Steady Wind-Driven Coastal Circulation on a β-Plane

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ABSTRACT

In tropical regions, and for applications where the alongshore scale k^{-1} of the forcing is large, the assumption of constant Coriolis parameter f in Csanady's Arrested Topographic Wave (ATW) model is invalid. Here we generalize the ATW model for steady wind-driven coastal circulation by allowing f to vary according to the β-plane approximation $f = f_0 + \beta y$, and by deriving solutions for finite width shelves. Bottom friction is assumed to be linear in the depth-averaged velocity with coefficient r and the depth $h(x) = sx$ is assumed to increase linearly with distance x offshore. The generalization includes the ATW solutions as a subset; however, theoretical and numerical calculations show that the dimensionless parameter β/f_0k plays a key role in the flow structure. In particular, for infinitely wide shelves and nonzero values of β/f_0k, enhanced trapping occurs for coastal circulation off an east coast while trapped solutions cease to exist for circulation off a west coast. For finite width shelves, specification of zero sea level anomaly at the shelf break allows solutions for wind-driven circulation on both east and west coasts. Inclusion of the β effect results in a smaller trapping scale for coastal flows on east coasts (western ocean boundaries) and a larger trapping scale for coastal flows on west coasts. Asymptotic solutions for geographically varying wind stress with oscillatory form are presented as examples.

1. Introduction

Many steady state circulation models describe the depth-averaged flow resulting from a balance between wind stress, bottom friction, pressure gradient and Coriolis forces in shallow coastal seas. Such models are comprehensively reviewed in a detailed monograph by Csanady (1982). The theory presented here is an extension of that developed by Csanady (1978) following the works of Birchfield (1967, 1973) and of Gill and Schumann (1974). Birchfield (1973) presented a solution for circulation in a circular parabolic basin and Csanady's problem appears similar except for a different coordinate geometry. Csanady also noted the close analogy of his equations with those of Gill and Schumann (1974) governing topographic wave generation. Since the decay of waves through friction can result in the generation of a mean flow, Csanady called his model the Arrested Topographic Wave (ATW).

Csanady's steady state ATW solution for the surface elevation field has the form of the one-dimensional heat conduction equation, with alongshore distance playing the role of time. This equation is unforced, and the wind stress enters the problem through the boundary condition at the coast, analogous to an applied heat flux. The extensive literature on heat conduction (e.g., Carslaw and Jaeger, 1959) facilitates solutions to the ATW problem that requires the Coriolis parameter f to remain constant in the region of interest. Winant (1979) provided further insight into the scaling associated with a step function forcing, with particular reference to the ATW model.

The assumption of a constant Coriolis parameter is acceptable in mid- to high-latitude applications, but becomes inappropriate in the tropics. In this paper, we develop solutions which allow a variation of Coriolis parameter with alongshore distance y through the β-plane approximation (Rossby, 1939). These solutions apply in tropical regions where the coastline is oriented parallel to a meridian, but may be used (with an appropriate change of β) when the coastline is oriented in any direction. The present solutions also apply when the forcing scale is comparable to f_0/β.

Csanady's ATW model is applicable only to shelves of infinite width. Here we generalize the ATW model to allow for shelves of finite width separated from the deep ocean by a vertical continental slope. This generalization is made by assuming the sea level anomaly is nonexistent at the shelf break or, equivalently, that the sea level in the deep ocean is unaffected by the shelf circulation.

The paper is organized as follows. In Section 2 we develop the governing equations, while in Section 3 we solve the governing equations in general form. Solutions for coastal circulation on midlatitude β-planes for oscillatory wind stress are described in Section 4 using the boundary condition that the sea level anomaly tends to zero as $x \to \infty$. In Section 5, solutions are found for finite width shelves by requiring the sea level anomaly to be zero at the shelf break, while Section 6 concludes with a discussion.

2. The governing equations

In this section we develop the governing equations for steady wind-driven circulation on a β-plane. The coordinate system consists of an x axis oriented offshore and y axis oriented alongshore as shown in Fig. 1. We assume that the depth h is solely a function of x and that the wind stress has an alongshore component τ_w that is independent of location across-shelf and is therefore a function of y only. Bottom friction is assumed to be linear in alongshore velocity and to be parameterized by friction coefficient r.

Scaling considerations indicate that time dependent effects are unimportant in the alongshore momentum equation provided that the frequency of the forcing $\omega \ll r/H$, where H is a scale depth for the coastal ocean. Time dependent and bottom friction effects may be neglected in the across-shelf momentum equation provided the scale of alongshore variations greatly exceeds the scale of across-shelf variations. Although the across-shelf component of wind stress cannot, in general, be neglected on scaling grounds, consideration of the flow resulting from across-shelf wind stress is beyond the scope of the present work. Under the above assumptions the depth averaged, linearized equations of motion for nondivergent flow may be written in terms of the surface elevation ζ, the across-shelf velocity u and the alongshore velocity v as

$$fv = g\zeta_x \tag{2.1}$$

$$fu = -g\zeta_y + \tau_w/\rho h - rv/h \tag{2.2}$$

$$(uh)_x + (vh)_y = 0 \tag{2.3}$$

where the subscripts x and y denote partial derivatives.

