NOTES AND CORRESPONDENCE

Initialization of Equatorial Waves in Ocean Models

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ABSTRACT

The relative information content of mass and velocity measurements for initializing low-frequency equatorially trapped waves is considered using analytical arguments and a numerical model. For the Kelvin wave, mass and velocity data are equally useful, but this is not true for the Rossby waves. It is the relative amount of potential energy and kinetic energy possessed by equatorial waves that determines the relative usefulness of mass and velocity data for initializing these waves. The effects of dissipation on the adjustment process is considered. It is shown for the Kelvin wave that mass and velocity data are equally useful even when dissipation is present, but the Rossby wave adjustment is sensitive to the level and form of dissipation used.

1. Introduction

The problem we want to consider is the following. We have a set of measurements of the velocity or density (mass) field in the ocean. From this incomplete set, we want to try to reconstruct the full velocity and mass field. Which type of observation contains most information, i.e., are measurements of velocity more useful than those of density or vice versa? We will confine attention in this note to the free surface equations,

$$u_t - fv = g'h_x + X$$
$$v_t + fu = -g'h_y + Y$$
$$h_t + \frac{c^2}{g'}(u_x + v_y) = Q \qquad (1)$$

where u and v are the components of the velocity field, h is the free surface displacement, c is the speed of short gravity waves equal to $(g'H)^{1/2}$, H is the depth of the upper layer of the ocean, g' is the reduced gravity; and f is the Coriolis parameter. Equations (1) do not contain dissipative terms, but the effects of model dissipation are considered later. It is found that Rossby wave adjustment is sensitive to model dissipation in contrast to Kelvin wave adjustment which is insensitive to dissipative processes.

Equations (1) are forced: X and Y represent wind forcing and Q buoyancy forcing. For the purpose of this paper, we will assume that the forcing terms (X, Y and Q) are known, and that the equations do correctly represent the processes of interest. Neither statement is likely to be completely true in reality of course but consideration of such complications is reserved for a later paper. Here, we have only the more limited objective of determining which type of observation contains the most useful information. The answer is likely to be dependent on the physical process under consideration: we restrict our attention to slowly varying equatorially trapped Kelvin and planetary waves as these are frequently of interest in low latitude processes. Further, any observations are likely to be irregularly distributed in space. To isolate the problem of interpolation from that of the adjustment problem we will consider here only complete fields of either velocity or mass field data.

If we assume the forcing is known, then we can introduce another set of equations identical to (1) in all respects except that the initial conditions are not completely known. Subtracting these equations from (1) leads to a set identical to (1), but unforced and with variables now representing the difference between the true state as given by (1) and the set with the incompletely observed initial conditions.

The problem of the relative importance of mass (h) or velocity information has been considered frequently in the meteorological literature. Temperton (1973) gives a clear argument for an f-plane showing the relative importance of mass and velocity as a function of space scale and latitude. He shows that the final (balanced) state of a fluid will be related to the initial (unbalanced) state by

$$\psi_f = \alpha\psi_i + \frac{(1-\alpha)}{f} g'h_i \qquad (2)$$

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where ψ is the streamfunction, h is the free surface displacement and subscript i or f indicates the initial and final states respectively. The expression for α is $\alpha = \{1 + f^2/[c^2(k^2 + l^2)]\}^{-1}$ where k and l are the wave numbers in the zonal and meridional directions. This argument shows that at small scale ($\alpha \sim 1$), the final state is determined by the initial velocity field, and at large scales ($\alpha \sim 0$), by the mass (or height) field.

For the midlatitude f-plane, there is a steady solution and there are gravity waves excited which carry out the adjustment from the initial conditions to the steady solution. If β-effects were included, then (2) would not be the equilibrium solution but the argument is still useful because there is a separation between the time scales on which the gravity waves work and those on which Rossby waves further adjust the solution, and the results of (2) are likely to carry over to the Rossby waves, since at large scale these waves have most of their energy in potential form, whereas at smaller scale they have more kinetic energy.

Although the argument is for an f-plane, by letting f become smaller it can be used to suggest a growing importance of velocity data at low latitudes. It is unlikely that it can be applied in the equatorial belt however given that equatorial Kelvin waves and planetary waves are important at low latitudes, but are not included in the above theory and there is no clear timescale separation between the inertia gravity wave and planetary and Kelvin wave responses. In this note we will show that for initializing Kelvin waves, velocity or mass data are equally important. The planetary wave case is more complicated. Further, it is shown that some results are sensitive to dissipation.

