proportional to the wave amplitude. This scale is so much larger than the other scales [O(1) jet width and O(1/λ) wave scales in the exterior] that the convergence of the coordinates is not important. With this linear formalism, we could also consider more complex exterior fields such as those produced in a bounded basin; the study of Harrison and Robinson (1979) indicates that in this case, eastward propagating meanders can generate waves that are not damped away from the jet but fill the basin.

Notice that the exterior solution (9) has comparable scales in both downstream and cross-stream directions. Now we match the interior barotropic flow in the region where $V \to 0$,

$$\int_{-1}^{0} dzp \to \frac{a}{\lambda} \int^{X} dX' Y_t(X', t) + \lambda \int_{-1}^{0} dzp_m$$

$$= \frac{ac}{\lambda} Y_0 e^{ik(X-ct)} + \lambda p_0 e^{ik(X-ct)},$$

(the sum of axis velocity and meander field) to the flow field of (9). Continuity of p and $\partial p/\partial\eta$ at $\eta = d/2$ gives

$$p_{0\eta}|_{\pm d/2} = \pm \frac{ac}{\lambda} Y_0 \left(\frac{\tilde{\beta}}{ac} + k^2\right)^{1/2},$$

which can be substituted into (8) to give

$$2\frac{a^2c^2}{\lambda}\left(\frac{\tilde{\beta}}{ac} + k^2\right)^{1/2} + a^2dk^2c^2 - 2ak^2c \int d\eta \int dzV$$

$$+ k^2 \int d\eta \int dzV^2 - \tilde{\beta} \int d\eta \int dzV + adc\tilde{\beta} = 0.$$

Clearly the boundary term is not negligible but will dominate the equation completely if a is order 1 and, when a is chosen properly, still contributes importantly. This equation requires us to choose a such that the first term is only order 1: $a^{1.5}/\lambda = 1$ or $a = \lambda^{2/3}$. The lowest order dispersion relation is then

$$2c^2\left(\frac{\tilde{\beta}}{c}\right)^{1/2} + k^2 \int d\eta \int dzV^2$$

$$- \tilde{\beta} \int d\eta \int dzV = 0. \quad (10)$$

This dispersion relationship, valid for meanders much smaller in amplitude than either the stream width, downstream scale or deformation radius, will be discussed further in Section 5. Notice, however, that it does have the satisfying properties of being independent of d and of the detailed structure of V.

4. Standard linear calculation

The long wave limit for the instability of a barotropic jet has been studied by Howard and Drazin (1964) who obtained a dispersion relation like (10) for waves on a barotropic jet in the long wave limit ($\lambda \ll 1$ but $\tilde{\beta} \sim 1$). They avoided the nonuniformity as $\eta \to \infty$ (arising from the large scale externally) by constructing two solutions that are well behaved at $+\infty$ or $-\infty$. Forcing the Wronskian of these to vanish gives a dispersion relation, which they approximated by a λ expansion. Here we shall consider sinuous mode long waves on a baroclinic jet and obtain a similar result using quasi-geostrophic instability theory. The equation for infinitesimal perturbations on a jet,

$$\left(a\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right)\left(\frac{\partial^2}{\partial \eta^2} + \frac{\partial}{\partial z}\frac{1}{N^2}\frac{\partial}{\partial z} + \lambda^2\frac{\partial^2}{\partial X^2}\right)p$$

$$+ \left(\tilde{\beta}\lambda^2 - V_{\eta\eta} - \frac{\partial}{\partial z}\frac{1}{N^2}\frac{\partial}{\partial z}V\right)p_X = 0,$$

is written in an η, X notation to ease the comparison; however, this notation change should not obscure the familiarity of this equation (e.g., Pedlosky, 1979). The nondimensional function $N^2(z)$ arises from the density equation and is of order the square of the deformation radius divided by the square of the jet width. Oceanically, this is an order-one function.