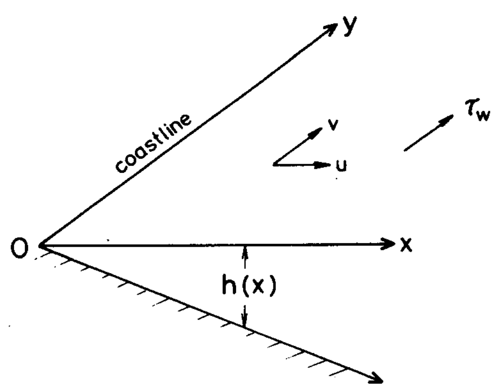

FIG. 1. A schematic diagram of a coastal ocean in $x > 0$ having depth $h(x)$. The x and y axes point offshore and alongshore, respectively, while the current components are denoted by u and v, and the alongshore wind stress by τ_w.

In contrast to Csanady's ATW model, which is valid only for constant Coriolis parameter f, the present model allows for wind-driven flows over scales where there are significant changes in Coriolis parameter with latitude.

Since $f = 2\Omega$ sin (latitude), where Ω is the earth's rotation about its axis, it is convenient to make use of the β-plane approximation

$$f = f_0 + \beta y \tag{2.4}$$

for cases where the extent of applicability of the model in the alongshore direction is large, but somewhat less in magnitude than the earth's radius. Using (2.1), (2.3) and (2.4), a single equation may be found for the surface elevation ζ by eliminating u and v from (2.2). Thus

$$r\zeta_{xx} + \beta h\zeta_x + h_x(f_0 + \beta y)\zeta_y = 0 \tag{2.5}$$

an unforced equation which is somewhat difficult to solve for arbitrary $h(x)$. As noted by a reviewer, (2.5) is essentially a vorticity equation with the first term proportional to the vorticity, the second term the meridional volume flux due to the β-effect and the third term the topographic term describing the effects of vortex stretching associated with flow across depth contours.

Following Csanady, we seek solutions to (2.5) applicable to coastal flow on a shelf with constant bottom slope s such that

$$h = sx. \tag{2.6}$$

The wind stress enters the problem through the boundary conditions which are, assuming $h(x) \to 0$ as $x \to 0$,

$$\zeta_x = \frac{f\tau_w}{\rho g r}, \quad x = 0 \tag{2.7}$$

$$\zeta \to 0, \quad x \to \infty. \tag{2.8}$$

These conditions indicate a balance between wind stress and bottom friction at the coast through (2.7), and restrict the solution to coastally trapped flows through (2.8).

Solutions for wind-driven flows in which the initial conditions do not enter the problem are asymptotic in nature and are sought from (2.5) by the method of separation of variables. For constant A and

$$\zeta(x, y) = \text{Re}[AZ(x)\phi(y)] \tag{2.9}$$

(2.5) becomes

$$\frac{r}{f_0 s}\left(\frac{Z_{xx}}{Z} + \frac{\beta s x}{r}\frac{Z_x}{Z}\right) = -\left(1 + \frac{\beta y}{f_0}\right)\frac{\phi_y}{\phi} = -\mu \tag{2.10}$$

where the separation constant μ may be either imaginary or real. Solutions to (2.10) are treated in detail in Sections 4 and 5 following the general solution method described in Section 3.

Equation (2.1) indicates that the formulation is inappropriate near the equator. To determine an approximate limit of validity we note that (2.1) can maintain its present form only when the Coriolis force greatly exceeds the across-shelf component of bottom friction. Denoting the alongshore and across-shelf scales by $l = 2\pi k^{-1}$ and L respectively, continuity implies $u \sim (L/l)v$. With $f_0 \approx 0$ and $f = \beta y$, (2.1) remains valid provided $y \gg kLr/2\pi\beta H$. Choosing $kL \approx 10^{-1}$, $r \simeq 10^{-3}$ m s^{-1}, $\beta = 10^{-11}$ m^{-1} s^{-1} and $H = 10^2$ m shows that $kLr/2\pi\beta H \approx 25$ km and the equations remain applicable except within a few hundred kilometers of the equator.

3. General method of solution

We proceed to outline a general solution method for (2.10) using dimensionless variables of form $X = x/L$, $Y = ky$ such that

$$\zeta = \mathrm{Re}[AZ(X)\phi(Y)].$$

Each of the equations for the across-shelf structure may be written in the form

$$\frac{d^2Z}{dX^2} + aX\frac{dZ}{dX} + bZ = 0 \qquad (3.1)$$

and, as we will show later, useful solutions can be found for real a, regardless of whether b is real or imaginary.

For real nonzero values of the constant a, Roger Grimshaw (personal communication, 1983) notes that use of the transformation of the dependent variable to $U = Z\exp(aX^2/4)$ and independent variable to t where $t^2 = aX^2$ allows solution in terms of parabolic cylinder functions or Whittaker functions (Abramowitz and Stegun, 1970). For nonzero values of a the coastally trapped solutions satisfying $Z \to 0$ as $X \to \infty$ are of the form (easily verified by substitution)

$$Z \sim X^{-b/a}, \quad Z \sim X^{b/a-1}\exp(-aX^2/2); \quad X \to \infty. \qquad (3.2)$$

It follows that if b is imaginary the only trapped solutions occur for $a > 0$. For $a = 0$, coastally trapped solutions exist provided b is imaginary or negative as may be easily verified by investigating solutions of $Z_{xx} + bZ = 0$.

For general values of b and a, numerical solutions to (3.1) may be found using a power series method. Assuming

$$Z = \sum_{m=0}^{\infty} C_m X^m \qquad (3.3)$$

the coefficients are governed by

$$C_{m+2} = \frac{-(b + am)}{(m + 2)(m + 1)} C_m \quad m = 0, 1, 2 \cdots. \qquad (3.4)$$

Because the recursion relation relates C_{m+2} to C_m it follows that there are two independent series solutions, one involving odd numbered coefficients, and one involving even numbered coefficients. For any given values of a and b, the odd and even series may be written, respectively, as

$$\left.\begin{aligned} Z_0 &= C_1X + C_3X^3 + C_5X^5 + \cdots \\ Z_e &= C_0 + C_2X^2 + C_4X^4 + \cdots \end{aligned}\right\}. \qquad (3.5)$$

By the ratio test, each of these infinite series is convergent for all X.