2. The equatorial Kelvin wave

For an equatorial Kelvin wave, the energy is everywhere equipartitioned between kinetic and potential energy. A heuristic argument then suggests that the solution obtained by specifying only velocity information and that obtained by specifying only height information will converge to the same solution and therefore that they are both equally important for determining the full solution. This can be confirmed by expanding the unforced version of Eqs. (1) in terms of parabolic cylinder functions and calculating the solution for the Kelvin wave amplitude, first when the initial conditions are that the height is given correctly and the velocity is zero, and then when the velocity is given correctly and the height field perturbation is zero. Although the solutions are obviously different at $t = 0$ and at short times thereafter, they do indeed converge to the same solution. From this expansion one can also show that the amplitude of the Kelvin wave produced is only one half that of the true solution.

Figure 1a shows the height field for the control or

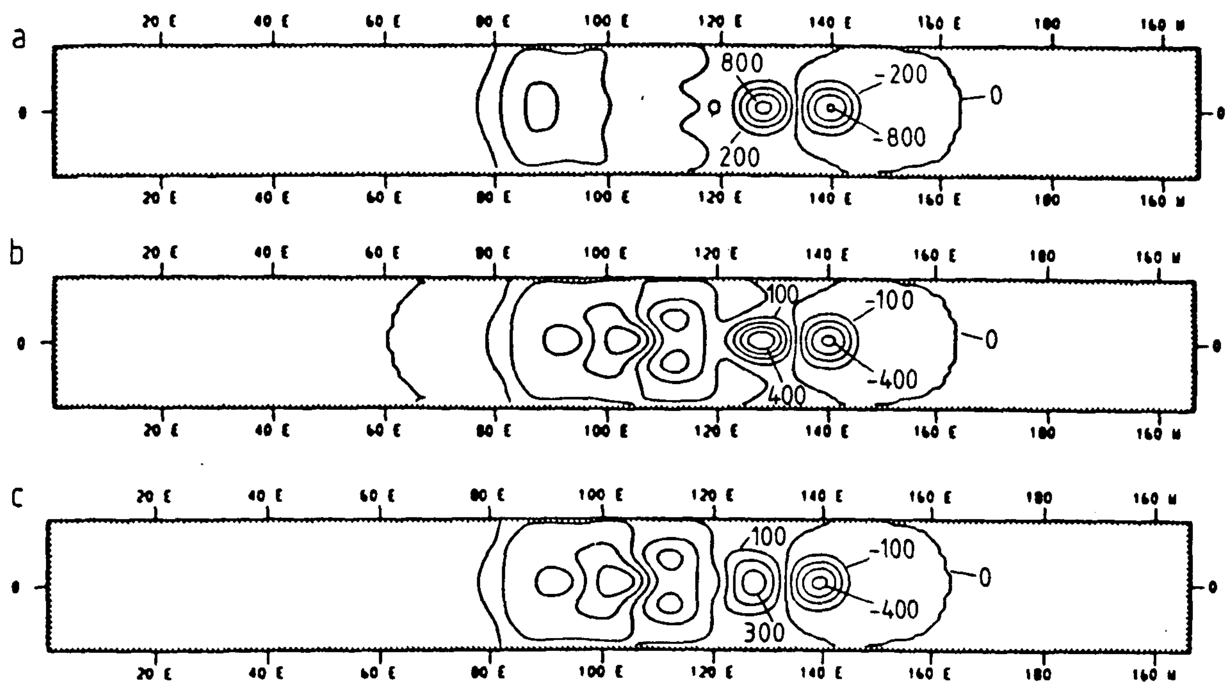

FIG. 1. Contours of the height field (units arbitrary) for three experiments 15 days after the start of the numerical integration. The experiments were (a) a control in which the model is supplied with height and velocity information corresponding to an analytic inviscid sinusoidal Kelvin wave; (b) an integration in which the initial h-field was as for (a) but the initial velocity was zero; (c) an integration in which the initial u-field was as for (a) but the initial height field perturbation was zero. The numerical model was dissipative. Laplacian friction was applied to the momentum equations with a coefficient of eddy viscosity $K = 2 \times 10^4\ m^2\ s^{-1}$. There was no damping on h.

true solution, and Figs. 1b and 1c the height field when a model (see appendix A) is given only height field or velocity field information. The results are from a numerical integration 15 days after initialization with dissipation included but the forcing zero. It is important to note that although Figs. 1b and 1c are similar to Fig. 1a in the Kelvin wave part, the assimilation runs have recovered only 25% of the energy of the original wave, so one initialization is insufficient. If a series of initializations is performed approximately 25% of the energy deficit is recovered from the control after each initialization (not shown). The solutions differ from the control solution west of the Kelvin wave because inertia gravity and Rossby waves are excited following initialization. Because the analytic form of the Kelvin wave is not an exact solution of the finite difference equation some spurious wave activity is also visible in the control run (Fig. 1a).