If we make the usual normal mode assumption $p(\eta, X, z, t) = P(\eta, z)\exp[ik(X - ct)]$, we have

$$(V - ac)\left(\frac{\partial^2}{\partial \eta^2} + \frac{\partial}{\partial z}\frac{1}{N^2}\frac{\partial}{\partial z} - \lambda^2k^2\right)P$$

$$+ \left(\tilde{\beta}\lambda^2 - V_{\eta\eta} - \frac{\partial}{\partial z}\frac{1}{N^2}\frac{\partial}{\partial z}V\right)P = 0, \quad (11)$$

with appropriate boundary conditions; at $\eta = 0$, we take a symmetric profile for V and apply symmetry boundary conditions upon P (selecting the sinus mode, since the varicose mode does not look like a wiggling of the jet axis). The downstream flow will be taken to be zero for $\eta > \frac{1}{2}$ and to vanish quadratically $V \sim (\eta - \frac{1}{2})^2$ in the neighborhood of $\eta = \frac{1}{2}$. Because of the vanishing mean flow in the far field, the perturbation structure can be written explicitly for $\eta > \frac{1}{2}$

$$P = \sum A_n F_n(z) \exp\left[\frac{\lambda^2\tilde{\beta}}{ac} + \lambda^2k^2 + \gamma_n{}^2\eta\right]^{1/2},$$

where γ_n and F_n are the eigenvalues and eigenfunctions of the vertical normal mode problem. Using this solution, we can move the condition of boundedness at ∞ onto $\eta = \frac{1}{2}$:

$$\frac{\partial}{\partial \eta} P_{(n)} = -\left[\gamma_n^2 + \lambda^2\left(\frac{\tilde{\beta}}{ac} + k^2\right)\right]^{1/2} P_{(n)}$$

$$\text{at} \quad \eta = \frac{1}{2}, \qquad (12)$$

where $P_{(n)}$ is the nth mode component of P.

In the domain $\eta \in [0, \frac{1}{2}]$ we can distinguish two regions: an inner one where $V \gg ac$ (since we now expect $a \sim \lambda^{2/3}$) and an outer one where $V \approx ac$. For $a = \lambda^{2/3}$ and V vanishing quadratically at $\eta = \frac{1}{2}$, this inner region becomes $0 \leqslant \eta < \frac{1}{2} - \lambda^{1/3}$. We expand

$$c = c^{(0)} + \lambda^{2/3}c^{(2/3)} + \lambda^{4/3}c^{(4/3)}.$$

(The $\lambda^{1/3}$ terms do not seem to be necessary in the matching process.) The sinuous mode solution to (11) in the inner region is then

$$P = V - ac + \lambda^2 Q + \cdots. \qquad (13)$$

Substituting into (11) gives

$$V\left(\frac{\partial^2}{\partial\eta^2} + \frac{\partial}{\partial z}\frac{1}{N^2}\frac{\partial}{\partial z}\right)Q - Q\left(\frac{\partial^2}{\partial\eta^2} + \frac{\partial}{\partial z}\frac{1}{N^2}\frac{\partial}{\partial z}\right)V = k^2V^2 - \tilde{\beta}V, \qquad (14)$$

where

$$Q_\eta = 0, \qquad \eta = 0;$$

$$VQ_z = V_zQ, \quad z = 0, -1.$$

In the outer region $\frac{1}{2} - \lambda^{1/3} \leqslant \eta \leqslant \frac{1}{2}$, we rewrite the equation using $\eta' = (\frac{1}{2} - \eta)\lambda^{-1/3}$ and $V' = \lambda^{-2/3}V$ as our order 1 variables. We find

$$\left[(V' - c)\frac{\partial^2}{\partial\eta'^2}P - P\frac{\partial^2}{\partial\eta'^2}V'\right] = -\lambda^{2/3}\left[(V' - c)\frac{\partial}{\partial z}\frac{1}{N^2}\frac{\partial}{\partial z}P - P\frac{\partial}{\partial z}\frac{1}{N^2}\frac{\partial}{\partial z}V'\right] - \lambda^2\tilde{\beta}P + \lambda^{8/3}k^2(V' - c)P$$

with boundary conditions

$$(V' - c)P_z = V'_zP \quad \text{at} \quad z = 0, -1$$

and

$$\frac{\partial}{\partial\eta'}P_{(0)} = \lambda\left(\frac{\tilde{\beta}}{c}\right)^{1/2}\left(1 + \lambda^{2/3}\frac{c}{\tilde{\beta}}k^2\right)^{1/2} \quad \text{barotropic,}$$

