On Mean Flow Instabilities within the Planetary Geostrophic Equations

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ABSTRACT
This note draws attention to the natural instabilities of a mean zonal flow that arise with the planetary geostrophic equations increasingly used in theories of large-scale oceanic circulation. Baroclinic instability is not excised by the absence of the relative acceleration terms in the momentum equations. The growth rate is shown to increase linearly with wavenumber, yielding an ill-posed mathematical problem. A small amount of lateral friction cures the problem, however, as shown in a three-layer model, which possesses the minimal vertical structure to exhibit the instability.

1. Introduction

For slow oceanic motions of planetary scale on a rotating earth the momentum balance is geostrophic. Both the variations of the Coriolis parameter with latitude and the important vertical excursions of the observed isopycnal surfaces require the consideration of the divergent part of the horizontal flows. Because the oceanic internal Rossby radius of deformation is small (on the order of 50 km), it is appropriate to emphasize the role of the stretching terms respective to the relative vorticity terms in the planetary-scale vorticity balance. The result is a system of equations, called planetary geostrophic (PG), which presents very attractive features for the analysis of the largest oceanic scales in regions free of complicated dynamical interactions with the mesoscale. This system has been perceptively described by Phillips (1963) under his "geostrophic of type 2" scaling and has been widely used in early theories of the abyssal circulation (Stommel and Arons, 1958) and of the main thermocline (see Veronis, 1969, for a review). A revival followed the analysis of Luyten et al. (1983) who constructed a steady solution forced by Ekman pumping in a semi-infinite ocean using a crude parameterization of the surface buoyancy fluxes. It is probably this property of being able to handle both the mechanical and thermodynamical forcings that makes the PG system so attractive for studies of planetary motions.

Because baroclinic instability has proven to be one of the cornerstones of eddy theories in both oceans and atmospheres, it is appropriate to investigate the natural instabilities encompassed by the PG set. This turned out, as well, to be necessary for the construction of a numerical model based explicitly on the above dynamics. Aspects of baroclinic instability influenced by beta plane approximations of the spherical geometry of the earth were pointed out originally by Charney (1947). More recently, Hollingsworth (1975) has retained the full spherical geometry to show that the beta plane gives a good description of the essential dynamics of the waves but that lateral momentum transports occur with heat transports when the basic flow is a differential solid body rotation. The present study departs from these works by neglecting horizontal advection of momentum and relative vorticity at the onset of the analysis, an approximation useful for horizontal scales large compared to the Rossby radius. It is shown that baroclinic instability is indeed possible, as the long-wave limit of the classical Charney problem, and attention is drawn to the mathematical difficulties of the inviscid set by presenting some detailed eigenvalue calculations within the framework of a three-layer model.

2. General properties

A unified and informative derivation of both planetary geostrophic and quasi-geostrophic sets of equations has recently been given by Pedlosky (1984). When the scale separation between the planetary and the synoptic scale becomes large, the PG equations describing the planetary motions can be written on a cartesian beta plane as

\[
\begin{align*}
\omega \mathbf{k} \times \mathbf{u} &= -\nabla H \frac{P}{\rho_0} \\
-P_2 - g \rho &= 0 \\
\nabla \cdot \mathbf{u} &= 0 \\
\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho &= 0
\end{align*}
\]

where \(\omega\) is the Coriolis parameter = \(f_0 + \beta v, f_0 = 2\Omega \times \sin \theta_0, \beta = 2\Omega \cos \theta_0 / R_e\) with \(\theta_0\) a reference latitude.
and \( R \) the earth radius, \( k \) is a vertical unit vector, \( \nabla \) is the three-dimensional gradient operator, \( \nabla_{B} \) being its horizontal projection. After scaling \( w \) by \((H/L)U\), \( P \) by \( \rho_{0}\delta_{0}ULU/gH \), \( \rho \) by \( \rho_{0}\delta_{0}ULU/gH \) and by \( L/U \), one may reduce the set (1) to a coupled \( P, w \) system:

\[
\frac{\partial}{\partial t} Q + \int_{0}^{\infty} J(P, \cdots) + w \frac{\partial}{\partial z} \cdots P_{z} = 0
\]

