# On the Propagation of Isolated Multilayer and Continuously Stratified Eddies

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ABSTRACT

Integral expressions are derived for the east–west velocity of propagation of isolated eddies on a beta plane. It is assumed that the eddies have no surface or floor expression, i.e., that both surface and floor are isopycnals. The results of Nof and Mory are generalized and demonstrate the crucial necessity for all such results that, on the bounding density surfaces, the linearized Bernoulli function depends only on the depth of that surface. Thus there are examples of isolated eddies satisfying the assumptions but which are not directly amenable to the analyses presented hitherto. Results for multiple layers (including a simple rule for the direction of propagation) and for continuously stratified eddies, subject to some assumptions, are given. A simple model fit to salt lenses observed by Armi and Zenk gives westward motion of order 1 cm $s^{-1}$, which is not unreasonable.

## 1. Introduction

Nearly circular, isolated eddies and lenses occur in many parts of the world ocean, and are notable for their varying internal structures, sizes and directions of propagation. In the Atlantic, for example, there are warm and cold-core Gulf Stream rings (Joyce, 1984; Richardson, 1983); large saline lenses (Armi and Zenk, 1984); and nearly isothermal eddies in the Sargasso Sea (Dugan et al., 1982). The flow in these eddies can be cyclonic or anticyclonic, both inferred from hydrographic data and by direct observation. The eddies usually contain a finite water mass whose *T–S* properties differ strongly from the surrounding water, although this is seldom the case for the density field, which differs little from its surroundings. The eddies vary both in radius, from around 100 km or more for Gulf Stream rings (Joyce, 1984) to 50 km or less for salt lenses (Armi and Zenk, 1984) down to 25 km for the Sargasso eddies (Dugan et al., 1982). There are similar variations in height, from the full ocean depth to under 100 m, respectively. When a direction of propagation can be determined, there is usually but not always a westward tendency (Joyce, 1984; Nof, 1985).

The existence of such eddies provides several theoretical problems: their formation, propagation, stability and eventual fate; McWilliams (1985) gives a review of efforts in these directions. This note will concentrate on the propagation of such eddies. Hitherto, analytical modelers have concentrated on simple, dynamical layered models for quasi-radially symmetric eddies (Warren, 1967; Shen, 1981; Nof, 1981; Killworth, 1983; Davey and Killworth, 1984), together with modon solutions of nonsymmetric form (e.g., McWilliams et al., 1981). The former studies have involved few—typically one—active layers, with a resting deep layer beneath. They find steady westward motion for the eddy; noting that N–S motion would involve a change in potential vorticity of the eddy—impossible for steady flow.

Numerical studies also have used layered models. Regrettably, the results of such models seem to depend critically upon the physics of the model: two-layer quasi-geostrophic experiments (McWilliams and Flierl, 1979) differ in results qualitatively from two-layer primitive equation experiments (Mied and Lindemann, 1979), which in turn differ from one-layer primitive equation results (Davey and Killworth, 1984), with a tendency toward more rapid eddy destruction as the number of degrees of freedom in the model—e.g., number of layers—is increased.

Recently, Nof (1985), and Mory (1985) have sought to extend the analytical work to include a second active layer, in the spirit of the numerical approaches. Nof's results for an eddy with a trapped volume of fluid density not equal to any of those in the resting fluid outside show that propagation can be west *or* east. However, it is unclear how many eddies have any fluid with this property, even though the *T–S* properties may differ radically, except in the top 100 m. It turns out to be straightforward to allow for layers not to be trapped within the eddy. Since there is no reason to restrict attention to two active layers, one of the aims of this note is to derive a formula for eddy propagation in a fluid with *N* active layers, each of which may or may not be confined either to the eddy or to the outer environment, together with a simple rule for the direction of eddy propagation.

Layered models, especially those with few layers, are quite difficult to apply to a continuously stratified fluid, if one wants to make estimates from observations. It therefore seems relevant to produce the equivalent integral formula for the propagation speed of a continuously stratified blob. This can be done, under slightly restrictive circumstances, but the analysis demonstrates clearly that not all eddies can be treated. Finally, as an example, the formula is applied to an approximate version of Armi and Zenk's (1984) data.