$$\frac{\partial}{\partial\eta'}P_{(n)} = \lambda^{1/3}\left(\gamma_n^2 + \lambda^{4/3}\frac{\tilde{\beta}}{c} + \lambda^2k^2\right)^{1/2}P_{(n)}$$

$$\text{baroclinic } n \geqslant 1$$

at $\eta = \frac{1}{2}$. The solution to the outer region problem is

$$P = \lambda^{2/3}[V' - c + \lambda Q'], \qquad (15)$$

where

$$[V' - c^{(0)}]Q'_{\eta'\eta'} = V'_{\eta'\eta'}Q'.$$

This last equation can be rewritten as

$$\frac{\partial}{\partial\eta'}[V' - c^{(0)}]^2\frac{\partial}{\partial\eta'}\frac{Q'}{V' - c^{(0)}} = 0.$$

The boundary conditions are

$$Q'_{\eta'} = -c^{(0)}\left(\frac{\tilde{\beta}}{c^{(0)}}\right)^{1/2} \quad \text{at} \quad \eta' = 0$$

from matching to the $\eta > \frac{1}{2}$ region and

$$[V' - c^{(0)}]Q'_z = V'_zQ' \quad \text{at} \quad z = 0, -1.$$

The baroclinic part of Q' must vanish as $\eta' \to 0$ in order to satisfy the baroclinic part of the boundary condition. We can solve for Q'

$$Q' = [V'(\eta, z) - c^{(0)}]c^{(0)2}(\tilde{\beta}/c^{(0)})^{1/2} \times \int_0^{\eta'} \frac{dx}{[V'(x, z) - c^{(0)}]^2} + G(z)[V'(\eta', z) - c^{(0)}],$$

where G is not yet determined. The only way the upper and lower boundary conditions can be satisfied without involving additional boundary layers is to consider $V'_z = 0$, $z = 0$ and -1. Apparently, the instabilities associated with surface or bottom density gradients will not be manifested as meandering motions (defined more precisely below). With the simplifying assumption that $V'(\eta', z) = \eta'^2H(z)$ we can integrate this formula explicitly

$$Q' = [\eta'^2H - c^{(0)}]c^{(0)2}(\tilde{\beta}/c^{(0)})^{1/2}\left\{\frac{\eta'}{2c^{(0)}(c^{(0)} - H\eta'^2)} - \frac{1}{4}c^{(0)-1}(c^{(0)}H)^{-1/2}\ln\left|\frac{c^{(0)} - \eta'\sqrt{c^{(0)}H}}{c^{(0)} + \eta'\sqrt{c^{(0)}H}}\right|\right\} + G(z)[\eta'^2H - c^{(0)}]. \qquad (16)$$

As usual, some care must be taken in choosing the proper branch of the logarithm for $\eta' > (c^{(0)}/H)^{1/2}$: for a viscous critical layer $\ln z = \ln|z| - i\pi$ (Lin, 1955), while for a nonlinear critical layer $\ln z = \ln|z|$ (Benny and Bergeron, 1969). For oceanic flows, the latter criterion seems more appropriate.

We now match (15–16) to (13–14) using standard asymptotic matching methods (e.g., Van Dyke, 1964). The inner limit of the outer solution is

$$P \to V - \lambda^{2/3}c - \lambda^2\frac{1}{3}\frac{c^{(0)2}}{(1/2 - \eta)H}\left(\frac{\tilde{\beta}}{c^{(0)}}\right)^{1/2},$$

while the asymptotic form of the inner solution is

$$P \to \lambda^{2/3}[V' - c] + \lambda^{5/3}\left[\frac{A(z)}{\eta'} + B(z)\eta'^2\right]$$

(using the asymptotic solutions of (14)) or

$$P \to V - \lambda^{2/3}c + \frac{\lambda^2A(z)}{1/2 - \eta} + \lambda B(z)\left(\frac{1}{2} - \eta\right)^2.$$

In order to connect these solutions, we must have

$$A = -\frac{1}{3}\frac{c^{(0)2}}{H(z)}\left(\frac{\tilde{\beta}}{c^{(0)}}\right)^{1/2}, \quad B(z) = 0. \tag{17}$$