The boundary condition (2.8) is satisfied by a linear sum of the odd and even series (using $C_0 = C_1 = 1$) such that

$$Z = Z_0 - \gamma Z_e = |Z|e^{i\delta} \qquad (3.6)$$

provided

$$\gamma = \lim_{X\to\infty} \frac{Z_0}{Z_e} = \lim_{X\to\infty} \frac{C_1X + C_3X^3 + C_5X^5 + \cdots}{C_0 + C_2X^2 + C_4X^4 + \cdots}. \qquad (3.7)$$

In the numerical calculations done to determine (3.7), values of γ were evaluated using 500 and then 1000 terms to ensure convergence, the criterion for convergence being that estimates of γ for values of X differing by 0.2 should be identical to the 8th decimal place.

Of the two boundary conditions, it is the boundary condition (2.8) requiring $Z \to 0$ as $X \to \infty$ that determines the value of γ and the across-shelf structure $Z(X)$. The value of β affects γ by virtue of the fact that the parameter a is proportional to β, and the coefficients C_m are dependent upon a through (3.4).

The alongshore structure governed by (2.10) has the general form

$$\phi = \left(1 + \frac{\beta}{f_0 k} Y\right)^{\mu f_0/\beta}. \qquad (3.8)$$

To satisfy boundary condition (2.7) as $X \to 0$ we note that

$$\frac{fL\tau_w}{\rho g r} = \mathrm{Re}\{AC_1\phi\}$$

so that asymptotic solutions for a wind stress of form

$$\tau_w = \mathrm{Re}\left\{\frac{\tau_0\phi}{1 + [\beta/(f_0k)]Y}\right\} = \mathrm{Re}\left[\tau_0\left(1 + \frac{\beta}{f_0k}Y\right)^{\mu f_0/\beta - 1}\right] \qquad (3.9)$$

exist provided $A = f_0L\tau_0/\rho g r$ where we have chosen $C_1 = 1$.

In any real application, expansion of an arbitrary wind stress as a power series comprised of terms of form (3.9) such that

$$\tau_w = \sum_{\mu} \tau_w(\mu)$$

allows solution for the alongshore structure of the circulation as a power series comprised of terms of form (3.8).

The alongshore structure ϕ is dependent upon β by virtue of the inclusion of $(f_0 + \beta y)$ in the third term of the governing partial differential equation (2.5). Although a scale analysis shows that inclusion of the complete term is appropriate for applications of this theory, conventional β-plane approximations (Rossby, 1939) ignore variations of f with latitude except where f appears in differentiated form. Such a conventional approach would render ϕ independent of β here.

In terms of the dimensionless variables $X = x/L$, $Y = ky$, the general solutions for the surface elevation and the streamfunction $\psi = \int hvdx$ are

$$\zeta = \mathrm{Re}\left[\frac{f_0 L \tau_0}{\rho g r}\left(1 + \frac{\beta}{f_0 k} Y\right)^{\mu f_0/\beta} \sum_{m=0}^{\infty} C_m X^m\right] \quad (3.10)$$

$$\psi = \mathrm{Re}\left[\frac{L^2 s \tau_0}{\rho r}\left(1 + \frac{\beta}{f_0 k} Y\right)^{\mu f_0/\beta - 1} \times \sum_{m=0}^{\infty} \frac{m}{m+1} C_m X^{m+1}\right]. \quad (3.11)$$

Here the alongshore scale k^{-1} and the separation parameter μ are related by $\mu = ik$ (k real) for oscillatory alongshore structure and $\mu = k$ (k real) for nonoscillatory alongshore structure. The nonoscillatory structure is exponentially dependent on alongshore distance and is likely to be a reasonable model only in unusual circumstances in limited coastal regions. For this reason oscillatory solutions only are discussed here.

The ATW solutions are similar in many respects to the present solutions because the basic equations of motion are almost identical, and differ only in that the Coriolis parameter is a function of alongshore distance in the present model. In particular, flow near the coast must consist of a balance between wind stress and bottom friction, while flow far offshore reduces to a wind-driven Ekman flux. Details of the transition are described by the solutions. As we show in Section 4, solutions to the wind-driven ATW problem are a subset of the solutions to the present wind-driven problem with periodic alongshore structure and $\beta = 0$.

Allowance for a meridionally varying Coriolis parameter means that the heat conduction analogy is lost and we may no longer use the extensive literature on solutions of the heat conduction equation (e.g., Carslaw and Jaeger, 1959). This renders the initial–boundary value problem (with no wind stress) representing the relaxation of a surface elevation mound (caused by wind stress elsewhere, for example) considerably more difficult. This unforced problem is beyond the scope of the present work and only the asymptotic solutions are discussed in the following sections.

4. Circulation due to a spatially oscillatory alongshore wind stress on a shelf of infinite width

Using the methods outlined in Section 3, we seek asymptotic solutions to (2.10), describing the steady coastal circulation resulting from a spatially oscillatory alongshore wind stress acting on an infinitely wide shelf.