## 2. Propagation speeds for multilayer eddies

### a. Solution

We consider a fluid made up of $N + 1$ layers, not all of which need be present at any given location. The density of the $n$th layer from the surface is $\rho_n$, with depth $h_n(x, y, t)$ where axes are $x$ (east), $y$ (north), and $t$ (time). We assume that the $(N + 1)$th layer is at rest; the consequences of this will be discussed later. The hydrostatic relation, plus the assumption of the resting layer, gives the pressure $p_n$ in the $n$th layer as

$$\rho_0{}^{-1}p_n = \sum_{m=1}^{N} g_{\max(m,n)}h_m, \tag{2.1}$$

where $\max(m, n)$ refers to the larger of $m$ and $n$, and the $g_n$ are reduced gravities given by

$$g_n = g(1 - \rho_n/\rho_{N+1}), \tag{2.2}$$

and $\rho_0$ is a reference density. Equation (2.1) holds even if some $h_n$ are zero.

The dynamical equations in each layer are

$$u_{nt} + \mathbf{u}_n\cdot\nabla u_n - fv_n + p_{nx}/\rho_0 = 0 \tag{2.3}$$

$$v_{nt} + \mathbf{u}_n\cdot\nabla v_n + fu_n + p_{ny}/\rho_0 = 0, \tag{2.4}$$

where $(u_n, v_n)$ are the velocity components and $f = f_0 + \beta y$ is the Coriolis parameter. Mass conservation yields

$$h_{nt} + (u_nh_n)_x + (v_nh_n)_y = 0. \tag{2.5}$$

We now follow Ball (1963), as in Killworth (1983) and Davey and Killworth (1984), to derive a simple expression for propagation speed. (The method relies on integral balances which do not depend on any specific depth profile.) The eddy is assumed localized, so that

$$h_n \to H_n, \quad (u_n, v_n) \to 0, \quad |x| \to \infty, \tag{2.6}$$

where $H_n$ is uniform, possibly zero. Thus for each layer we may define the mass anomaly

$$Q_n = \int (h_n - H_n)dA, \tag{2.7}$$

where the integral is over the entire area. We note that

$$\frac{d}{dt}\int h_n\phi dA = \int h_n\frac{D_n\phi}{Dt}dA, \tag{2.8}$$

for any $\phi$, where

$$\frac{D_n}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}_n\cdot\nabla, \tag{2.9}$$

is the advective operator. The center of mass $(X_n, Y_n)$ of the $n$th layer can be defined by

$$Q_nX_n = \int (h_n - H_n)\mathbf{x}dA. \tag{2.10}$$

From (2.8), we thus have

$$\frac{dQ_n}{dt} = 0 \tag{2.11}$$

$$Q_n\frac{d\mathbf{X}_n}{dt} = \int h_n\mathbf{u}_ndA. \tag{2.12}$$

If the motion is *steady*, then $dY_n/dt = 0$ for all $n$, as meridional motion would involve potential vorticity changes (see Nof, 1984 for a case of unsteady meridional motion). Further differentiation of (2.12) gives

$$0 = Q_n\frac{d^2Y_n}{dt^2} = \int h_n\frac{D_nv_n}{Dt}dA = -\int fu_nh_ndA + \int h_np_{ny}\Big/\rho_0dA = -f_0Q_n\frac{dX_n}{dt} - \beta\int yu_nh_ndA + \int h_np_{ny}\Big/\rho_0dA, \tag{2.13}$$

or

$$f_0Q_n\frac{dX_n}{dt} = -\beta\int yu_nh_ndA + \frac{1}{\rho_0}\int h_np_{ny}dA. \tag{2.14}$$

Now a steady east–west propagation *requires* the $dX_n/dt$ to equal $c$, the east–west speed of the eddy. This is not an assumption, but a necessity if such a flow is to exist. There are scaling arguments (Killworth, 1983; Nof, 1985) to show that since $\epsilon = \beta af_0{}^{-1}$ is a small parameter for isolated eddies, where $a$ is a typical length scale, then $c$ is of order $\epsilon$ times a typical velocity. Thus in coordinates moving at speed $c$, $(u_n - c, v_n) \approx (u_n, v_n)$ is describable by a streamfunction $\phi_n$ by (2.5), and (2.14) gives

$$f_0Q_n\frac{dX_n}{dt} = \beta\int \phi_ndA + \frac{1}{\rho_0}\int h_np_{ny}dA, \tag{2.15}$$

as an alternative formulation. To remove the unknown pressure gradients, we sum (2.14) over the layers, giving