Note also that we are not considering here trying to recover small-scale Kelvin waves whose dispersion properties are similar to inertia gravity waves. Later, when we consider dissipative effects we also exclude very long waves.

3. Equatorial planetary waves

In principle, we can repeat the argument of the previous section for planetary waves. The ratio of kinetic energy to potential energy averaged over a wavelength in the zonal direction and over all latitudes is given by

$$R = 1 + \frac{2}{\beta c}(k^2c^2 - \omega^2)[(m+1)(kc+\omega)^2 + m(kc-\omega)^2]^{-1} \quad (3)$$

where all symbols have their usual meaning (see Gill 1982), and ω is the wave frequency. Equation (3) shows that for waves of high meridional mode number (m) and or long wavelength ($2\pi k^{-1}$), the wave energy is again equipartitioned, and so one might expect both mass and velocity data to be equally useful for initialization of such waves. However, this argument has not been found to be very useful, because it does not take into account the fact that, locally, kinetic and potential energy vary differently from the integrated values. For example, Fig. 2 for an $m = 5$ planetary wave, shows that the latitudinal distributions of kinetic and potential energy are quite different in contrast to the Kelvin wave where the potential and kinetic energy are locally the same everywhere. For the planetary wave, the kinetic energy is concentrated near the equator, with the potential energy concentrated off the equator.

To illustrate the relative importance of velocity versus mass field data, a numerical integration similar to that of Fig. 1 was performed for the Rossby wave of Fig. 2 (meridional wave number $m_0 = 5$, zonal wavelength = 4000 km) (see appendix A). Figures 3a and 3b (short-dashed curves) show a comparison of the root mean square sea surface height and zonal velocity components averaged over the equatorial band, (defined as $-5°$S to $+5°$N over all longitudes) when the model is initialized with height-field data. Figures 3c and 3d show the same comparison for the case when the model is initialized with the velocity field. The analytic form of the control wave (i.e., that obtained after the simultaneous initialization of h, u and v) is not a perfect solution of the model finite difference equations, so the rms height and velocities (solid curve) oscillate slightly instead of remaining constant in time. The values of k and m used for this experiment are such that from Eq. (3) one would expect that velocity data would carry more information near the equator than height field data. This is partly confirmed by comparing Figs. 3b and 3c which show that more of the h field is recovered following a velocity initialization than vice versa.

It can be shown that just as for the Kelvin wave, the amplitude of the primary wave ($m = m_0$) is recovered equally well from either velocity or height field information and the amplitude is again half of the control wave. However $m_0 \pm 2$ waves, (i.e., $m = 3$ and $m = 7$ in this example) are also excited and what one sees in physical space is the sum of these waves. In the case where the velocity field is specified this sum is quite different from that when the height field is specified.

4. Dissipation effects

In section 2 the theory of the relative importance of mass field versus velocity field data was based on inviscid theory. Yet when we used the numerical model to test the theory we used quite large values of dissipation (see appendix A). Figure 1 is in fact from a calculation with Laplacian dissipation terms in the momentum equations (with a coefficient of eddy viscosity $K = 2 \times 10^4$ m^2 s^{-1}). The result, that velocity and mass field data were equally important, was found to be true regardless of the values of K used (not shown). Two other forms of dissipation were also used (namely Rayleigh damping on the momentum equation or a Newtonian damping term on the h equation simulating vertical diffusion of heat).