For imaginary separation parameter $-ik$, the across and alongshore structures in (2.10) obey the dimensional equations

$$\frac{r}{f_0 k s}\frac{d^2 Z}{dx^2} + \frac{\beta}{f_0 k}\frac{x dZ}{dx} + iZ = 0,$$

$$\frac{d\phi}{dy} - \frac{ik}{(1 + \beta y/f_0)}\phi = 0. \quad (4.1)$$

The alongshore structure $\phi = (1 + \beta y/f_0)^{ikf_0/\beta}$ allows solutions for wind stress of form

$$\tau_w = \tau_0(1 + \beta y/f_0)^{ikf_0/\beta - 1}. \quad (4.2)$$

Csanady found solutions for $\beta = 0$ and we note that our solutions are consistent with his since as $\beta \rightarrow 0$

$$\phi \rightarrow \lim_{\beta \to 0} (1 + \beta y/f_0)^{ikf_0/\beta} = \lim_{\beta \to 0} (1 + \beta y/f_0)^{ikf_0/\beta - 1} = e^{iky}.$$

As explained in the previous section, expansion of any real wind stress as a power series of terms each having different k allows an analogous power series solution for ϕ.

The across-shelf structure is dependent upon the sign of $f_0 k$. We choose k positive to simplify our arguments, then defining variables $Y = ky$ and $X = x/L$ allows transformation of (4.1) to dimensionless form. The sign of f_0 controls the definitions of L, the constants a and b and the streamfunction ψ as follows:

(i) $f_0 > 0$

$$L = \left(\frac{r}{f_0 k s}\right)^{1/2}, \quad a = \frac{\beta}{f_0 k}, \quad b = i \quad (4.3)$$

$$\frac{\zeta \rho g r}{f_0 L \tau_0} = \mathrm{Re}\left\{(1 + aY)^{i/a} \sum_{m=0}^{\infty} C_m X^m\right\} \quad (4.4)$$

$$\frac{\psi \rho f_0 k}{\tau_0} = \mathrm{Re}\left\{(1 + aY)^{i/a - 1} \sum_{m=0}^{\infty} \frac{m}{m+1} C_m X^{m+1}\right\} \quad (4.5)$$

(ii) $f_0 < 0$

$$L = \left(\frac{-r}{f_0 k s}\right)^{1/2}, \quad a = \frac{-\beta}{f_0 k}, \quad b = -i \quad (4.6)$$

$$\frac{\zeta \rho g r}{f_0 L \tau_0} = \mathrm{Re}\left\{(1 - aY)^{-i/a} \sum_{m=0}^{\infty} C_m X^m\right\} \quad (4.7)$$

$$\frac{\psi \rho f_0 k}{\tau_0} = -\mathrm{Re}\left\{(1 - aY)^{-i/a - 1} \sum_{m=0}^{\infty} \frac{m}{m+1} C_m X^{m+1}\right\}. \quad (4.8)$$

For the across-shelf structures defined by (4.3) and (4.6), values of γ found numerically from (3.8) by the methods outlined in the previous section are shown in Table 1 for selected values of the constant a. Values of the

TABLE 1. Values of γ such that the solution $Z = Z_0 - \gamma Z_e$, comprising a linear sum of odd and even series satisfying the equation $Z'' + aXZ' + bZ = 0$, obeys the boundary condition $Z \to 0$ as $X \to \infty$.

	b	
a	i	$-i$
0.6	0.80747 + i 0.58086	0.80747 − i 0.58086
0.5	0.79256 + i 0.60673	0.79256 − i 0.60673
0.4	0.77603 + i 0.63008	0.77603 − i 0.63008
0.3	0.75893 + i 0.65113	0.75893 − i 0.65113
0.2	0.74180 + i 0.67062	0.74180 − i 0.67062
0.1	0.72459 + i 0.68918	0.72459 − i 0.68918
0.0	0.70711 + i 0.70711	0.70711 − i 0.70711

constant a chosen for calculations here are limited to values of $a \leqslant 0.6$ since the value of $a = 0.6$ would appear to be appropriate for the largest scale low frequency wind forcing system caused by the succession of high pressure systems which dominate the midlatitudes from the tropics to the "roaring forties".

Figure 2 shows both the magnitude $|Z|$ and the phase δ of the across-shelf structure (3.7) for selected values of a. As explained in the caption to Fig. 2, the magnitudes $|Z|$ for each value of a are independent of whether $b = i$ or $b = -i$, while the arguments δ for $b = -i$ have opposite sign to those for $b = i$. The structure is effectively coastally trapped only for $a \geqslant 0$, and becomes more strongly trapped as a increases.

To physically interpret these results, consider the chosen conventions that k is positive and X represents distance offshore. In the Northern Hemisphere ($f_0 > 0$), on an east coast ($\beta > 0$), the flow is more strongly trapped than that for $\beta = 0$ while on a west coast ($\beta < 0$), the flow is not trapped since $a < 0$. Similarly in the Southern Hemisphere ($f_0 < 0$), on an east coast ($\beta > 0$), the flow is more strongly trapped while on a west coast the flow is not trapped. In general, the strongest trapping occurs when the alongshore length scale of forcing k^{-1} becomes comparable to f_0/β when $\beta > 0$. Therefore, enhanced trapping due to the β effect occurs only on east coasts, while the β-effect on west coasts acts to prohibit coastally trapped flows.