$$f_0c\sum_{n=1}^{N} Q_n = -\beta\int y\sum_{n=1}^{N} u_nh_ndA, \tag{2.16}$$

since the pressure gradients cancel in pairs of form $\int g_{\max(m,n)}(h_mh_{ny} + h_nh_{my})dA$ from (2.1). The most useful form for (2.16) is by assuming approximate radial symmetry. Then, using $r$ as radius, (2.16) becomes

$$c = \frac{\beta \int_0^\infty r^2 dr \sum_{n=1}^{N} h_n V_n}{2f_0 \int_0^\infty r dr \sum_{n=1}^{N} (h_n - H_n)}, \qquad (2.17)$$

where $V_n$ is the azimuthal velocity in layer $n$, although forms using the integrated streamfunction $\phi$ are possible from (2.15). Using the cyclostrophic balance

$$f_0 V_n + \frac{V_n^2}{r} \approx \frac{1}{\rho_0} \frac{\partial p_n}{\partial r}, \qquad (2.18)$$

(2.17) can be written

$$c = \frac{-\beta}{2f_0^2 \int_0^\infty r dr \sum_{n=1}^{N} (h_n - H_n)} \int_0^\infty r dr \{ \sum_{n=1}^{N} [h_n V_n^2 + g_n(h_n^2 - H_n^2)] + 2 \sum_{m>n} g_m(h_n h_m - H_n H_m) \}, \qquad (2.19)$$

proportional to (energy/mass), as used by Davey and Killworth (1984). Using their methods, limits on $c$ may be evaluated from (2.19), although this is not our purpose here.

### b. The pressure gradients

It is of importance to understand why the pressure gradients cancelled in (2.16), since their removal is vital. We relax temporarily the requirement that layer ($N + 1$) be at rest, so that the pressure $p_n$ becomes

$$p_n = g \sum_{m=1}^{N+1} \rho_{\min(m,n)} h_m. \qquad (2.20)$$

Repeating the sum in (2.15), (2.16) then gives

$$\begin{aligned} \sum_{n=1}^{N} h_n p_{ny} &= g \sum_{n=1}^{N} \sum_{m=1}^{N+1} \rho_{\min(m,n)} h_n h_{my} \\ &= g \frac{\partial}{\partial y} (\sum_{n=1}^{N+1} \sum_{m=1}^{N+1} \rho_{\min(m,n)} h_n h_m) - g h_{N+1} p_{N+1,y} \\ &= g \frac{\partial}{\partial y} \left( \sum_{n=1}^{N+1} \sum_{m=1}^{N+1} \rho_{\min(m,n)} h_n h_m + \frac{1}{2} \rho_{N+1} h_{N+1}^2 \right) - h_{N+1} B_y, \end{aligned} \qquad (2.21)$$

where we define the (linear) Bernoulli function $B$ at height $h_{N+1}$ by

$$B = p + \rho_{N+1} g h_{N+1}. \qquad (2.22)$$

Although not the dynamical Bernoulli function, $B$ serves as the equivalent of pressure when using density as a vertical coordinate. Now, integration over $dA$ removes the first $y$-derivative, leaving only the term in $h_{n+1} B_y$, or, by parts, $B h_{N+1,y}$.

It is now clear that for this term to vanish on integration,

$$B = fn(h_{N+1}). \qquad (2.23)$$

In the case of no flow in the lower layer, $B \equiv$ constant, which is a special case of the above. Thus, *if there is motion in all layers, no formula of form (2.17) is, in general, possible; the assumption of a resting layer is important to the success of any integral method.*

It is tempting to circumvent this difficulty by adding in a weighted term involving the ($N + 1$)th layer designed to cancel the outstanding terms. However, the denominator, instead of being a baroclinic mass deficit, which involves a pycnocline displacement, becomes a weighted barotropic deficit, which involves a surface displacement. This yields propagation speeds beyond the range of validity of the method unless very large surface displacements are assumed (e.g., Mied and Lindemann's, 1979, barotropic eddy 5, which dispersed and did not propagate, had a surface displacement an order of magnitude larger than for their baroclinic eddies).

### c. Propagation directions and speeds

Nof (1985) has evaluated propagation velocities for special cases when $N = 2$, and when either $H_1$ or $H_2$ vanishes, i.e., there is a trapped lens of fluid within the eddy. Equation (2.17) or its equivalents is considerably more general, as it allows any combination of layers to be trapped or continuous as required. Since Nof (1985) has shown cases of both east and west propagation, there is little point in extending those calculations, though specific solutions with three or more layers can trivially be written down.