Finally, the coefficients A and B can also be related to the symmetry boundary condition at $\eta = 0$ by integrating (14) with respect to z and η to find

$$\int_{-1}^{0} dz\left[V(\eta, z)\frac{\partial Q}{\partial \eta} - Q\frac{\partial V}{\partial \eta}\right] = \int_{0}^{\eta} dx \int_{-1}^{0} dz[k^2V^2(x, z) - \tilde{\beta}V(x, z)].$$

Evaluating this for η close to $\frac{1}{2}$, we find

$$3\int_{-1}^{0} dzA(z)H(z) = \frac{1}{2}\int_{-1/2}^{1/2} dx \int_{-1}^{0} dz[k^2V^2(x, z) - \tilde{\beta}V(x, z)].$$

Combining this with (17) gives a dispersion relation

$$c^{(0)2}\left(\frac{\tilde{\beta}}{c^{(0)}}\right)^{1/2} = -\frac{1}{2}\int_{-\infty}^{\infty} dx \int_{-1}^{0} dz[k^2V^2(x, z) - \tilde{\beta}V(x, z)],$$

which is identical to the result (10) found by the thin jet dispersion relation.

If we examine the shapes of the perturbations in this model, certain characteristic structures of meandering motions show up readily. The dominant term in the perturbation pressure in the interior is

$$P \sim \text{constant}[V - ac^{(0)}]e^{ik(X-ct)}. \tag{18}$$

The two terms listed here dominating the perturbation pressure have simple intuitive interpretations: they correspond respectively to zonal flow perturbations due to shifts in the latitude of the axis and to meridional flow perturbations due to the north–south translation of the axis. If the jet axis is shifted northward a distance $Y(X, t) = Y_0 \exp(ik(X - ct))$, the downstream velocity is

$$\mu \approx V(Y + \eta, z)$$

or

$$\mu \approx V(\eta, z) + YV_\eta(\eta, z),$$

giving a contribution to the perturbation geostrophic pressure

$$P' \approx V(\eta, z)Y_0e^{ik(X-ct)},$$

as in the first term in (18) (see also Howard and Drazin, 1964). The second contribution is from the pressure associated with the displacement of a material surface at latitude $Y(X, t)$: e.g., $\nu = -aY_t = -p_X$, giving

$$p = a\int^{X} dX'Y_t(X', t) = -ac^{(0)}Y_0e^{ik(X-ct)}.$$

In any jet perturbation problem, then, the meandering perturbations can be identified as those with the dominant contribution to the pressure within the jet being $P \approx V - ac$. In two cases, the barotropic and baroclinic top hat jets, where the eigenfunctions can be calculated explicitly (cf. Flierl, 1975, or Talley, 1982), we find that there is a long wave mode having meandering form. [For more general linear motions, we can recover the general linear thin-jet equation (7) by substituting the analogue to $V - ac^{(0)}$

$$p = \frac{1}{\lambda}\left[VY(X, t) - a\int^{X} dX'Y_t(X', t)\right] + \lambda p_m,$$

$$\omega = \lambda\omega_m,$$

into the quasi-geostrophic perturbation vorticity equation.]

5. Results

The simplest prediction of the thin-jet model is the dispersion relation (10) for the linearized case (in dimensional form)

$$c^{3/2} = \frac{1}{2}\beta^{1/2}\int_{-\infty}^{\infty} d\eta \frac{1}{H}\int_{-H}^{0} dzV(\eta, z) - \frac{1}{2}\frac{k^2}{\beta^{1/2}}\int_{-\infty}^{\infty} d\eta \frac{1}{H}\int_{-H}^{0} dzV^2(\eta, z)$$

shown in Fig. 2. Here we shall present c as a complex function of real k, although representations of complex wavenumber as a function of real frequency could also be of interest. The necessary transport and momentum transport values