To show the dependence of the solutions on the value of a, streamfunction and surface elevation plots are shown in Fig. 3 for $a = 0.0$, and Fig. 4 for $a = 0.4$. These plots are appropriate for east coast oceans in the Northern Hemisphere, and plots for the Southern Hemisphere may be obtained by reflection about $Y = 0.0$. The plots for $a = 0.0$ are the same as those drawn by Csanady (1978, Figs. 4 and 5) while the plots for $a = 0.4$ show trapping to be confined slightly more strongly to the coast, while recirculation offshore be-

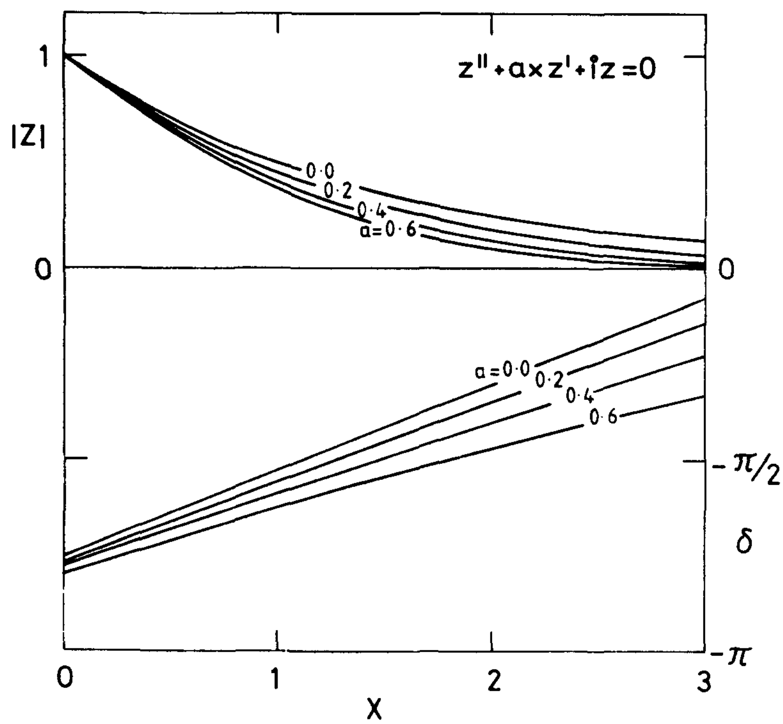

FIG. 2. The across-shelf structure $Z(X) = |Z(X)| \exp(i\delta)$, being the solution to $Z_{xx} + aXZ_x + bZ = 0$, plotted for selected values of the constant a for $b = i$. The imposed boundary conditions are $Z_x \to -1$ as $X \to 0$ and $Z \to 0$ as $X \to \infty$. The structure for $b = -i$ is identical except that the phase δ has opposite sign. Trapped solutions obeying the second boundary condition do not exist for $a < 0$ as discussed in Section 3.

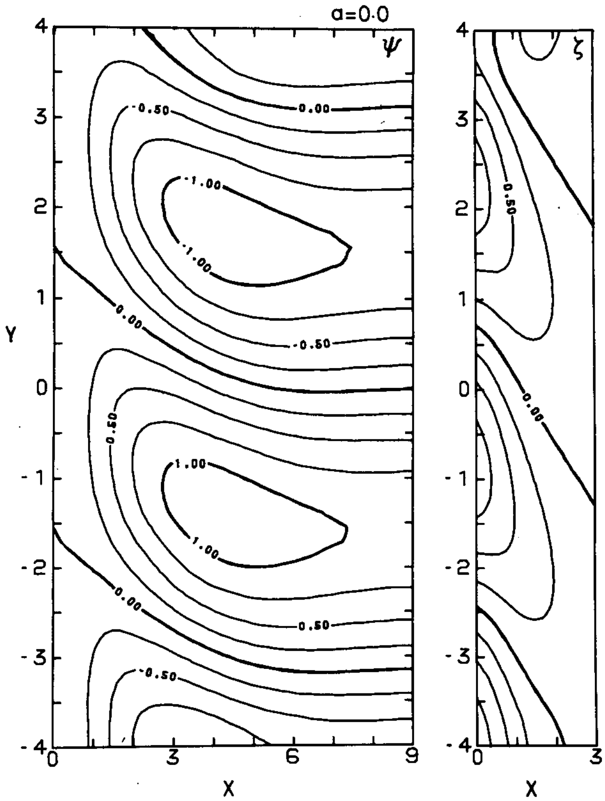

FIG. 3. The dimensionless streamfunction ψ and surface elevation ζ contours resulting from the wind stress $\tau = \tau_0 \cos Y$ for the case $a = 0$ in (4.2) (Northern Hemisphere) obeying the boundary conditions $\zeta \rightarrow 0$ as $X \rightarrow \infty$. Contours for the Southern Hemisphere are found by mirror-imaging these plots about $Y = 0$. The contour interval is 0.25.

comes weaker for larger values of a. The asymptotic solutions for the alongshore structure with nonzero a are stretched (compressed) at locations poleward (equatorward) of the y origin compared with those for $\beta = 0$.

In summary, the effects of inclusion of the β-plane variation in f for steady coastal circulation driven by a geographically varying wind stress of sinusoidal form are as follows. Off east coasts (western ocean boundaries) the effect of the β parameter is to tighten the trapping of the wind-driven coastal circulation, while off west coasts the β parameter acts to eliminate the trapping, leading only to oscillatory solutions in across-shelf structure. The absence of trapped solutions for $a < 0$ is primarily due to (2.8), and Section 5 examines solutions for a finite width shelf for both $a > 0$ and $a < 0$ when (2.8) is replaced by $\zeta \rightarrow 0$ at the shelf break.