Instead, we can deduce a simple rule to determine the direction of propagation. Provided merely that each pressure gradient $p_{nr}$ is one-signed, it follows that the denominator in (2.17) has sign

$$\begin{aligned} \mathrm{sgn}(\sum_{n=1}^{N} (h_n - H_n)) &= -\mathrm{sgn}(\sum_{n=1}^{N} h_{nr}) \\ &= -\mathrm{sgn} p_{Nr} = -\mathrm{sgn}(V_N), \end{aligned} \qquad (2.24)$$

from (2.18), since the eddies have nonnegative absolute vorticity. Thus (2.17) gives

$$\mathrm{sgn}(c) = -\mathrm{sgn}(V_N) \cdot \mathrm{sgn}\left( \int r^2 dr \sum_{n=1}^{N} h_n V_n \right).$$

This yields the simple rule that the eddy moves *westward* unless (put a little loosely) the depth-averaged azimuthal velocity is in the opposite direction from the flow in the lowest active layer.

This result helps to explain Nof's findings (his Table 1) concerning eastward flow for strong cyclones above anticyclones. Whether eddies with such a strong vertical shear would be baroclinically stable involves lengthy calculations which are not attempted here (Ikeda, 1981, finds two active quasi-geostrophic layers to be unstable, for example).

Two other points may be noted. First, by making the denominator of (2.17) small (i.e., flattening the eddy bottom), the eddy can be made to move as fast as desired, subject to the restraints imposed by the approximations as Nof (1985) demonstrates. Second, as Davey and Killworth (1984) point out, (2.17) is only of use if the eddy does move as a coherent entity, and this involves sufficient nonlinearity in the dynamics. If motions are weak enough to be geostrophic, for example, then analysis given in the Appendix shows there to be no steadily propagating solution.

## 3. Propagation speeds for subsurface continuously stratified fluids

### *a. Solution*

We consider the equations of motion using $\rho$ as a vertical coordinate, and write $X = x$, $Y = y$, $T = t$. Capital letters will be used in $\rho$ space, with lower case retained for physical space.

$$z_{\rho T} + (uz_\rho)_x + (vz_\rho)_y = 0 \tag{3.1}$$

$$B_\rho = gz \tag{3.2}$$

$$\frac{Du}{DT} - fv + \frac{B_x}{\rho_0} = 0 \tag{3.3}$$

$$\frac{Dv}{DT} + fu + \frac{B_y}{\rho_0} = 0, \tag{3.4}$$

where $B$ remains the Bernoulli function

$$B = p + \rho g z, \tag{3.5}$$

and

$$\frac{D}{DT} = \frac{\partial}{\partial T} + u\frac{\partial}{\partial X} + v\frac{\partial}{\partial Y} \tag{3.6}$$

is the horizontal advective operator on a density surface.

The physical situation considered is shown in Fig. 1. The fluid lies between a rigid lid (although a free surface can trivially be included) and a rigid floor, each of which is assumed to be a surface of constant density, for reasons discussed below. Warm core rings with a surface and floor density expression (Joyce, 1984) are not directly amenable to the analysis here. (Note that it is not closed density contours that create the problem, but that the surface be isopycnal.) While a strong restriction, it is stressed that such a restriction also applies automatically to the layered models in the literature.

We now choose a bounding contour, $\rho = \rho_b$, to delineate the "bottom" of the eddy; restrictions on the choice of contour will be given below. The surface lies on $\rho = \rho_t$. Then the volume anomaly

$$Q = \int (z - z_\infty)_\rho dXdYd\rho \tag{3.7}$$

is defined, where $z_\infty(X, Y, \rho)$ is the depth of the density contour far from the eddy. Thus

$$Q = \int (z - z_\infty)dXdY, \quad \rho = \rho_b \tag{3.8}$$

corresponds exactly with the layered case in section 2. The centroid $(\bar{X}, \bar{Y})$ is defined by

