Reply

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1. Introduction

The comment by Wilkin and Chapman (WC) on our paper (HB) raises a number of fundamental questions regarding the nature of the "zero-divergence" approximation.

The standard nondimensional equations of motion for barotropic shallow water are

$$u_t - v = -\eta_x, \tag{1a}$$

$$v_t + u = -\eta_y, \tag{1b}$$

$$(hu)_x + (hv)_y + \epsilon^2\eta_t = 0, \tag{1c}$$

where the standardization factors are f^{-1} for time, the shelf width Λ for the spatial variables, a typical depth h_0 for the depth, U for the horizontal velocity and the scaled pressure η is given by

$$\eta = (g/f\Lambda U)\zeta, \tag{2}$$

with ζ the surface elevation, and

$$\epsilon^2 = f^2\Lambda^2/(gh_0). \tag{3}$$

Given that g, f and h_0 are fixed, the parameter ϵ is effectively a length scale. The type of solution of (1) that is obtained depends on the choice of length scale. Without entering a detailed explanation of the asymptotic matching procedure (e.g., Van Dyke, 1975), we separate the plane into an inner and an outer region. Solutions for the inner region are obtained from (1) with $\epsilon \to 0$, and the equation corresponding to (1) in the outer region is obtained by the coordinate change $(x, y) = \epsilon^{-1}(X, Y)$. The solution as $(x, y) \to \infty$ in the inner region is then matched with appropriate solutions in the outer region as $(X, Y) \to 0$. Solutions in the inner region are obtained by using the expansion

$$(u, v, \eta) = (u_0, v_0, \eta_0) + \epsilon(u_1, v_1, \eta_1) + \cdots. \tag{4}$$

Substitution of this expansion into (1), and equating powers of ϵ yields an iterative procedure for determining (u_n, v_n, η_n). In particular, the equation for (u_0, v_0, η_0) is (1) with $\epsilon^2 = 0$, so that the equations for the zero order terms in (4) are the same as those for the "rigid-lid" case in which $\epsilon = 0$. It must be emphasized, though, that although the equations are formally the same, the conditions as $(x, y) \to \infty$ are not the same, and that, in general, the asymptotic solution of the system as $\epsilon \to 0$ is *not* the same as for $\epsilon \equiv 0$.

Thus, while the rigid-lid case $\epsilon \equiv 0$ excludes Kelvin and Poincaré waves, in a real ocean ϵ is small, but not zero, so that we must look for inner solutions which are excluded if $\epsilon \equiv 0$, but which for $0 < \epsilon \ll 1$, are local manifestations of Kelvin or Poincaré waves in the outer region; in other words, they are the limits of these waves as $(X, Y) \to 0$.

In a real ocean, given $h = 1$ and a coastline $Y = 0$, a Kelvin wave in the outer region is given by

$$\eta_k = A\exp[i\omega X - Y - i\omega t], \tag{5}$$

so that $u_k = \epsilon\eta_k$, $v_k \equiv 0$, and that 1(a)–(c) are satisfied by η_k.

As $R = (X^2 + Y^2)^{1/2} \to 0$,

$$\eta_k = A[1 + i\omega X - Y + O(R^2)]e^{-i\omega t} \tag{6}$$

$$u_k = \epsilon A[1 + O(R)]e^{-i\omega t}. \tag{7}$$

If we now consider the inner region and neglect $O(\epsilon^2)$, Eq. (1c) is satisfied by using the streamfunction ψ defined by

$$hu = \psi_y, \quad hv = -\psi_x. \tag{8}$$

The zero mode defined by HB is

$$e^{i\omega t}\psi_0 = B_0\int_0^y hdy' \Big/ \int_0^L hdy', \tag{9}$$

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for the region $0 < y < L$, where h is a function of y only and B_0 is a constant. This ensures that $\psi_0 = 0$ at $y = 0$, and $\psi_0 = B_0 e^{-i\omega t}$ at $y = L$. The corresponding velocities are

$$u_0 = A' e^{-i\omega t}, \quad v_0 \equiv 0, \tag{10}$$

where A' is a constant. As pointed out by WC [Eq. (13)], the corresponding surface elevation is given by

$$e^{i\omega t}\eta_0 = A'(i\omega x - y) + A'', \tag{11}$$

where A'' is another arbitrary constant.