5. Circulation due to a spatially oscillatory alongshore wind stress on a shelf of finite width

The absence of trapped solutions (obeying $Z \rightarrow 0$ as $x \rightarrow \infty$) for a coastal ocean off a west coast is disconcerting since we intuitively expect a geographically varying, but steady, wind stress to produce a steady circulation in any coastal ocean. In this section we investigate solutions for a finite width shelf, these being found simply by imposing the condition

$$Z \rightarrow 0 \quad \text{at} \quad X = \chi \tag{5.1}$$

where χ is the dimensionless shelf width. This condition may be applied for both east and west coast oceans. Other possible conditions and their drawbacks are described in Section 6.

The methods outlined in Section 3 may be used with a simple modification to (3.7), which by virtue of (5.1) becomes

$$\gamma = \lim_{X \rightarrow \chi} \frac{Z_0}{Z_e}. \tag{5.2}$$

Solutions (4.3)–(4.5) and (4.6)–(4.8) are then appropriate for $f_0 > 0$ and $f_0 < 0$, respectively, but with values of γ found through (5.2). Table 2 gives values of γ for both Northern and Southern hemispheres over a range of shelf widths and values of the parameter a. These were calculated from (5.2) using sums with 500 and 1000 terms to ensure convergence.

The across-shelf structure $Z = |Z| \exp(i\delta)$ is plotted

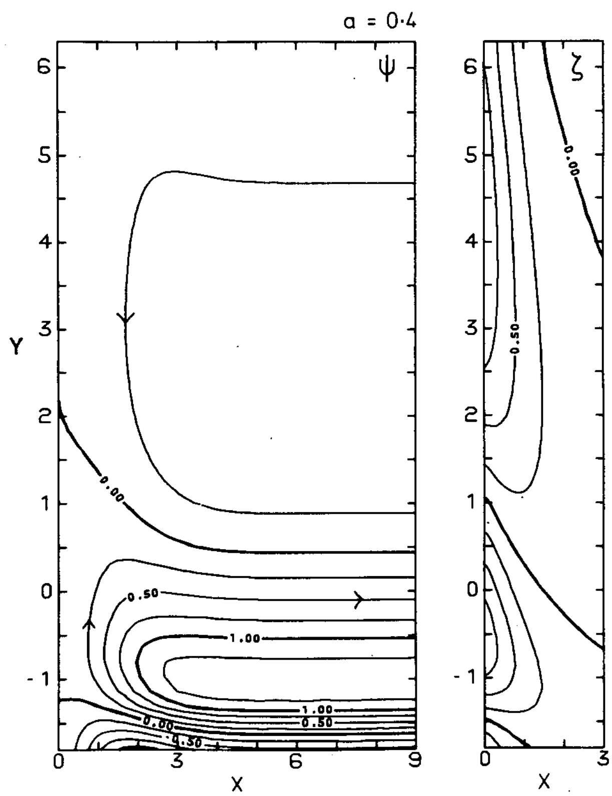

FIG. 4. The dimensionless streamfunction ψ and surface elevation ζ contours resulting from the wind stress $\tau_w = \tau_0(1 + aY)^{i/a-1}$ with $a = 0.4$ (Northern Hemisphere) obeying the boundary condition $\zeta \rightarrow 0$ as $X \rightarrow \infty$. Contours for the Southern Hemisphere are found by mirror imaging these diagrams about $Y = 0$. The contour interval is 0.25.

TABLE 2. Values of γ such that the solution $Z = Z_0 - \gamma Z_e$, comprising a linear sum of odd and even coefficient series satisfying the equation $Z'' + aXZ' + bZ = 0$, obeys the boundary condition $Z = 0$ at $X = \chi$. This table is appropriate for $b = i$ which corresponds to coastal flows in the Northern Hemisphere. Values of γ for $b = -i$ which correspond to coastal flows in the Southern Hemisphere are complex conjugates of the tabulated values.

	χ					
a	0.25	0.50	0.75	1.00	1.25	1.50
0.6	0.248 + *i* 0.005	0.484 + *i* 0.039	0.685 + *i* 0.118	0.826 + *i* 0.234	0.895 + *i* 0.358	0.908 + *i* 0.459
0.4	0.249 + *i* 0.005	0.488 + *i* 0.040	0.696 + *i* 0.123	0.845 + *i* 0.250	0.917 + *i* 0.391	0.924 + *i* 0.509
0.2	0.249 + *i* 0.005	0.492 + *i* 0.040	0.708 + *i* 0.128	0.865 + *i* 0.268	0.938 + *i* 0.428	0.937 + *i* 0.563
0.0	0.250 + *i* 0.005	0.496 + *i* 0.041	0.720 + *i* 0.134	0.885 + *i* 0.287	0.959 + *i* 0.467	0.947 + *i* 0.621
−0.2	0.250 + *i* 0.005	0.500 + *i* 0.042	0.732 + *i* 0.140	0.906 + *i* 0.307	0.979 + *i* 0.511	0.951 + *i* 0.684
−0.4	0.251 + *i* 0.005	0.504 + *i* 0.043	0.745 + *i* 0.146	0.928 + *i* 0.329	0.998 + *i* 0.557	0.951 + *i* 0.749
−0.6	0.251 + *i* 0.005	0.508 + *i* 0.044	0.758 + *i* 0.152	0.949 + *i* 0.353	1.014 + *i* 0.608	0.945 + *i* 0.817

in Fig. 5 as a function of X for selected values of a, for $b = i$ and for a dimensionless shelf width of $\chi = 1$. As explained in the caption, the magnitudes $|Z|$ for each value of a are independent of whether $b = i$ or $b = -i$, while the arguments δ have opposite sign for $b = -i$.