(2)

where \( \gamma = \beta_{0}L/f \) and \( J \) is the Jacobian operator. A rigid lid approximation and a flat ocean floor provides two boundary conditions:

\[
w = 0 \quad \text{at} \quad z = 0, -1.
\]

It is particularly simple to study the stability properties of a purely zonal mean flow \( U(z) \) because the mean vertical velocity vanishes identically in that case. Assuming a constant \( \gamma \), and perturbation varying as \( e^{i(kx - \omega t)} \), the linearized pressure field satisfies

\[
(U - c) \frac{\partial}{\partial Z} [B^{-2}P_{Z}] + \left[ \frac{\gamma}{\epsilon} - \frac{\partial}{\partial Z} (B^{-2}U_{Z}) \right] P \equiv 0
\]

(3)

where \( B = N(z)\delta_{0}L \), the Burger number, and \( \epsilon = U\delta_{0}L \), the Rossby number, are both much smaller than one while \( \gamma \) is on the order of one. Under this form the instability properties can be deduced easily and represent the long-wave limit of the quasi-geostrophic Charney problem (see Pedlosky, 1979, for a review). If \( c_{i} \) is the imaginary part of \( c \), the following condition must be satisfied by the perturbations:

\[
c_{i} \int_{-1}^{0} \frac{\partial Q}{\partial y} \left| \frac{P^{2}}{U - c} \right| dz + \left[ B^{-2}U_{Z} \frac{P^{2}}{U - c} \right]_{Z = -1} = 0
\]

in which the meridional potential vorticity gradient

\[
\frac{\partial Q}{\partial y} = \gamma - \frac{\partial}{\partial Z} B^{-2}U_{Z}.
\]

Necessary conditions for instability \( (c_{i} \neq 0) \) follow:

(i) \( \partial Q/\partial y \) must change sign within the fluid if \( B^{-2}U_{Z} \) vanishes at horizontal boundaries;

(ii) if \( \partial Q/\partial y \) keeps the same sign, then at least one of the vertical shears at horizontal boundaries must be of opposite sign to \( \partial Q/\partial y \).

It is possible to provide bounds for the complex eigenvalues in a manner analogous to what has been done for the Charney problem. Introducing the variable \( \phi = P/(U - c) \), one obtains

\[
\frac{\partial}{\partial Z} [(U - c)^{2}B^{-2}\phi_{Z}] + \frac{\gamma}{\epsilon} \phi(U - c) = 0
\]

with \( \phi_{z} = 0, \quad z = 0, -1. \)

Multiply the above by \( \phi^{*} \) and integrate in \( z \) to obtain

\[
\int_{-1}^{0} [(U - c)^{2}B^{-2}\phi_{z}^{2} - \frac{\gamma}{\epsilon} \phi^{2}(U - c)]dz = 0.
\]

Identifying real and imaginary parts gives

\[
\int_{-1}^{0} [(U - c)^{2}B^{-2}\phi_{z}^{2} - \left( \frac{\gamma}{2\epsilon} \right) \phi^{2}]dz = 0 \quad (4a)
\]

\[
\int_{-1}^{0} [(U^{2} - (C_{1}^{2} + C_{2}^{2}))B^{-2}\phi_{z}^{2} - \left( \frac{\gamma}{\epsilon} \right) U \phi^{2}]dz = 0. \quad (4b)
\]

Consider the obvious inequality:

\[
\int_{-1}^{0} (U - U_{\max})(U - U_{\min})B^{-2}\phi_{z}^{2}dz \leq 0.
\]

Developing and using (4a) and (4b) yields

\[
\left\{ \left[ C_{1} - \left( \frac{U_{\max} + U_{\min}}{2} \right) \right]^{2} + C_{2}^{2} - \left( \frac{U_{\max} - U_{\min}}{2} \right)^{2} \right\}
\]

\[
\times \int_{-1}^{0} B^{-2}\phi_{z}^{2}dz + \frac{\gamma}{\epsilon} \int_{-1}^{0} \left( U - \frac{U_{\max} + U_{\min}}{2} \right) \phi_{z}^{2}dz \leq 0. \quad (5)
\]