Comparing (6) and (7) with (10) and (11) yields

$$A = A'\epsilon^{-1}, \quad A'' = A. \tag{12}$$

The fact that $A = O(\epsilon^{-1})$ as $\epsilon \to 0$ is not surprising, since any Kelvin wave will have large surface elevation changes compared with shelf waves.

The mistake made by WC is to assume in (WC-6) that $\hat{p}$ is O(1) as $\epsilon \to 0$. By the foregoing analysis $\hat{p}$ is $O(\epsilon^{-1})$, and then u is of the correct order. This mistake is repeated after (WC-13), where the erroneous conclusion that $\hat{p}\epsilon \to 0$ as $\epsilon \to 0$ is taken as a necessary consequence.

There remains the question of the role of the constant streamfunction

$$\psi_0' = C_2 e^{-i\omega t} \tag{13}$$

as in (WC-9). This clearly yields $u = v = 0$ everywhere and therefore makes no contribution to shelf-wave-dynamics as such. As far as the inner region is concerned, C_2 is quite arbitrary, but the constant is determined after matching with the outer solution has taken place. This has important implications in our discussion of energy propagation as will be seen presently.

2. The singularity at (0, L)

The instantaneous fluid volume flux across a line segment x = constant, $y_1 < y < y_2$ is given by

$$Q(y_2, y_1) = \int_{y_1}^{y_2} hu dy, \tag{14}$$

which by HB-(2.2) reduces to

$$Q = \psi(y_2) - \psi(y_1). \tag{15}$$

If we refer to WC Fig. 1b, the volume flux in the model proposed by WC is discontinuous cross $x = 0$. This is because as $x \to 0-$, the volume flux across $x = 0$, $0 \leqslant y \leqslant L$, is $A\Phi_m^{\mathrm{I}}(L)$ (using the notation of HB). However, the boundary conditions assumed by WC for region II are $\psi(0) = \psi(L) = 0$, which implies zero flux as $x \to 0+$. Inspection shows that for $0 < x < x_0$ each of the eigenfunction $\Phi_n^{\mathrm{II}}(y)$ gives zero volume flux, and so any sum of these must also give zero volume flux. The question is: what has happened to the volume flux at $x = 0$? The explanation offered by WC after (WC-19) in response to our original Reply (which pointed out the presence of a singularity in their solution), is not helpful.

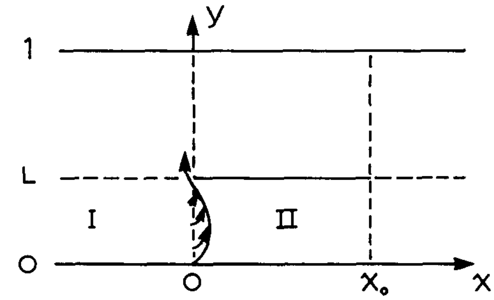

FIG. 1. Possible streamlines for water exiting region II through the singularity at (0, L) in the WC model.

Consider first the behavior of ψ in WC along the line $y = L$. Clearly $\psi = A\Phi_m^{\mathrm{I}}(L)$ for $x < 0$, and $\psi = 0$ (assumed) for $x > 0$. Thus ψ is discontinuous at $x = 0$, and v is infinite. It seems, therefore, that there is a point source at $y = L$, given by

$$(u, v) = [0, A\Phi_m^{\mathrm{I}}(L)\delta(x)/h(L)]e^{-i\omega t} \tag{16}$$

which ensures that the incident mass flux in the x-direction is directed to an equivalent mass flux in the y direction at $x = 0-$. The effect of such a source can be analyzed by the method of Buchwald and Kachoyan (1987) and raises further problems which will not be discussed here.

Further, despite appeals by WC to Lighthill, it is common knowledge that the Fourier series used by WC in region II represents a discontinuous function which yields a sink at $x = 0$, $y \to L$, of the same strength as in (16). Thus all the water that enters region II at $x = 0$, $0 \leqslant y < L$, exits as a source flowing around the point (0, L) as illustrated in Fig. 1. Note also the presence of sources on the left-hand side of (1) implies that the surface elevation η is discontinuous at (0, L) in the WC model.