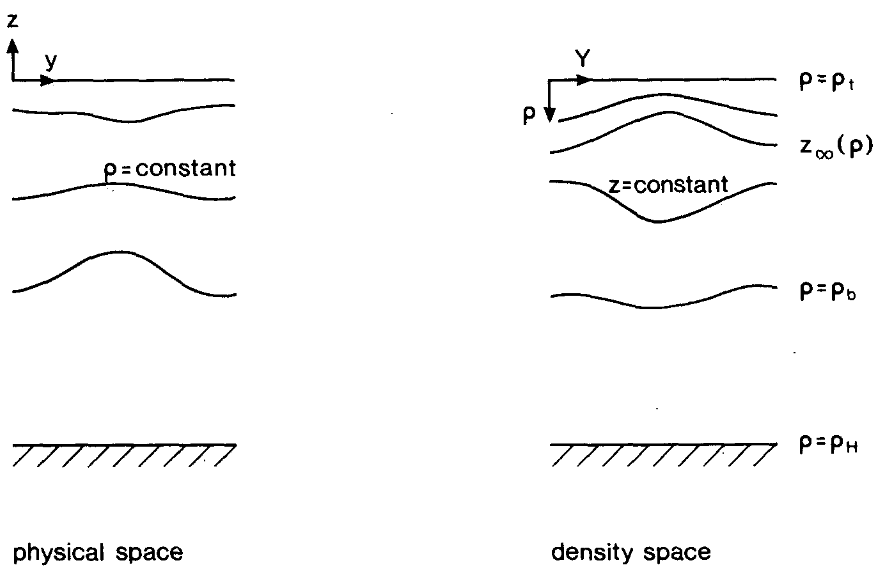


FIG. 1. The two coordinate systems used for continuously stratified eddies. $z_\infty(\rho)$ is the depth of the surface of density $\rho$ far from the eddy.

$$Q\bar{\mathbf{X}} = \int (z - z_\infty)_\rho \mathbf{X} dX dY d\rho. \quad (3.9)$$

We note that

$$\frac{d}{dT}\int z_\rho \phi dX dY d\rho = \int z_\rho \frac{D\phi}{Dt} dX dY d\rho, \quad (3.10)$$

so that

$$Q_T = 0 \quad (3.11)$$

$$Q\bar{\mathbf{X}}_T = \int z_\rho \mathbf{u} dX dY d\rho. \quad (3.12)$$

Seeking again $\bar{Y}_T$ equal to zero, and further differentiating (3.12) gives

$$0 = Q\bar{Y}_{TT} = \int z_\rho \frac{Dv}{Dt} dX dY d\rho$$

$$= -f_0 Q\bar{X}_T - \beta \int Y u z_\rho dX dY d\rho$$

$$- \frac{1}{\rho_0}\int B_y z_\rho dX dY d\rho. \quad (3.13)$$

The first two terms correspond precisely to those for the layered case. The last (pressure) term needs a little care. We have

$$\int B_Y z_\rho dX dY d\rho = -\int B z_{\rho Y} dX dY d\rho \quad \text{(by parts, } Y\text{)}$$

$$= -\int [B z_Y]_{\rho=\rho_t}^{\rho=\rho_b} dX dY$$

$$+ \int g z z_Y dX dY d\rho \quad (3.14)$$

and the last term is a $Y$-derivative and so vanishes. Thus for the pressure contribution

$$-\int [B z_Y]_{\rho=\rho_t}^{\rho=\rho_b} dX dy$$

to vanish, either

$$z_y = 0, \qquad \rho = \rho_b \quad \text{or} \quad \rho = \rho_t \quad (3.15\text{a})$$

or

$$B = fn(z), \quad \rho = \rho_t \quad \text{or} \quad \rho = \rho_t \quad (3.15\text{b})$$

is *required*. The existence of such a surface is a *necessity* for the theory; there is no reason such a surface should exist in practice. At the upper surface $\rho = \rho_t$, $z = 0$, so that (3.15a) is satisfied there. However, we cannot require (3.15a) to hold on $\rho = \rho_b$, since this would yield $Q = 0$ and a breakdown of the argument. We are thus forced to assume (3.15b) holds on $\rho = \rho_b$.

A special case, corresponding to the layered case, is

$$B = \text{constant}, \quad \rho = \rho_b. \quad (3.16)$$

Provided (3.15b) is satisfied, (3.13) becomes

$$c = \bar{X}_T = \frac{-\beta}{f_0 Q}\int Y u z_\rho dX dY d\rho \quad (3.17)$$

or

$$c = \frac{+\beta}{f_0 Q}\int r^2 dr V z_\rho dX dY d\rho = \frac{-\beta}{f_0 Q}\int r^2 dr V dx dy dz \quad (3.18)$$

after reexpression, first into polar coordinates, and second into physical space. Alternatively, in coordinates moving eastward at speed $c$, a streamfunction $\phi$ exists from (3.1) such that

$$u z_\rho = -\psi_y, \quad v z_\rho = \psi_x \quad (3.19)$$

so that

$$c = \frac{\beta}{f_0 Q}\int \psi dX dY d\rho. \quad (3.20)$$

The solution (3.18) or its reexpressions is the aim of this note, but its derivation raises some interesting issues that are now discussed.