Specializing to the case of uniform stratification \( (B = \text{constant}) \) for simplicity, it is possible to derive a useful inequality obtained by expansion of the arbitrary function \( \phi(z) \) in a Fourier cosine series over the interval \(-1, 0\):

\[
\int_{-1}^{0} \phi_{z}^{2}dz \geq \frac{\pi^{2}}{2} \int_{-1}^{0} \phi^{2}dz.
\]

It is then straightforward to transform (5) into

\[
\left( C_{1} - \frac{U_{\max} + U_{\min}}{2} \right)^{2} + C_{2}^{2} \leq \left( \frac{U_{\max} - U_{\min}}{2} \right)^{2} + \frac{\gamma B^{2}}{\epsilon} \left( \frac{U_{\max} - U_{\min}}{2} \right)^{2} = R^{2}. \quad (6)
\]

This proves that all eigenvalues \( c \) are bounded and contained within a semicircle of radius \( R \) defined above. It is expected that the complex eigenvalues will lie within the range of the zonal flow velocities because \( \gamma B^{2}/\epsilon \) is on the order of one in the planetary limit. Since the boundary value problem (3) is independent of the zonal wavenumber, this result implies that the growth rate increases linearly with the zonal wavenumber when the flow is unstable. This shows immediately the difficulty inherent in the inviscid PG set; for a given initial distribution of Fourier amplitudes, the smallest scales of the flow grow faster than exponentially. This appears to invalidate this set of equations as a practical one; yet since the growth rate varies only linearly with wavenumber, one may expect that the addition of a traditional friction term should resolve the problem at the high wavenumber end.
Thus far we have shown necessary conditions for instabilities and provided upper bounds for the complex phase speeds, but we have not demonstrated that the instability occurs. To provide an example, this will be carried out explicitly in the next paragraph. The nature of the instability, however, is perfectly clear; it is a baroclinic instability in the classical sense with potential energy of the perturbations feeding upon the potential energy of the mean flow, the energy release operating through a downgradient heat transport. It is also clear from (2) that the beta effect is a crucial ingredient for the instability. As \( \gamma \) goes to zero, \( W_Z \) and \( W \) (because of boundary conditions) vanish everywhere, which inhibits releases of potential energy of the mean flow by the perturbations.

3. An example

It is best to compute eigenvalues in a simple model of low resolution to demonstrate the actual instability and the mollifying effects of lateral friction. The two-layer model is always stable because the nonlinear terms in the density equation vanish identically. The next simplest model is, therefore, the three-layer model whose governing equations, derived from (1) to which a Laplacian friction term is added, are

\[
\begin{align*}
fk \times (u_1 - u_2) &= -\gamma_1 \nabla h_1 + A \nabla h^2 (u_1 - u_2) \\
fk \times (u_2 - u_3) &= -\gamma_2 \nabla h_3 + A \nabla h^2 (u_2 - u_3) \\
\beta v_i h_i &= f \left( \frac{\partial h_i}{\partial t} + u_i \cdot \nabla h_i \right), \quad i = 1, \ldots, 3
\end{align*}
\]

\[
\sum_{i=1}^{3} u_i h_i = 0
\]

\[
h = \sum_{i=1}^{3} h_i
\]

where \( h_i \) is the depth of the \( i \)th layer, \( \gamma_i = g(\rho_2 - \rho_i)/\rho_0 \) and \( \gamma_2 = g(\rho_3 - \rho_2)/\rho_0 \).

A system of two coupled equations for the variables \( h_1, h_3 \) can be derived to leading order, assuming small friction and the usual \( \beta \) plane approximation:

\[
\begin{align*}
\frac{\partial h_1}{\partial t} + \frac{\gamma_2 h_3}{f_0 h} j(h_1, h_3) + \frac{\beta h_3}{f_0^2} \left[ \gamma_1 \left( \frac{h_1}{h} - 1 \right) h_{1x} + \gamma_2 \frac{h_3 h_{3x}}{h} \right] &= \frac{4h_1}{f_0^2} \left[ \gamma_1 \left( \frac{h_1}{h} - 1 \right) \nabla^4 h_1 + \gamma_2 h_3 \nabla^4 h_3 \right] \\
\frac{\partial h_3}{\partial t} + \frac{\gamma_1 h_1}{f_0 h} j(h_3, h_1) + \frac{\beta h_3}{f_0^2} \left[ \gamma_2 \frac{h_3}{h} - 1 \right] h_{3x} + \gamma_1 h_1 \frac{h_{1x}}{h} \right] &= \frac{4h_3}{f_0^2} \left[ \gamma_1 \frac{h_1}{h} \nabla^4 h_1 + \gamma_2 \left( \frac{h_3}{h} - 1 \right) \nabla^4 h_3 \right]
\end{align*}
\]