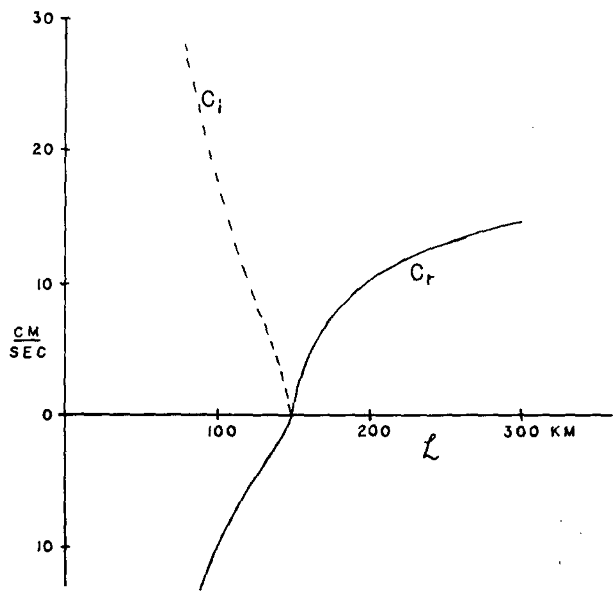

FIG. 2. The dimensional thin-jet dispersion relationship for the parameter values given in the text. The real and imaginary parts of the phase speed are plotted against the length scale $\mathcal{L} = \lambda/2\pi$ of the wave.

$$\left.\begin{aligned}&\int_{-\infty}^{\infty} d\eta \frac{1}{H}\int_{-H}^{0} dz V(\eta, z) = 3.4 \times 10^4 \text{ m}^2 \text{ s}^{-1}, \\ &\qquad\qquad [150 \text{ Sv for } H = 4.5 \text{ km}] \\ &\int_{-\infty}^{\infty} d\eta \frac{1}{H}\int_{-H}^{0} dz V^2(\eta, z) = 1.3 \times 10^4 \text{ m}^3 \text{ s}^{-1}\end{aligned}\right\}$$

have been taken from Robinson, Luyten and Fuglister (1974) and $\beta = 1.8 \times 10^{-11}$ m^{-1} s^{-1}. For wavelengths shorter than the critical scale

$$\frac{1}{k} < \left[\frac{\int d\eta \int dz V^2}{\beta \int d\eta \int dz V}\right]^{1/2} = 150 \text{ km},$$

the meandering motion is unstable. One prediction of this dispersion relationship which requires some discussion is that the phase speed for the unstable modes is retrograde. From the theoretical point of view, this result appears possibly to violate the semicircle theorem (cf. Pedlosky, 1979); in the observations, systematic retrograde motion is rarely seen in the Stream, although such motion does occur during the formation of large meanders as they pinch off into rings. We can also check this result against that obtained from the dispersion relationship for the barotropic top hat jet (cf. Howard and Drazin, 1964):

$$c^2(1 + \tilde{\beta}/c)^{1/2} + (V_0 - c)^2\left(1 - \frac{\tilde{\beta}}{V_0 - c}\right)^{1/2} \times \tanh\left[\frac{1}{2}\lambda\left(1 - \frac{\tilde{\beta}}{V_0 - c}\right)^{1/2}\right] = 0. \quad (19)$$

Fig. 3 shows that the top hat jet's dispersion relation is quite similar to (10) in the limit that the width of the jet l is small compared to both the wave scale $\mathcal{L}$ and the Rossby scale $(V_0/\beta)^{1/2}$ (recall that $\lambda = l/\mathcal{L}$). If, however, we choose the top hat velocity V_0 and width l by matching the long wave limit of (19),

$$c^2\left(\frac{\tilde{\beta}}{c}\right)^{1/2} + (V_0^2 - \tilde{\beta}V_0)\frac{1}{2}\lambda \approx 0,$$

to the general expression (10), we find

$$\left.\begin{aligned}V_0 &= \int d\eta \int dz V^2 \Big/ \int d\eta \int dz V = 0.40 \text{ m s}^{-1} \\ l &= \int d\eta \frac{1}{H}\int dz V/V_0 = 84 \text{ km}\end{aligned}\right\},$$

giving the dispersion relation shown in Fig. 4. Clearly in this case the width is sufficiently near to the scale $(V_0/\beta)^{1/2}$ that the long wave approximation cannot represent the phase speeds of the unstable modes properly. For a smooth profile, we do not know when the long wave approximation breaks down; however we suspect that the rapid retrograde motion predicted for short scale meanders might be an artifact of the approximation technique.