### *b. The water below $\rho = \rho_b$*

It is traditional to ignore water below the eddy. Yet either this water is motionless, or it too must propagate at the eddy speed $c$. Flierl (1984) has shown, for example, that a finite lower layer supposedly at rest leaks Rossby waves away from the eddy. Thus the lower water merits closer attention.

If the water is at rest, the horizontal pressure gradient must vanish. A little consideration shows that this can only be so if

$$B = \text{constant}, \quad \rho = \rho_b, \quad \text{and}$$

$$\rho \equiv \rho_b \quad \text{below the eddy,} \quad (3.21)$$

so that the fluid is unstratified below the eddy. For eddies like the "Meddy," (3.21) would not be a very good model, but under some circumstances it would be quite useful. The alternative is that the lower water also moves at speed $c = \bar{X}_T$. There is now no reason why a similar argument cannot be applied to this water:

$$\hat{Q} = \int_{\rho_b}^{\rho_H} (z - z_\infty)_\rho dX dY d\rho \equiv -Q, \quad (3.22)$$

where the integral is from $\rho_b$ to $\rho_H$, the density at the bottom. Thus we eventually obtain

$$c' = \bar{X}'_T = \frac{-\beta}{f_0 Q}\int_{\rho_b}^{\rho_H} r^2 dr V z_\rho dX dY d\rho \quad (3.23)$$

as a second expression for $c$, which must be equal for a steady state, so that

$$0 = \int_{\rho_t}^{\rho_H} d\rho \int dX dY r^2 dr V z_\rho = \int dx dy dz V, \quad (3.24)$$

converting to physical space. This can be thought of as a different proof of the zero angular momentum theorem of Flierl et al. (1983), and provides strong bounds on which density surfaces are permissible for (3.18).

*c. Directions of propagation*

Some weak restrictions and "rules of thumb" can be placed on the direction of propagation. Suppose first that $B$ = constant, $\rho = \rho_b$. Since

$$f_0 V + \frac{V^2}{r} = \frac{B_r}{\rho_0} \quad (3.25)$$

is the cyclostrophic equation,

$$\mathrm{sgn}(V) = \mathrm{sgn}(B_r). \quad (3.26)$$

If $V$ is uniform in sign above $\rho = \rho_b$, and using

$$B = g \int_{\rho b}^{\rho} z d\rho + \text{constant}, \quad (3.27)$$

we find that

$$\mathrm{sgn}(V) = \mathrm{sgn}(Q), \quad (3.28)$$

or

$$c < 0, \quad (3.29)$$

i.e., westward propagation. Hence flows uniform in azimuthal direction above a surface of no motion propagate westward. Eastward motion requires $V$ to change sign above $\rho = \rho_b$.

The requirement of a surface of no motion can be relaxed for the simple case of solid body rotation in a cylinder of radius $a$ (Killworth, 1983; Nof, 1985). Subject to geometrical restrictions, let

$$V = \mu(\rho) f_0 r \quad (3.30)$$

$$B = \alpha(\rho) r, \quad \alpha = \rho_0 f_0^2 \mu(1 + \mu) \quad (3.31)$$

$$z_{\infty\rho} = \gamma < 0, \quad \gamma = \text{constant}, \quad (3.32)$$

and $\mu$ be linear in $\rho$:

$$\mu = \mu_t + (\rho - \rho_t)\hat{\mu}, \ \mu_t, \ \hat{\mu} \text{ constants}. \quad (3.33)$$

Then (3.18) yields (with no assumptions at $\rho = \rho_b$)

$$\mathrm{sgn}(c) = \mathrm{sgn}(\hat{\mu})\ \mathrm{sgn}\left[\mu_t + \frac{1}{2}\hat{\mu}(\rho_b - \rho_t)\right] \quad (3.34)$$

after some algebra, while (3.24) implies

$$\mu_t + \frac{1}{2}\hat{\mu}(\rho_H - \rho_t) = 0. \quad (3.35)$$

Since $\rho_H > \rho_t$, the signs of $\mu_t$, $\hat{\mu}$ are opposite, and $\mathrm{sgn}(\mu_t) = \mathrm{sgn}[\mu_t + \frac{1}{2}\hat{\mu}(\rho_b - \rho_t)]$. Thus

$$\mathrm{sgn}(c) = \mathrm{sgn}(\mu_t)\ \mathrm{sgn}(\hat{\mu}) < 0 \quad (3.36)$$

so that the eddy again moves westward.

