Comment on "The Scattering of a Continental Shelf Wave by a Long Thin Barrier Lying Parallel to the Coast"*

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1. Introduction

The analytical solution presented by Hsieh and Buchwald (1985, hereafter referred to as HB) for the scattering of a low-frequency barotropic continental shelf wave by a long thin island located parallel to the coast (Fig. 1) does not conserve energy flux. Hsieh and Buchwald attributed this result to the elimination of short reflected shelf waves through making the long-wave (or alongshore geostrophy) approximation. In actuality, the failure to conserve energy flux will be shown to be the result of an incorrect representation of the degenerate Kelvin wave under a rigid lid. Therefore, HB's principal conclusion that, in the channel between the island and the coast, much of the incident shelf wave energy is transferred to a zero (or Kelvin) mode, is invalid.

In this comment, we reexamine the scattering problem of HB. First, we show that energy flux must be conserved (section 2). Then we discuss the form of the Kelvin wave under a rigid lid (section 3) and present a corrected solution to the scattering problem (section 4). In section 5, it is demonstrated that the corrected solution conserves energy flux and that the other conclusions of HB remain qualitatively correct; namely, that there may be a substantial transfer of energy to modes lower than the incident wave seaward of the island, and that a mode different from the incident mode may dominate downstream of the barrier.

2. Energy flux conservation

In the absence of dissipation, any consistent approximation to the most general equations of motion must conserve energy. This fact is not altered by the omission of any particular dynamics within the approximation procedure. For example, consider an object which is moved impulsively in a real fluid. The motion generates acoustic waves as well as other fluid motions. For typical geophysical scales, the energy in the acoustic waves is small relative to that in the other motions. To model the geophysical scale flow, the fluid may be assumed to be incompressible, thus eliminating the acoustic waves. Nevertheless, the remaining simplified equations of motion must still conserve energy. Similarly, the barotropic long-wave equations neglect backscattered shelf waves (as well as other waves such as edge and internal waves), yet they must still conserve energy.

To demonstrate, consider the streamfunction equation derived from the nondimensional barotropic long-wave momentum equations [HB Eq. (2.1) and (2.2)] without assuming a wavelike structure:

$$\left(\frac{1}{h}\psi_y\right)_{yt} - \frac{h_y}{h^2}\psi_x = 0 \qquad (1)$$

where subscripts denote partial differentiation and the depth h is a function of y only. After multiplying (1) by ψ, manipulating terms, and integrating over some area R, we can express the result as

$$E_t = \int\int_R \left(\frac{1}{2h}\psi_y{}^2\right)_t dR = -\int\int_R \nabla\cdot\mathbf{S}dR \qquad (2)$$

where E is the total kinetic energy in R,

$$\mathbf{S} = \mathbf{i}\left(\frac{h_y}{2h^2}\psi^2\right) - \mathbf{j}\left(\frac{1}{h}\psi\psi_{yt}\right),$$

and $\mathbf{i}$, $\mathbf{j}$ are unit vectors in x, y. Using the divergence theorem, (2) becomes

$$E_t = -\oint_c \mathbf{S}\cdot\mathbf{n}dl \qquad (3)$$

where c represents a closed path around R. Taking R to be a rectangle bounded by $y = 0$, $y = L$, $x = x_1$, and $x = x_2$ and employing the boundary conditions used by HB ($\psi = 0$ at the coast and $\psi_y = 0$ at the offshore boundary), (3) may be written as

$$E_t = -\frac{1}{2}\int_0^1 \frac{h_y}{h^2}\psi^2 dy\bigg|_{x=x_1} + \frac{1}{2}\int_0^1 \frac{h_y}{h^2}\psi^2 dy\bigg|_{x=x_2}. \qquad (4)$$