As with the infinite shelf case, for the Northern Hemisphere ($f_0 > 0$) on an east coast ($\beta > 0$) where $a > 0$, the sea level response is slightly reduced and confined slightly more to the coast than that for $a = 0$. However, solutions now exist for the west coast ocean where $a < 0$, and these have a slightly larger sea level response and are somewhat less confined than those for $a = 0$. Thus enhanced trapping occurs on east coast oceans and reduced trapping occurs on west coast oceans. Associated with the reduced trapping on a west coast ocean is a larger sea level response.

To further investigate the effects of the finite width shelf condition (5.1) consider the case of $a = 0$, $b = i$, $\chi = 1$. This corresponds to circulation with no β effect on a Northern Hemisphere shelf having width equal to the scale width. Streamfunction and elevation contours for this case are shown in Fig. 6. Comparison with Fig. 3 shows the shelf flow to be of a similar strength in each case for the coastal strip $0 < X < 1$, but with the maximum alongshore current component geographically in phase with the wind stress for the finite width shelf. The offshore volume flux does not match the deep ocean Ekman flux at the shelf break. The sea level elevation field is also somewhat different to that for Fig. 3, as might be expected from the streamfunction contours.

Figures 7 and 8 are also relevant to the Northern Hemisphere ($b = i$) for shelf widths of $\chi = 1$ with the values of $a = 0.4$ and $a = -0.4$ corresponding to east

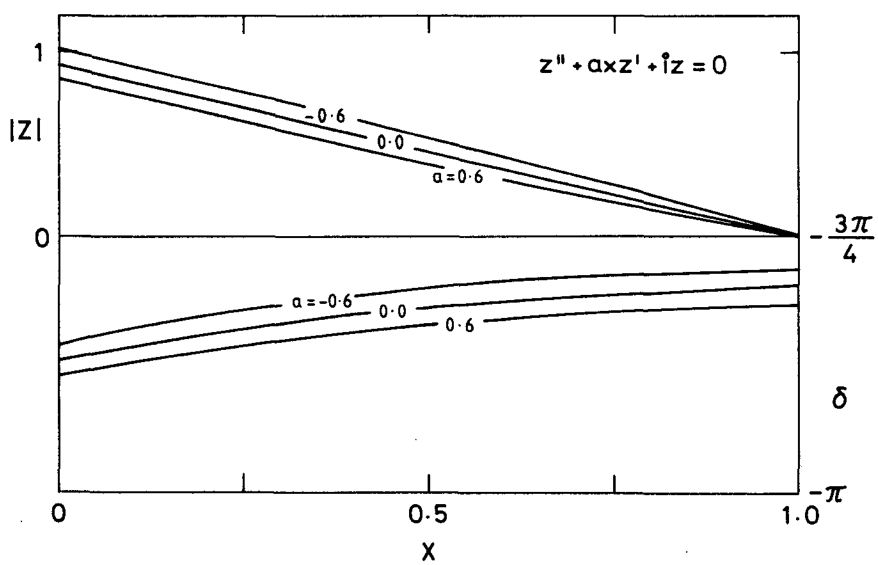

FIG. 5. The across-shelf structure $Z(X) = |Z(X)| \exp(i\delta)$, being the solution to $Z_{xx} + aXZ_x + bZ = 0$, plotted for selected values of the constant a with $b = i$. The imposed boundary conditions are $Z_x \rightarrow -1$ as $X \rightarrow 0$ and $Z \rightarrow 0$ as $X \rightarrow 1$. The structure for $b = -i$ is identical except that the phase δ has opposite sign.

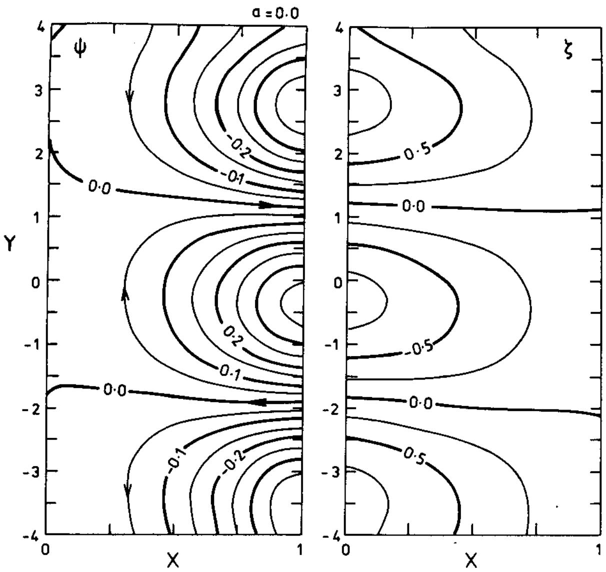

FIG. 6. The dimensionless streamfunction ψ and surface elevation ζ contours resulting from the wind stress $\tau_w = \tau_0 \cos Y$ with $a = 0.0$ (Northern Hemisphere) obeying the boundary condition $\zeta \rightarrow 0$ at $X = 1$, the shelf break. Contours for the Southern Hemisphere are found by mirror imaging these diagrams about $Y = 0$. The contour interval is 0.05 for ψ and 0.25 for ζ.

and west coast oceans, respectively. As with Fig. 4 the asymptotic solutions are stretched in the poleward alongshore direction. Apart from the opposite geographical orientation of the Y axis, the major differences between Figs. 7 and 8 are that the east coast ocean contours (with $a = 0.4$ and the Y axis oriented north) show stronger currents and larger sea level anomalies and gradients than the west coast ocean contours (with $a = 0.4$ and the Y axis oriented south).