Indeed, a little experimentation shows that for an eddy to move east implies strong flows in both cyclonic and anticyclonic directions above (and below) $\rho = \rho_b$. It is again unclear how baroclinically stable such an eddy would be.

## 4. Approximate solution for the Armi and Zenk salt lens

We apply (3.18) here to lens 3, described by Armi and Zenk (1984), whose paper gives both dynamic height data relative to 1900 db and background stratification, which suffice for the calculation. An approximate value for $c$ is easily estimated. Assuming an isopycnal displacement of 50 m, an azimuthal velocity of 10 cm$^{-1}$, a radius of 40–50 km and a vertical depth of 1 km, (3.18) gives $c \approx 3$ cm s$^{-1}$ westward. However, the structure of the flow is fairly crucial to the calculation, as we shall now see. Guided by Armi and Zenk's Fig. 10, a quadratic in $r$ is fitted for $D$, the dynamic height, and $\rho$, the density, inside a cylinder of radius $a$ = 50 km:

$$D = -g \int_{-1900\,\mathrm{m}}^{z} \rho dz \approx D_0(z)\left(\frac{r^2}{a^2} - 1\right) + D_\infty(z) \quad (4.1)$$

$$\rho = \rho_\infty(z) + \gamma(z)\left(\frac{r^2}{a^2} - 1\right). \quad (4.2)$$

Since density excursions are small (Armi and Zenk, 1984; Fig. 1) we can move freely between $z$ and $\rho$ coordinates, in particular

$$z - z_\infty|_{\rho=\rho b} \approx \frac{-\gamma(z_{b\infty})(r^2/a^2 - 1)}{\rho_{\infty z}(z_{b\infty})}, \quad (4.3)$$

so that

$$Q = \int_0^a r dr (z_b - z_{b\infty}) = \frac{\gamma(z_{b\infty}) a^2}{4\rho_{\infty z}(z_{b\infty})}. \quad (4.4)$$

Now

$$V = f_0 r \mu(\rho) \quad (4.5)$$

once more, so that

$$\int_{zb}^{0}\int_0^a r^2 dr V dz \approx \frac{f_0 a^4}{4}\int_{zb\infty}^{0} \mu dz, \quad (4.6)$$

and (3.24) gives

$$\int_{-H}^{0} \mu dz = 0 \quad (4.7)$$

for no net circulation. Cyclostrophy and the assumption of no motion at $\rho = \rho_b$ (as yet undefined) give

$$f_0^2 \mu(1 + \mu) = \frac{2p_0(z)}{a^2 \rho_0} \quad (4.8)$$

$$p_{0z} = D_{0z} \quad (4.9)$$

$$p_0 = 0, \quad z = z_{b\infty}. \quad (4.10)$$

Guessing the value of $z_{b\infty}$ (or, equivalently, $\rho_b$) gives

$$p_0 = D_0 - D(z_{b\infty}) \tag{4.11}$$

for some unknown $z_{b\infty}$, and we require $V = \mu = 0$ there. Now (4.8) implies $\mu(z)$, and (4.7) will provide a restriction which enables $z_{b\infty}$ to be found.

Armi and Zenk (1984) give fits in $D$ linear in $r$; it is straightforward to convert these to the quadratic fits required here. The fits are extended linearly from 400 db to the surface, and $D$ is assumed zero below 1900 db, lacking data (Zenk, private communication, 1985), though this is naturally questionable.

Equation (4.7) then yields a surface of no motion at about 1500 m, which gives anticyclonic motion above this level and weak cyclonic flow below it. Substitution into (3.18) then gives

$$c \approx -0.9 \text{ cm s}^{-1}, \tag{4.12}$$

which is not unreasonable.