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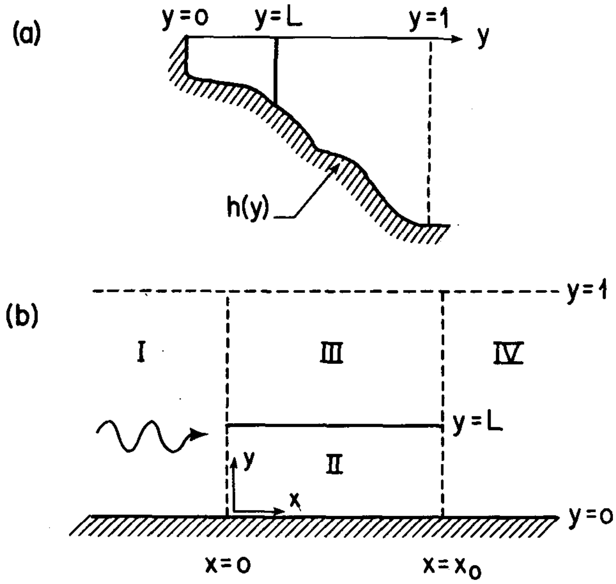

FIG. 1. Problem geometry. (a) Side view; (b) plan view. The shelf has a monotonically increasing depth profile $h(y)$. An infinitely thin barrier is located at $y = L$, $0 < x < x_0$. The regions I through IV are labeled in (b) with the incident wave shown approaching from region I.

For the periodic problem under consideration, the average of each side of (4) over a wave period must be zero. Thus, the incident wave energy flux (negative of the first term on the right) must equal the transmitted energy flux (second term on the right). No energy flux occurs through the coast or across the offshore boundary to the deep ocean.

For a wavelike streamfunction [i.e., $\psi = \phi(y)e^{i(kx-\omega t)}$], each integral in (4) reduces to the orthogonality statement for $\phi(y)$ [Eq. (2.10) in HB]. Thus, energy flux separates by modes (as proven originally by Huthnance, 1975), and the energy flux in each mode (using HB's normalization) is equal to one-half of the squared amplitude of that mode. Thus, a correct solution to the exponential topography example considered by HB must obey (in their notation)

$$\frac{1}{2}A^2 = \sum_{n=1}^{\infty} \frac{1}{2} D_n^2. \tag{5}$$

That is, the total energy flux in region IV must equal the incident energy flux. As will be shown in section 5, the solution presented by HB does not satisfy (5) and, therefore, must be incorrect. Next we investigate the cause of this discrepancy.

3. The Kelvin wave under a rigid lid

Hsieh and Buchwald assert that the rigid lid reduces the barotropic Kelvin wave to a spatially uniform, time periodic, *velocity* fluctuation [their Eq. (2.13)] with a nonzero energy density. To examine this assertion, we consider the general form of a barotropic Kelvin wave (i.e., making neither the rigid-lid nor the long-wave approximation) propagating in x along a straight coast at $y = 0$ with y increasing offshore in an ocean of constant depth. The solution may be written as

$$\begin{aligned} p &= \hat{p}e^{-\epsilon y}e^{i\omega(\epsilon x - t)} \\ u &= \hat{p}\epsilon e^{-\epsilon y}e^{i\omega(\epsilon x - t)} \\ v &= 0 \\ \epsilon^2 &= f^2\lambda^2/gH \end{aligned} \tag{6}$$

where $\hat{p}$ is the nondimensional Kelvin wave amplitude, ω the wave frequency scaled by the Coriolis parameter f, λ a length used to scale x and y, g the gravitational acceleration and H the (constant) ocean depth. Note that for this solution the energy density [as defined in HB Eq. (3.11)] is of order ϵ^2. Under a rigid lid, $\epsilon^2 \to 0$ and (6) reduces to a spatially uniform, time periodic, *pressure* fluctuation with no associated currents and a vanishing energy density.