The insensitivity of the initialization of the Kelvin wave to dissipation can be understood from the following. The damped equations for a plane equatorial Kelvin wave of the form

$$u = U(y)e^{i(kx-\omega t)}, \quad h = G(y)^{i(kx-\omega t)}$$

are

$$-i\omega U + ikg'G = -\Sigma U$$

$$\beta y U + g'G_y = 0$$

$$-i\omega G + ikHU = -\gamma G$$

where Σ and γ are the coefficients of Rayleigh friction and Newtonian damping respectively. For $\omega \gg \Sigma$ or γ

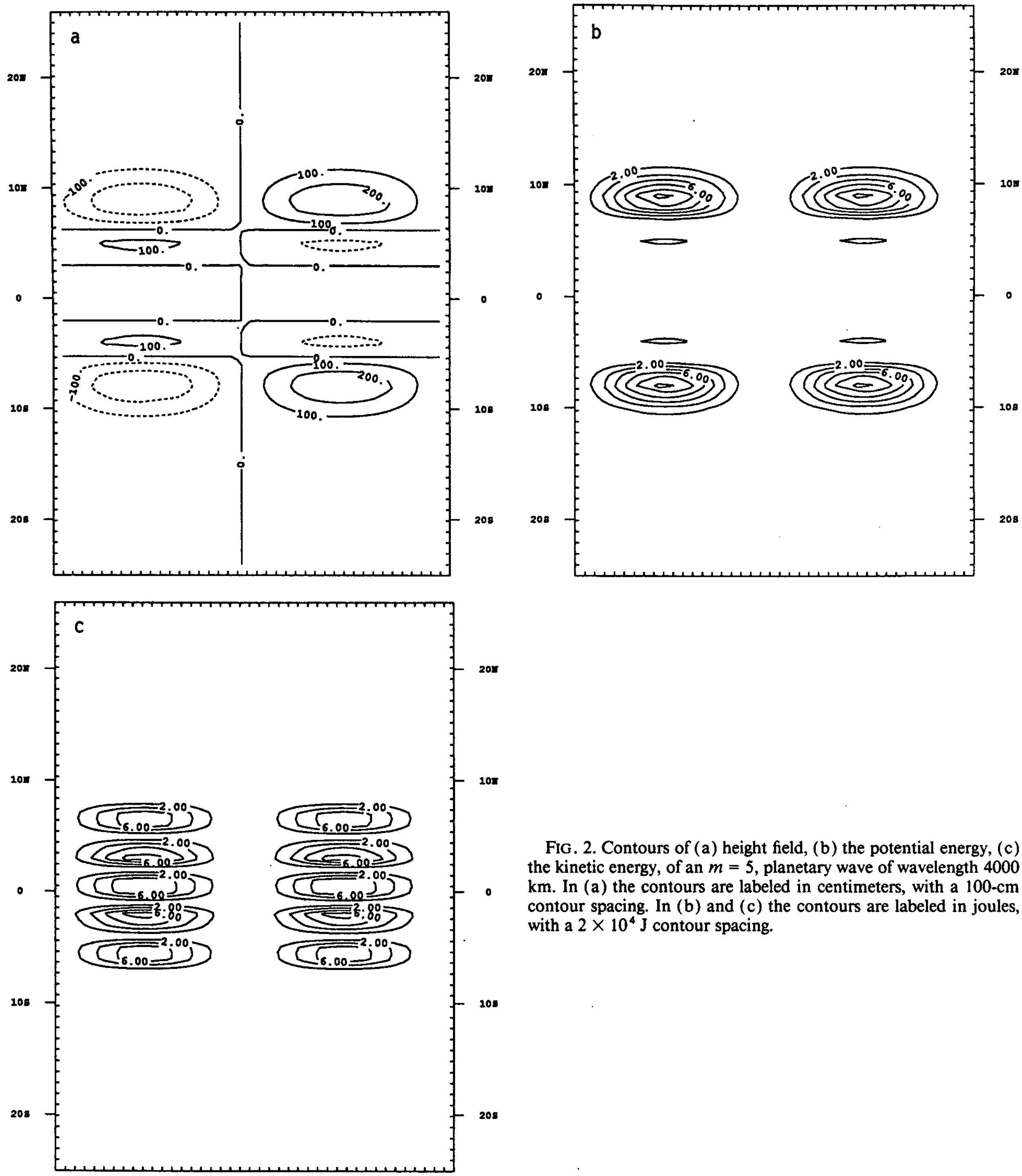

FIG. 2. Contours of (a) height field, (b) the potential energy, (c) the kinetic energy, of an $m = 5$, planetary wave of wavelength 4000 km. In (a) the contours are labeled in centimeters, with a 100-cm contour spacing. In (b) and (c) the contours are labeled in joules, with a 2×10^4 J contour spacing.