Linearizing the above set around a zonal shear flow and introducing perturbations as \( e^{ik \cdot x - \omega t} \) provides a linear second-order eigenvalue problem whose eigenvalues are easily determined. Figure 1 shows the imaginary part of the phase speeds in the inviscid case, which varies independently of the upper- and lower-layer shears. In the three-layer system, the meridional potential-vorticity gradient is of one sign. For the parameters chosen, the second necessary condition for instability is fulfilled, with the negative upper-layer shear (westward shear) opposing the positive potential vorticity gradient. The unstable phase speeds are on the order of a few centimeters per second, as expected. With friction added, the instability selects a preferential scale as seen on Fig. 2. At the wavenumber \( k \) maximizing the growth rate, the inviscid growth rate (say \( U_{\max} \)) and the frictional decay rate (\( A \lambda^2 k^4 \)) balance, giving

\[
k \sim \left( \frac{U_{\max}}{A \lambda^2} \right)^{1/3},
\]

where \( \lambda_i \) is the internal Rossby radius. In large-scale oceanic numerical models, given a horizontal resolution, the value of the friction coefficient is often chosen so that the barotropic Gulf Stream is resolved. For instance, with an horizontal resolution \( \Delta = 1000 \text{ km} \), the lateral friction needs to be \( O(\beta \Delta^3) = 2 \times 10^8 \text{ cm}^2 \text{ s}^{-1} \), according to this criterion. With parameters as in Fig. 2, the smallest e-folding time scale is on the order of 115 days and occurs for a horizontal wavelength on the order of 500 km. Figure 3 shows the growth rate in the wavenumber plane for a given unstable profile of the zonal flow and a given friction coefficient. As

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**FIG. 1.** The imaginary part of the phase speeds scaled by \( c_1 = \beta \lambda_{U_1} U_1^2 \) as a function of the upper- and lower-layer shears \( U_1 \) and \( U_2 \) in the three-layer system (\( H_1 = H_3 = 400 \text{ m}, H_2 = 3200 \text{ m}, \gamma_1/g = \gamma_2/g = 10^{-3}, C_1 = 0.78 \text{ cm s}^{-1} \)).
expected, the instability favors perturbations with meridional wavecrests optimizing the release of the potential energy associated with the zonal flow.

4. Conclusion

Baroclinic instability exists within the context of the planetary geostrophy equations. The inviscid form has bounded unstable phase speeds leading to growth rates increasing linearly with zonal wavenumber. This unbounded growth makes the inviscid system ill posed as compared to the quasi-geostrophic system, for example. The drastic filtering inherent in the PG set leads to mathematical difficulties in the continuous system, which would be magnified in a numerical model. Lateral friction has the required spectral properties to provide a high wavenumber cut off to the instability and induces a preferential scale of instability of order \( (U_{\text{max}} / A \lambda^2)^{-1/3} \). In an analogous way the inviscid equations suffer from the difficulties of satisfying lateral boundary conditions (see Huang, 1984) or develop discontinuities in the fluid interior (Luyten et al., 1983). It is proposed here that the addition of a frictional dissipative process makes the PG set attractive by allowing a complete solution within closed boundaries to be determined while keeping baroclinic instability within reasonable physical bounds. The fact, however, that the scale and the amplification of the most unstable wave are frictionally controlled indicates that this set of equations is not well suited to wave-mean flow interaction problems. The nature of the solution may change drastically as the spatial resolution increases, while diminishing the friction. Clearly, much physics is being left out at the scale of the internal Rossby radius shown to be so crucial for both linearized and finite amplitude solutions of the baroclinic instability problem.

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