The question of whether this "wave" can propagate energy can be resolved by first computing the energy flux in the divergent Kelvin wave (6) and then taking the rigid-lid limit. The energy flux is the time-averaged product of pressure and alongshelf velocity integrated over a plane perpendicular to the coast:

$$F = \int_0^{y_{\max}} h\overline{pu}dy. \tag{7}$$

Using (6) in (7) and scaling the depth by H yields

$$F = \frac{1}{4}\hat{p}^2(1 - e^{-2\epsilon y_{\max}}). \tag{8}$$

In a semi-infinite ocean ($y_{\max} \to \infty$) the energy flux is $F = H\hat{p}^2/4f$. This result is independent of ϵ and will therefore remain unchanged in the limit of a rigid lid. In a region of finite width $y_{\max}$, such as HB's region II between the island and the coast, the limit of $\epsilon \to 0$ results in zero energy flux, so the Kelvin mode under a rigid lid is unable to propagate energy. The degenerate Kelvin wave can only propagate energy in a semi-infinite ocean where the increasing offshore decay scale overcomes the effect of vanishingly small alongshelf currents. Therefore, the zero mode introduced by HB is *not* the Kelvin mode under a rigid lid.

4. Corrected solution

Since there are no currents associated with the degenerate Kelvin mode under a rigid lid (whether in a channel or in a semi-infinite ocean) there will be no signal in the streamfunction ψ. The question remains as to what the zero mode introduced by HB represents. This may be addressed by examining the signature of this zero mode in the pressure field. The zero mode has the form [HB Eq. (2.12)]

$$\phi_0 = C_1 \int_0^y hdy' + C_2 \tag{9}$$

where C_1 and C_2 are constants. The velocities associated with this mode, in the nondimensional variable set of HB, are

$$u = \frac{\psi_y}{h} = C_1 e^{-i\omega t} \tag{10a}$$

$$v = \frac{\psi_x}{h} = 0. \tag{10b}$$

Substituting (10) into the momentum equations [HB Eq. (2.1)] gives

$$p_x = i\omega C_1 e^{-i\omega t} \tag{11a}$$

$$p_y = -C_1 e^{-i\omega t}. \tag{11b}$$

Integrating (11) presents a dilemma. If p is assumed to have the wavelike form $p = p_0(y)e^{i(kx-\omega t)}$ but with vanishing alongshelf dependence (i.e., $k \to 0$) as assumed by HB, (11a) can only be satisfied if $C_1 = 0$ or $\omega = 0$ (i.e., a steady current). If, instead, p is allowed an arbitrary alongshelf variation, i.e., $p = p_0(x, y)e^{-i\omega t}$, then

$$\int p_{0_x} dx = i\omega C_1 x + (\text{a function of } y). \tag{12}$$

Integrating (11b) and comparing to (12) gives

$$p_0 = C_1(i\omega x - y) + C_3. \tag{13}$$

The solution (13) might be thought of as the $O(\epsilon)$ expansion of the divergent Kelvin wave; i.e., the first two terms of the Taylor series expansion of (6) for small ϵ. If this were so, then C_1 in (13) would equal $\hat{p}\epsilon$ and the zero mode contribution would vanish when ϵ goes to zero. However, inspection of HB's Eq. (4.12) shows that their C_1 is proportional to B_0 which is assumed $\neq 0$ and therefore $O(1)$. This contradiction can only be resolved by concluding that the zero mode of HB is spurious and should have zero amplitude in their solution ($C_1 = C_2 = 0$).