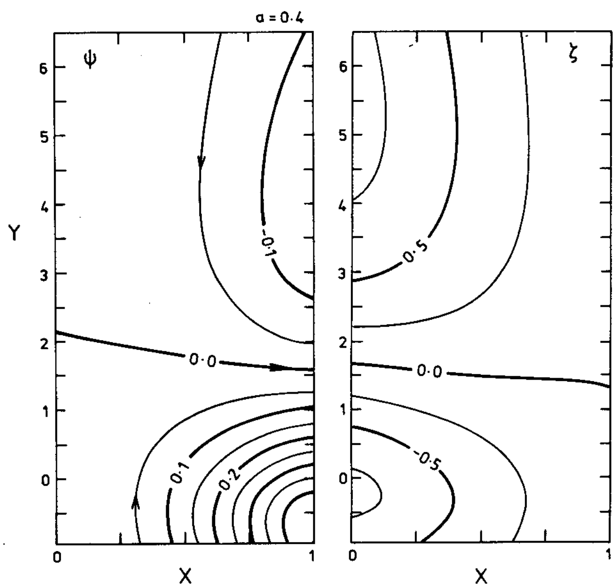

FIG. 7. The dimensionless streamfunction ψ and surface elevation ζ contours resulting from the wind stress $\tau_w = \tau_0(1 + aY)^{i/a-1}$ with $a = 0.4$ and $b = i$ corresponding to a shelf with Y axis poleward in the Northern Hemisphere. The applied boundary condition at the shelf break ($X = 1$) is $\zeta = 0$. The contour interval is 0.05 for ψ and 0.25 for ζ.

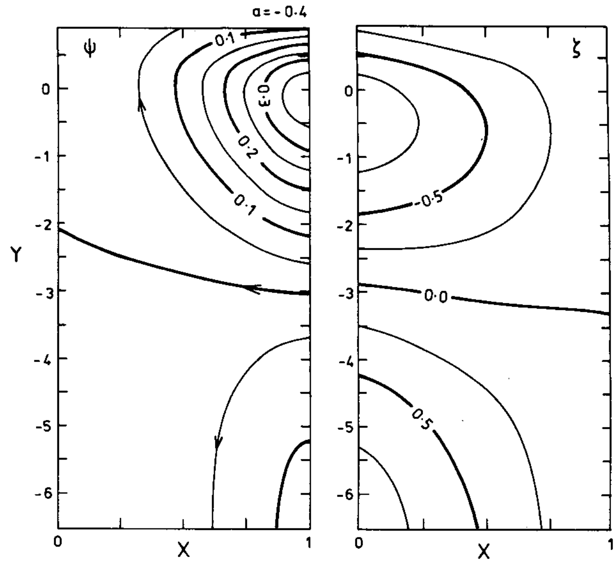

FIG. 8. As for Fig. 7 but for $b = -i$ corresponding to a shelf with Y axis equatorward in the Southern Hemisphere.

Thus we conclude that westward intensification of currents due to the β-effect appears to be a salient feature of wind-driven shelf circulation.

6. Discussion

In this paper we have developed the solutions for steady wind-driven circulation on a β-plane for the case of continental shelves of finite width, these being a generalization of the arrested topographic wave solutions developed by Csanady (1978). Our solutions are appropriate for wind-driven circulation on continental shelves where the spatial scale of alongshore forcing is comparable to the scale over which the Coriolis force changes. In particular, for meridionally oriented continental shelves near the equator the generalization provides solutions in cases where the arrested topographic wave model (having constant Coriolis parameter) breaks down.

For constant Coriolis parameter f, trapped flow fields (obeying $\zeta \rightarrow 0$ as $X \rightarrow \infty$) occur for wind stresses which vary geographically in the alongshore direction in an oscillatory fashion on either an east or a west coast. The generalized solutions indicate that, in circumstances where the wind scale k^{-1} is sufficiently large that the dimensionless parameter (β/f_0k) is significantly different from zero, the coastal trapping on an infinitely wide shelf becomes somewhat more pronounced on an east coast ocean, while trapped solutions on a west coast ocean no longer exist.

The lack of trapped solutions for circulation off a west coast is evidently a function of applying the boundary condition $\zeta \rightarrow 0$ as $X \rightarrow \infty$, since solutions are readily obtainable if $\zeta \rightarrow 0$ is applied at a finite shelf width (the shelf break). Using this condition solutions are obtainable for any realistic values of the parameter β/f_0k, including $\beta/f_0k = 0$. The effects of nonzero β include an intensification of east coast (western ocean boundary) currents compared with those on the west coast.

It might appear at first that boundary conditions other than $\zeta \rightarrow 0$ might be applicable at the shelf break, for example, a matching of deep sea with coastal alongshore velocities or a matching of open ocean Ekman flux with on–offshore shelf flux. Matching of the alongshore velocity v appears to be inappropriate since in this paper the assumed bathymetry is effectively assumed to be discontinuous at the shelf break, and discontinuities of alongshore velocity at a depth discontinuity are a common feature of time-dependent shelf circulation models (Middleton, 1983; Middleton and Cunningham, 1984). Matching of on–offshore values of volume flux at the shelf break through $\psi_y = -\tau_w/\rho f$ appears to give physically unrealistic results for steady flows on narrow finite width shelves since the offshore flux arising from the bottom Ekman layer and its interaction with the onshore flux in the surface Ekman layer (Gill and Schumann, 1974) cannot be taken into account in this depth-averaged model.

The role of stratification is not at all addressed in this model. Yet the regions where the model is most likely useful (i.e., the tropics) are also those regions where stratification plays a greater role as indicated by the inverse square dependence of the Burger Number on the Coriolis parameter. The theory is therefore restricted to well-mixed coastal oceans.

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