There are many uncertainties in the calculation; however the strongest effect is caused by a *lowering* of the $\rho_b$ surface, since the denominator (4.4) depends on $\gamma(z_b)$, which decreases rapidly below 1600 m. An increase in $c$ by a factor of 2 could be achieved by choosing $z_b$ at 1800 m, for example. Other imponderables (e.g., the deep and near-surface dynamic topography) have a weak effect on the calculation, but could easily provide an offset (via a mean flow) that could advect the lens eastward. While plausible, $z_b = 1500$ m does not agree with the current meter data at 1608 m (Armi and Zenk, 1984; Fig. 14) which indicates anticyclonic flow at that depth. Whether this reflects errors in the model fit, or in the assumption of a surface of no motion, remains unclear.

## 5. Summary

This note serves two functions. First, it extends and consolidates the various approaches in the literature to the steady propagation speed for an almost radially symmetric eddy on a beta-plane. Both multilayered and continuously stratified fluids are included. Second, it draws attention to assumptions that underpin the extant theories, in particular that the eddy be bounded by a lower density surface on which the linear Bernoulli function be a function only of the depth of that surface, correct to first order in the small beta parameter. There is also a requirement (Flierl et al., 1983; Nof, 1985) that the eddy decay sufficiently rapidly away from its center (although the requirement of finite mass anomaly $Q$ is the strongest condition, needing the depth anomaly to decay faster than $r^{-2}$, exactly as in the papers cited).

At no stage has it been proved that such structures can exist, nor that they are stable. If they exist and are stable, however, then they propagate at the speeds given in the text. The dependence of the theory on the structure of the bounding density surface suggests that it may be possible for eddies arbitrarily close to those studied here to propagate steadily, although with a different dynamical balance. It merely requires the Bernoulli terms in (3.14) to be of order $\epsilon$, not order 1, at each density level for which the theory is evaluated, in such a way as to make $c$ the same over a variety of density levels. Such a model is beyond the scope of this work.

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## APPENDIX

### Geostrophic Flow

We demonstrate for the layered fluid of section 2 that no steadily propagating geostrophic solution of any shape can normally exist. Taking axes moving eastward at speed $c$ (assumed small, as is the beta-effect) and measuring velocities relative to the frame, we have

$$(f_0 + \beta y)v_n = \frac{p_{nx}}{\rho_0} \tag{A1}$$

$$(f_0 + \beta y)(u_n + c) = \frac{-p_{ny}}{\rho_0} \tag{A2}$$

$$\nabla \cdot (\mathbf{u}_n h_n) = 0 \tag{A3}$$

together with (2.1) to define the $p_n$. We write

$$h_n = h_n{}^0 + \epsilon h_n{}^1 \tag{A4}$$

and similarly for other variables, where $\epsilon$ is the small parameter introduced in section 2. To leading order,

$$v_n{}^0 = \frac{p_{nx}{}^0}{\rho_0}, \quad u_n{}^0 = \frac{-p_{ny}{}^0}{\rho_0} \tag{A5}$$

$$\nabla \cdot (\mathbf{u}_n{}^0 h_n{}^0) = 0 \tag{A6}$$

are assumed satisfied; (A6) is nontrivial for more than one layer, although satisfied for radially symmetric flows. At $O(\epsilon)$, we find

$$v_n{}^1 = \frac{p_{nx}{}^1}{f_0\rho_0} - \frac{\beta y}{f_0} v_n{}^0 \tag{A7}$$

$$u_n{}^1 = \frac{-p_{ny}{}^1}{f_0\rho_0} - \frac{\beta y}{f_0} u_n{}^0 - c \tag{A8}$$

$$\nabla \cdot (\mathbf{u}_n{}^1 h_n{}^0 + \mathbf{u}_n{}^0 h_n{}^1) = 0. \tag{A9}$$

Solving for $v_n{}^1$, $u_n{}^1$ in terms of $p_n{}^1$ from (A7), (A8), and substituting into (A9), followed by a summation over the layers to eliminate pressure gradients gives

$$\sum_{n=1}^{N} \left( c h_{nx}{}^0 + \frac{\beta}{f_0} h_n{}^0 v_n{}^0 \right) = 0, \tag{A10}$$

or

$$c = \frac{-\dfrac{\beta}{f_0} \sum_{n=1}^{N} h_n^0 v_n^0}{\sum_{n=1}^{N} h_{nx}^0}. \qquad \text{(A11)}$$

The right-hand side of (A11) is, except in very special circumstances, a function of $x$ and $y$ and so cannot be constant. Thus *weak* flows cannot propagate as a coherent eddy, and can only disperse like Rossby waves.

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