Apart from the inclusion of the zero mode, the solution procedure of HB may be followed. The streamfunction in each of the four regions (see Fig. 1) is expressed as a sum over the set of local shelf wave eigenmodes with unknown amplitudes. The amplitude of each mode is determined by matching the streamfunction at cross-shelf sections at the beginning ($x = 0$) and end ($x = x_0$) of the island. Since the Kelvin mode has no signal in the streamfunction, it is straightforward to proceed through HB's analysis omitting the zero mode ϕ_0 by setting $B_0 = 0$ to obtain the corrected solution. The resulting new expressions for the scattered wave amplitudes corresponding to HB's equations (3.6), (3.8), and (3.10) are, respectively,

$$B_n = \int_0^L \frac{h'}{h^2} A\phi_m{}^{\mathrm{I}}(y)\phi_n{}^{\mathrm{II}}(y)dy \tag{14}$$

$$C_n = \int_L^1 \frac{h'}{h^2} A\phi_m{}^{\mathrm{I}}(y)\phi_n{}^{\mathrm{III}}(y)dy \tag{15}$$

$$D_n \exp[i(k_n{}^{\mathrm{IV}}x_0 + \delta_n)] = \int_0^L \frac{h'}{h^2} \sum_{j=1}^{\infty} B_j\phi_j{}^{\mathrm{II}} \exp(ik_j{}^{\mathrm{II}}x_0)\phi_n{}^{\mathrm{IV}}dy + \int_L^1 \frac{h'}{h^2} \sum_{j=1}^{\infty} C_j\phi_j{}^{\mathrm{III}} \exp(ik_j{}^{\mathrm{III}}x_0)\phi_n{}^{\mathrm{IV}}dy. \tag{16}$$

Over exponential topography, $h = h_0e^{2by}$, evaluation of the above expressions yields

$$B_n = \frac{2bL}{h_0} AA_m{}^{\mathrm{I}}A_n{}^{\mathrm{II}} \sin(\gamma_m{}^{\mathrm{I}}L) \times (-1)^n n\pi\left[\frac{1}{(\gamma_m{}^{\mathrm{I}}L)^2 - (n\pi)^2}\right] \tag{17}$$

$$C_n = \frac{-2b}{h_0} AA_m{}^{\mathrm{I}}A_n{}^{\mathrm{III}}\left[\frac{\gamma_n{}^{\mathrm{III}} \sin(\gamma_m{}^{\mathrm{I}}L)}{\gamma_m{}^{\mathrm{I}2} - \gamma_n{}^{\mathrm{III}2}}\right] \tag{18}$$

$$D_n \exp[i(k_n{}^{\mathrm{IV}}x_0 + \delta_n)] = \sum_{j=1}^{\infty} B_j \exp(ik_j{}^{\mathrm{II}}x_0)Q_{nj} + \sum_{j=1}^{\infty} C_j \exp(ik_j{}^{\mathrm{III}}x_0)R_{nj} \tag{19}$$

where the orthonormalization factors A_n, eigenvalues γ_n, and integrals Q_{nj} and R_{nj}, are exactly as presented by HB. [Note that the first two terms in the square bracket of HB's Eq. (4.14) actually cancel and have therefore been omitted in (18). Similarly, the first two terms in the curly bracket of the expression for R_{nj} cancel.]

Before proceeding to the results, some comments about the new solution are in order. The incident wave generally has a nonzero value of ψ at the tip of the island ($x = 0$, $y = L$) equal to $A\phi_m{}^{\mathrm{I}}(L)$. However, the eigenmodes which are summed in regions II and III to match this incident wave all equal zero at the island. This creates the disturbing illusion that the mass flux in region II is zero and does not match that incident from region I because the streamfunction is discontinuous at $y = L$. This result occurs because simply evaluating ψ^{II} at $y = L$ misses the true character of the singularity there. The reader can verify that for arbitrarily small δ, evaluation of the series for ψ^{II} at $y = L - \delta$ indeed converges to $A\phi_m{}^{\mathrm{I}}(L)$ showing that the transport is in fact continuous. [This behavior is well known in Fourier analysis (see, for instance, examples 39 and 40 of Lighthill, 1958).] Also, the B_n in (17) behave like $(-1)^n/n$ as $n \to \infty$, so that a velocity series computed by a term by term differentiation of ψ appears to diverge. This situation is not a problem if one recognizes that the apparent divergence is due solely to the aforementioned singularity which is embedded in the series solution. This simply means that the order of summation and differentiation cannot be interchanged, but says nothing about the validity of the solution for ψ.