As was pointed out in our original Reply, and not satisfactorily resolved by WC, the coefficients B_n of their series for ψ in region II behave like $(-1)^n/n$, as $n \to \infty$. Thus the series in II for u, v diverge (we are not sure what is the meaning of only "apparently divergent"). Moreover, lack of absolute convergence of the series for ψ means that term by term equating with other series at $x = 0$ and at $x = x_0$ is theoretically invalid. Perhaps in this case the formal manipulation gives a consistent result, but this requires proof.

3. Energy

There is a basic inconsistency in the argument leading to (WC-5) and the deductions from it. The "long-wave" model assumption implies that longshore derivatives are small compared with offshore derivatives, as may be seen from (18)–(20). Consider, however, the higher modes so obtained. For all regions the nth mode looks like

$$\psi_n = \alpha_n e^{-by} \sin\gamma_n y \cos(k_n x - \omega t) \tag{17}$$

where, from (HB-4.3) and HB-4.4), $\gamma_n = O(n\pi)$ and $k_n \sim \omega\gamma_n^2/2b$. It is evident that x-derivatives will exceed the y-derivatives for quite small values of n, thus contradicting the initial assumption. What saves the HB calculation is the rapid convergence of the coefficients α_n. For instance, in region II, $B_n = O(n^{-3})$ as $n \rightarrow \infty$. Thus the higher modes and all their first derivatives make only a small contribution. On the other hand, convergence in WC is only very slow, since B_n behaves as $(-1)^n n^{-1}$. Further evidence of the slow convergence of the D_n is given in WC Fig. 3a.

As further illustration of this point, the equation for ψ obtained from (1) by taking $\epsilon = 0$ is

$$\operatorname{div}(G\,\operatorname{grad}\psi_t) + G_y\psi_x = 0, \tag{18}$$

where $G = h^{-1}$. Following the steps in WC, Eqs. (1) to (4), the energy density per unit area is

$$E' = \frac{1}{2}G[\psi_x^2 + \psi_y^2], \tag{19}$$

and the flux per unit longshore length is

$$F' = G\psi\psi_{xt} + \frac{1}{2}G_y\psi^2. \tag{20}$$

This result may be obtained by using either the boundary condition $\psi = 0$ at $y = 0, 1$, or, as in WC, $\psi = 0$ at $y = 0$ and $\psi_y = 0$ at $y = 1$. The long-wave model neglects the x-derivatives in (19) and (20), whereupon the WC result is recovered. It is clear, though, that these derivatives cannot be ignored for the higher modes. Moreover, the orthogonality principle no longer applies, so that interactions must be taken into account when summing modes in (20). Bearing all this in mind, since the lefthand side of (WC-5) refers to small n, the incident wave energy is about $\frac{1}{2}A^2$. On the other hand, although the right-hand side of (WC-5) yields a formally interesting result, it is difficult to understand its physical interpretation.

More generally, in the asymptotic matching procedure, discussion of energy should be confined to the outer region, since it is difficult to interpret results for the inner region on its own. For the problem in hand, the outer region will contain an incident, together with a finite number of reflected shelf waves in region I, and transmitted shelf waves, together with a Kelvin wave in a semi-infinite ocean in region IV. A possible Kelvin wave in region IV could be derived from matching sea level at $x = 0$, x_0, as in Wilkin and Chapman (1987), resulting in a uniform sea level change, as for the term in A″ in (11). Some proportion of this term in the inner region could correspond to a Kelvin wave in the outer region, and thus convey energy to infinity, as in (WC-8).

4. Discussion

The rigid-lid approximation for long period oscillations should be regarded as the zero order inner solution of an asymptotic matching procedure. The singular solutions corresponding to zero eigenvalues in the inner region are required to complete the matching with the outer solution, which contains Kelvin waves as well as shelf waves. Since diffracted Kelvin waves may carry energy, discussion of energy balance can only be completed in the outer region. Note that linear interaction between shelf and Kelvin waves is consistent with Flather's (1987) numerical results.

The long-wave approximation rapidly breaks down for higher modes, and the HB paper must be regarded as yielding the approximate amplitudes of the first two or three modes of forward-scattered shelf waves. Conclusions regarding higher modes and energy propagation are inconsistent with the model assumptions.

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