(i.e., for physically reasonable damping timescales), the dispersion relationship is

$$k = \frac{\omega}{c}\left(1 + i\frac{(\Sigma + \gamma)}{2\omega}\right)$$

$$G = \frac{c}{g'}\left[1 + \frac{1}{2}i\frac{(\Sigma - \gamma)}{\omega}\right]U.$$

Thus, to first order, both u and h are damped equally [by a factor $\exp -(\Sigma + \gamma)t/2$], regardless of whether the damping comes from Rayleigh friction or Newtonian damping or some combination. The condition that $\omega \gg \Sigma$ implies that the argument does not apply to the very longest Kelvin waves. One can estimate what this value is likely to be. For a value of Σ of $5 \times 10^{-7}\ s^{-1}$,

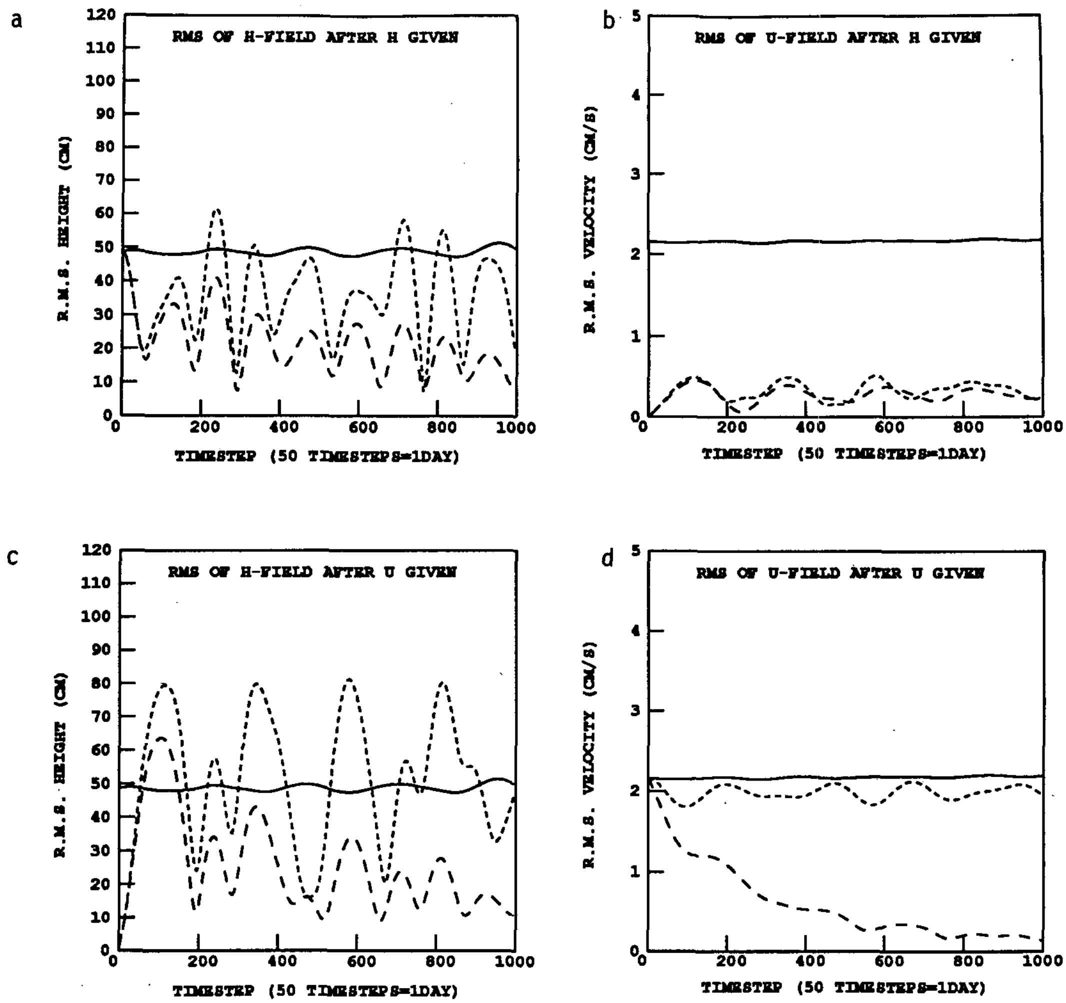

FIG. 3. The time evolution of the root mean square height field and zonal velocity field averaged over the equatorial band (defined as 5°S to 5°N over all longitudes) during numerical integrations relating to the planetary wave in Fig. 2. In (a) and (b) the planetary wave height field is supplied at time $t = 0$, with the velocity field set to zero. In (c) and (d) the planetary wave velocity field is supplied at $t = 0$, with the height field perturbations set to zero. The three curves shown in each case are for: (i) the control wave where h, u and v are all supplied at $t = 0$ (solid curve); (ii) low dissipation, $K = 1\ \mathrm{m^2\ s^{-1}}$ (short dashed curve); (iii) high dissipation, $K = 2 \times 10^4\ \mathrm{m^2\ s^{-1}}$ (long dashed curve).

the time scale Σ^{-1} is ~20 days. Thus, if we consider first or second baroclinic waves with $c \sim 2\ \mathrm{m\ s^{-1}}$ we exclude waves with a wavelength greater than 24 000 km from this analysis. The value of $5 \times 10^{-7}\ \mathrm{s^{-1}}$ for Σ is probably on the high side (it corresponds roughly to that for $K = 2 \times 10^4\ \mathrm{m^2\ s^{-1}}$ used in the numerical model). For smaller values of Σ the wavelength of the excluded waves is even longer.

The result does not hold in the case of planetary waves. Figure 3 shows the relative importance of velocity and height field data for a calculation similar to the short-dashed curve but for larger values of K (long dashed curve). The relative importance of velocity decreases at larger K. Since in the real equatorial ocean the solution will be a combination of planetary and Kelvin waves, the level of dissipation will influence the relative value of the observations. Further examples of the sensitivity of assimilation methods to dissipative processes in more realistic ocean basins are given by Moore et al. (1987) while Yamagata and Philander (1985) consider the effects of damping on other equatorial waves.

5. Summary and conclusions