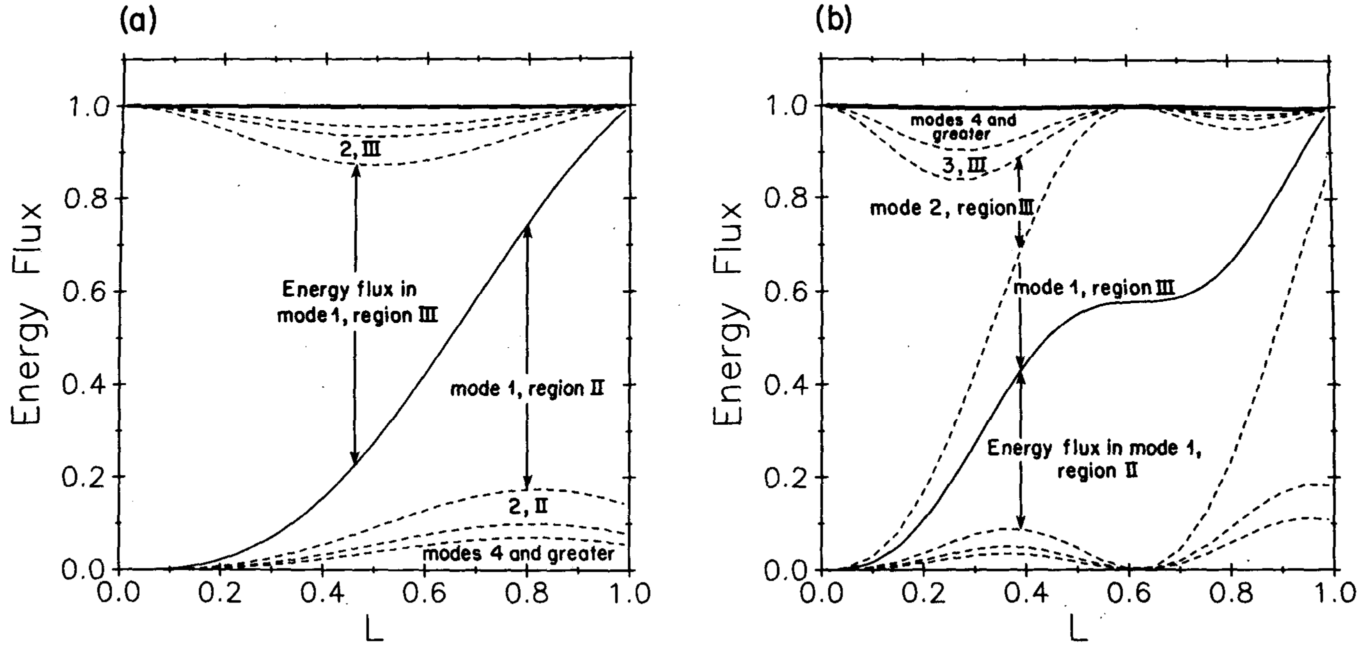

FIG. 2. Energy flux partitioning in regions II and III vs barrier location L ($h = h_0e^{4y}$, $\omega x_0 = 1$). Thin solid line is the energy flux in region II. Heavy solid line is the total energy flux in regions II and III. Dashed lines divide energy between the modes in each region. (a) Mode 1 incident; (b) mode 2 incident.

5. Results and discussion

To present their results, HB elected to plot scattered mode energy densities, choosing a normalization procedure which obscured their failure to conserve energy. To show how much of the incident energy is transported alongshelf by each scattered wave and to emphasize that the corrected solution conserves energy for all cases, the results shown here are presented in terms of energy fluxes normalized by the energy flux of the incident wave. As shown in section 2, the energy flux for each mode is given by one-half of its squared amplitude. Using the typical value $b = 2$ chosen by HB, the features of the solution may be illustrated by varying the remaining free parameters: L, the location of the barrier relative to the coast; ωx_0, the scaled length of the barrier; and m, the incident wave mode number. For comparison, some of HB's results are replotted in this manner.