In addition, we should comment that there may be long wave modes of a baroclinic jet which are not of meandering form. These show up in the two-layer jet models of Flierl (1975) and Talley (1982) as waves with phase speeds nearly independent of k for small k. Basically, the $\lambda = 0$ form of the stability equation (11),

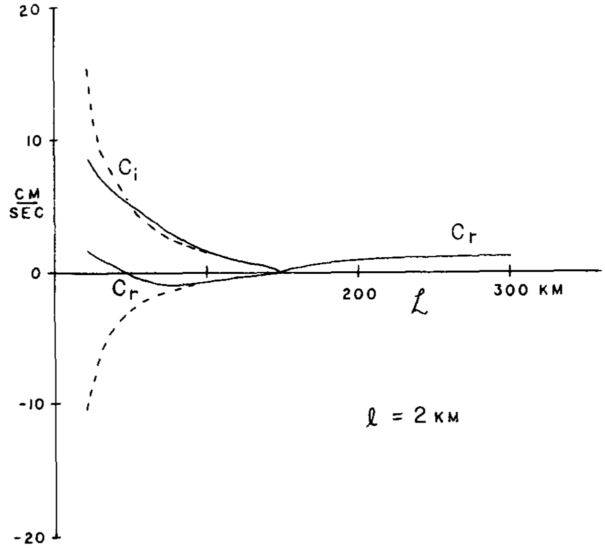

FIG. 3. Comparison of the sinuous mode dispersion relationship for the barotropic top hat to the thin-jet model (dashed). The width of the top hat is $l = 2$ km and the current speed is $V_0 = 0.4$ m s^{-1}.

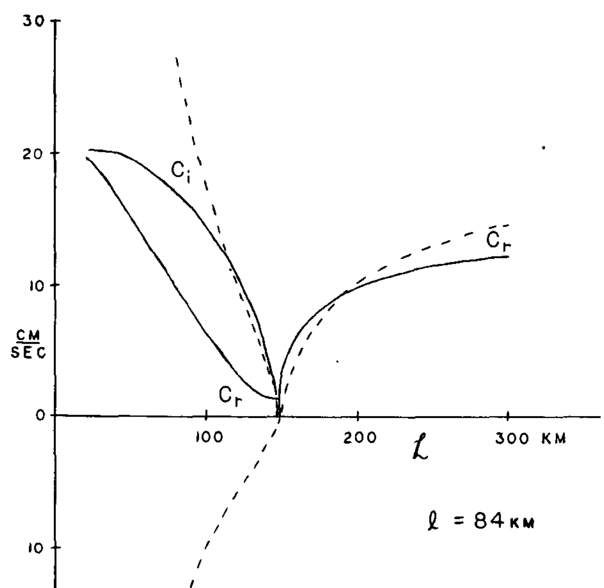

FIG. 4. As in Fig. 3 except with the correct parameter values in the long wave sense: $l = 84$ km and $V_0 = 0.4$ m s^{-1}.

$$(V - ac)\left(\frac{\partial^2}{\partial\eta^2} + \frac{\partial}{\partial z}\frac{1}{N^2}\frac{\partial}{\partial z}\right)P - P\left(\frac{\partial^2}{\partial\eta^2} + \frac{\partial}{\partial z}\frac{1}{N^2}\frac{\partial}{\partial z}\right)(V - ac) = 0,$$

not only has a meandering mode solution

$$P = V - ac,$$

but may have other solutions as well, with eigenvalues which, in the two-layer models, appear to be much more sensitive to the detailed structure of $V(\eta, z)$. The thin-jet approximation, therefore, filters out a number of modes, including these dispersionless roots and the varicose modes.

The energetics of the meandering instability indicate that it is drawing upon both kinetic and available potential energy. This follows from substituting $P = V - ac$ in the usual formulas for the energy conversion terms

$$[\mathrm{KE} \rightarrow \mathrm{KE}'] = \int d\eta \int dz V_\eta \frac{P^*P_\eta - PP_\eta^*}{2i} = ac_i \int d\eta \int dz V_\eta^2,$$

$$[\mathrm{APE} \rightarrow \mathrm{APE}'] = \int d\eta \int dz \frac{V_z}{N^2}\frac{P^*P_z - PP_z^*}{2i} = ac_i \int d\eta \int dz \frac{V_z^2}{N^2}.$$

For a simple Gaussian Stream profile,

$$V = [\alpha_0 + \alpha_1 F_1(z)] \exp\left(-\frac{1}{2}\frac{\eta^2}{l^2}\right),$$

the ratio of energy input from available potential energy to that from kinetic energy is about

$$\frac{\int dz \int d\eta \frac{V_z^2}{N^2}}{\int d\eta \int dz V_\eta^2} = \frac{R_d^2}{l^2}\frac{\alpha_1^2}{\alpha_0^2 + \alpha_1^2},$$

which is 2.4 for typical Gulf Stream widths and deformation radii.