Some equatorial wave initialization experiments have been described in this paper. For the Kelvin wave, initialization is relatively straight forward. Both mass and velocity data carry equal amounts of information since Kelvin wave energy is always equipartitioned, regardless of the level or form of wave dissipation, and irrespective of whether the velocity or mass field is damped provided that the frequency of the waves is higher than some minimum value ($\sim\Sigma$). Also by virtue of its large phase speed, a Kelvin wave can escape from the other wave modes excited during the initialization which propagate more slowly. The Kelvin wave response is therefore not masked by these other waves which constitute noise.

On the other hand, planetary wave initializations are more complicated. In an inviscid ocean, velocity data is found to be best for initializing a planetary wave near the equator, while off the equator, it is the mass field which is better. Planetary wave initializations are sensitive to the form and magnitude of wave dissipation, and hence it is important that the levels of potential and kinetic energy of a numerical model be consistent with that of the system under observation.

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APPENDIX A

The Numerical Model

The model used in the experiments described was a two layer, adiabatic reduced gravity model. The barotropic mode is suppressed by virtue of the infinite total depth of the ocean, leaving only one baroclinic mode. The lower layer is passive so there are no currents or pressure gradients in this layer. The model solves the shallow water equations, with Laplacian dissipation of momentum, on a staggered (c) grid using a leapfrog time differencing scheme. Unless otherwise specified, the following parameters are used: Coefficient of eddy viscosity $K = 2 \times 10^4$ m^2 s^{-1}, mean depth of upper layer $H = 100$ m, reduced gravity $g' = g(\Delta\rho/\rho) = 3 \times 10^{-2}$ m s^{-2}.

For the Kelvin wave initialization experiments described in section 2, the model grid resolution was 100 km in the zonal and meridional directions, and all model boundaries were solid with no-slip conditions applied. The zonal extent of the model was 20 500 km, and extended from 600 km south to 600 km north of the equator.

For the planetary wave initialization experiments described in sections 3 and 4, the numerical model had a zonal extent of 4000 km and extended from 2600 km south to 2600 km north of the equator. The coefficient of eddy viscosity (K) was 1 m^2 s^{-1} to make the model essentially inviscid in accordance with the theory of Eq. (3) or 2×10^4 m^2 s^{-1}. The model grid resolution was 100 km in the east–west direction but only 25 km in the north–south direction in order to resolve the meridional structure of the $m = 3$, 5 and 7 waves which will be excited by the initialization of the $m = 5$ wave. No-slip boundary conditions were applied at the northern and southern boundaries, with periodic boundary conditions to the east and west.

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