Upon encountering the barrier, the incident energy flux is split between regions II and III. How this partitioning of energy varies as L is varied is shown in Fig. 2a for an incident wave of mode $m = 1$. The region below the thin solid line is the energy flux in region II while that above the line is the energy flux in region III. The heavy solid line is the total energy flux in both regions obtained by summing the energy in the first 100 computed modes. The total energy is clearly unity (i.e., energy is conserved) for all values of L. The dotted lines show the partitioning of the energy between the various modes in each region. For small L (barrier near the coast), little energy enters region II; most of the energy is transferred to mode 1 in region III. As L increases, more of the energy enters region II and less enters region III with modes higher than the incident now being excited. When $L = 1$ (barrier at the shelf edge), there is no energy in region III but there is still

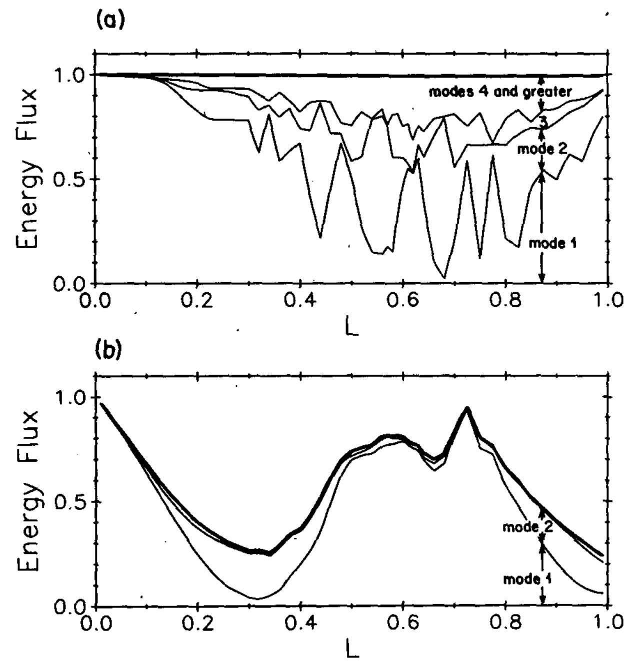

FIG. 3. Energy flux partitioning in region IV vs barrier location L ($h = h_0e^{4y}$, $m = 1$, $\omega x_0 = 1$). Heavy line is the total energy flux. Thin lines divide energy between the region IV modes. (a) Corrected solution; (b) Hsieh and Buchwald (1985) solution.