Figure 3 or 4 shows that the long wave dispersion relation (10) cannot predict the scale of the most rapidly growing mode since the shortest waves have the fastest growth. Rather we expect this scale will be of the order of R_d or the width scale of the jet and therefore cannot be resolved by the long wave approximation; however, it is not always the case that the most rapidly growing linear wave is the one seen at finite amplitude (e.g., Pedlosky, 1981). Perhaps the observed large-scale meanders are stable or weakly growing meanders forced by the inlet conditions and the smaller-scale, more rapidly growing instabilities equilibrate or dissipate at a rather small amplitude. In any event, observations clearly indicate the relevance of thin-jet theory for the large scale and large amplitude meandering.

Comparison of the thin-jet dispersion relationship to oceanic data is difficult: undoubtedly finite amplitude effects are of great importance in the observations and there are no studies covering synoptically a wide enough range of temporal and spatial scales. The dispersion relationship presented by Watts (1983) or Watts and Johns (1982) covers only the short scale meandering; however, his c_i values of 1 to 12 cm s^{-1} for $\mathcal{L}$ from 25 to 100 km are quite comparable to those we calculate and the c_r values also match well to those shown for the top-hat model in Fig. 4. (Remember that the thin-jet model may not be able to reproduce the c_r values in this regime.) Watt's (1983) summary of data from other investigators show length scales in the range 20 to 160 km, phase speeds 5 to 20 cm s^{-1} and growth rates corresponding to $c_i \sim 4$ cm s^{-1}. These are not inconsistent with our model results showing phase speeds of around 15 to 5 cm s^{-1}; however, the model growth rates tend to be too large.

If we wish to apply this model to large-amplitude meandering, the appropriate equations to solve would be the finite amplitude Rossby wave equations (without mean flow) to the north and south of the line $Y(X, t)$; there may be cases of interest where the external field is approximately linear. Outgoing or damped wave conditions must be applied far to the north and south or boundary effects explicitly dealt with (Harrison and Robinson, 1979). The two regions are matched together using the fact that $Y(X, t)$ is a material curve as viewed from either side. Finally, Eq. (6) allows the jump in tangential velocity to be determined, giving the motion of the stream, Y_t. Campbell (1980) has shown in the periodic barotropic f-plane case that this procedure appears to be well defined and the problem can be written as a single integro-differential equation for Y; however, the β-plane case is more complex because the exterior equations can no longer be solved so readily in terms of the path position Y as the Laplace equation which results for f-plane dynamics.

In conclusion, we would like to reiterate some major points. First, considerable care must be taken in a time-dependent thin-jet meandering model to choose scales consistently. The practice of integrating the equations may hide inconsistencies. It appears as if the role of the exterior fields is critical in determining the time-dependent motion of the Stream; there is a strong feedback between the generated waves and the motion of the axis. No model that neglects these fields will be able to reproduce the dispersion relationship (10). This dispersion relation does suggest the critical wavelength below which waves become unstable but does not distinguish the most rapidly growing linear wave; however, the finite amplitude behavior of the meanders is still unknown and must be explored. The thin-jet formalism does offer one approach to this problem. Although numerical solutions to these equations could be formidable since it involves matching two solutions together at an unknown, moving boundary, conventional numerical methods also have difficulties because of the large flow speeds and the wide range of space and times scales which must be resolved. Just as the infinitely thin discontinuity idealization for Kelvin-Helmholtz instabilities has proved useful even at finite amplitude (cf. Grimshaw, 1981), the thin-jet model may lead to new insights on the finite amplitude baroclinic and barotropic instability driving the meandering of the Gulf Stream. On the basis of this work, we believe the time-dependent thin-jet theory to be established as a credible approximation for the study of relevant dynamical processes of meandering, ring formation, and interaction between the Stream and the surrounding flow. As a complement to eddy-resolving ocean current and general circulation models, such process studies should play a useful and important role in the interpretation of observations and the design of experiments.

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