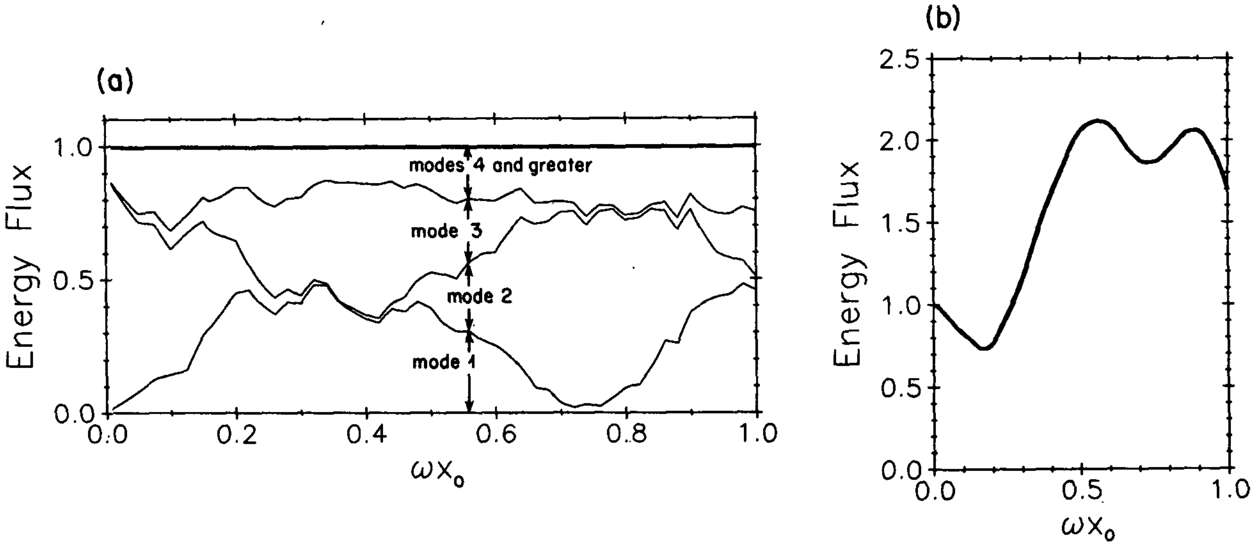

FIG. 4. Energy flux in region IV vs barrier length ωx_0 ($h = h_0 e^{4y}$, $m = 2$, $L = 0.45$). Heavy line is the total energy flux. (a) Corrected solution. Thin lines divide energy between region IV modes. (b) Hsieh and Buchwald's total energy flux which is greater than one for $\omega x_0 \gtrsim 0.3$; i.e., there is more energy in region IV than was incident.

some scattering in region II as a result of what is effectively a change in boundary condition at the shelf edge. It is the case here, as in a similar scattering problem (Wilkin and Chapman, 1987), that in general the mode most readily excited is that which best fits the cross-shelf structure of the incident wave. This is apparent in the mode 2 incident case (Fig. 2b) where for L near 0.6 ($x = 0.6$ being the zero crossing in the incident mode cross-shelf structure), mode 1 is excited in both regions II and III with little energy being transferred to any other modes.

The solution downstream of the barrier depends on both L and the relative length of the barrier ωx_0. For a barrier of scaled length $\omega x_0 = 1$, the partitioning of energy flux between the various modes in region IV is shown in Fig. 3a as a function of L. Again, the total transmitted energy flux (the heavy solid line) is always unity. This may be compared to Fig. 3b which presents HB's solution for the same case. The majority of their incident energy is missing.

In Fig. 4a the barrier length is varied while the other parameters are held constant at $L = 0.45$ and $m = 2$. An interesting result is that for ωx_0 near 0.4, the incident mode 2 wave is almost totally eliminated, whereas for a longer ($\omega x_0 \approx 0.7$) or shorter ($\omega x_0 \lesssim 0.1$) barrier the island appears to be a weaker scatterer. This results from the widely differing phase speeds of the scattered modes in regions II and III which produce a strong alongshelf modulation in the flow field. The flow pattern to which the set of scattered waves in region IV must match, therefore, depends on where the barrier terminates. Hsieh and Buchwald obtained a similar qualitative result, namely, that for certain barrier lengths the incident mode may vanish downstream. That this was a fortuitous outcome is evident from Fig. 4b which shows the total energy flux in region IV computed using their solution. For $\omega x_0 \gtrsim 0.3$, there is *more* energy downstream of the island than was incident, showing clearly that the details of their analysis are in error.

Finally, two points should be emphasized:

1) In the absence of any dissipative mechanism, energy flux conservation is a *necessary* condition for obtaining a sensible solution to wave scattering problems and should be used to validate such solutions.

2) Nothing can be learned here about barotropic Kelvin wave excitation due to the scattering of shelf waves by an offshore island because in the present formulation the zero (Kelvin) mode is absent from the solution. Examination of the scattering of energy into the Kelvin mode requires the inclusion of a free surface (e.g., Brink, 1986).

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