Oceanographic Implications of Laboratory Experiments on Diffusive Interfaces

HARINDRA J. S. FERNANDO

Department of Mechanical and Aerospace Engineering, Arizona State University, Tempe, Arizona

(Manuscript received 18 January 1989, in final form 17 May 1989)

ABSTRACT

This paper contains a summary of the results from some laboratory and theoretical studies on the diffusive interface in double diffusive convection, paying particular attention to the recent work of Fernando. A simple model is developed which predicts the thicknesses of the convecting layers in a thermohaline staircase structure. The laboratory buoyancy-flux measurements and the model results are extrapolated for oceanic situations and comparisons are made with field measurements.

1. Introduction

With the development of sophisticated CTD measuring instruments, the capability of resolving minute features of the oceanic microstructure has been enhanced. At the same time, the literature emphasizing the role of double diffusion as a major oceanic mixing mechanism has markedly increased (Mack 1985). Often, oceanic mixing events contain more than one physical process, e.g., double diffusion and shear instabilities, so that the identification of the detailed physics and the relative contribution of individual processes is difficult. Controlled laboratory experiments, which are designed to study a single mixing mechanism, have proven to be useful in interpreting and modeling oceanic phenomena (Federov 1970). In this paper, the results of some theoretical and laboratory studies on diffusive interfaces are summarized. Particular attention will be given to the recent theoretical and laboratory work of Fernando (1987, 1989), which will be extrapolated to predict oceanic parameters. These predictions are also compared with those of previous investigations.

2. A review of laboratory studies on thermohaline staircases

Turner and Stommel (1964) were the first to demonstrate the formation of a “thermohaline staircase,” a series of turbulently convecting layers separated by density interfaces, when a stable salinity-stratified fluid is heated from below. These interfaces are stably (unstably) stratified with respect to salt (heat), and the buoyancy transport through them, at high interfacial stabilities, occurs mainly by molecular diffusion.

The mixed layer development and the layered-structure formation in a linearly stratified fluid, subjected to a constant bottom heat flux, have been studied in a more quantitative manner by Turner (1968), Huppert and Linden (1979), and Fernando (1987). A schematic diagram of the flow configuration used in these studies is shown in Fig. 1. A nomenclature of symbols is given in the Appendix.

Turner (1968) argued that the breakdown of the thermal boundary layer, which develops above the growing convective mixed layer, is responsible for the formation of multiple layers. The thermal boundary layer was assumed to become unstable when the Rayleigh number, defined as \( R_a = g \alpha \Delta T \delta^3 / k_b \nu \), where \( \delta \) is the thermal boundary-layer thickness, \( \alpha \) is the thermal expansivity, \( g \) is the gravitational acceleration, \( \Delta T \) is the temperature jump at the edge of the mixed layer, \( k_b \) is the thermal diffusivity and \( \nu \) is the kinematic viscosity, exceeds a critical value \( R_c \). On this basis, it was shown that the critical-height of the lower layer, at which the second mixed layer appears, is

\[
h_c = \left( \frac{R_c}{4} \right)^{1/4} \left( \frac{v_0^3}{k_b^2 N_c^8} \right)^{1/4},
\]

where \( N_c \) is the buoyancy frequency, defined in terms of the initial salinity gradient \( dS/dz \) as \( N_c^2 = -g \beta (dS/dz) \), where \( \beta \) is the salinity contraction coefficient, \( z \) the vertical coordinate, and \( v_0 \) the buoyancy flux due to heat, defined in terms of the bottom heat flux \( Q \), a reference density \( \rho_0 \) and the specific heat \( c_p \), as \( q_0 = g \alpha Q / \rho_0 c_p \). By fitting laboratory data to (1), Turner (1968) found that \( R_c \approx 2.5 \times 10^4 \), noting that this is an order of magnitude larger than the value \( R_c \approx 10^3 \), expected from linear stability theory calculations. Federov (1970) and Newman (1976) have used (1) in

© 1989 American Meteorological Society
interpreting the data obtained during various field expeditions.
Fernando (1987) argued that the reason for high value of \( R_s \) obtained by Turner (1968), is a misinterpretation of the laboratory data. Turner (1968) used the thickness of the lower convecting layer of the laboratory staircases as \( h_c \) in calculating \( R_s \), from (1). Detailed temperature and salinity profile measurements showed that the thermal boundary layer breaks down at \( R_s \approx 10^3 \), but this does not lead to the formation of a “thermohaline” staircase structure. Rather the convective layer formed due to the break down was shown to be “engulfed” by the turbulent eddies of the lower layer, until the eddies are too “feeble” to do so. Based on the previous laboratory results of Fernando and Long (1985), it was argued that the bottom-layer eddies cannot engulf the layer above when the inertia and buoyancy forces of the eddies become of the same order, or
\[
\overline{w^2} = c_1 \Delta b h,
\]
where \( \Delta b \) is the buoyancy jump across the interface, \( (\overline{w^2})^{1/2} \) is the rms velocity in the convecting layer, \( h \) is the mixed-layer height, and henceforth \( c_1, c_2, \ldots \) are used to denote constants; the estimated values of these constants are given in Table 1.

**Table 1.** Estimated values of the constants used in the double diffusion equations.

<table>
<thead>
<tr>
<th>Constant</th>
<th>Value</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_1 )</td>
<td>0.063</td>
<td>Fernando and Long (1985)</td>
</tr>
<tr>
<td>( c_2 )</td>
<td>1.8</td>
<td>Hunt (1984)</td>
</tr>
<tr>
<td>( c_3 )</td>
<td>41.5</td>
<td>Fernando (1987)</td>
</tr>
<tr>
<td>( c_4 )</td>
<td>8.58 \times 10^{-3}</td>
<td>Marmorino &amp; Caldwell (1976)</td>
</tr>
<tr>
<td>( c_5 )</td>
<td>0.323</td>
<td>Huppert (1971)</td>
</tr>
<tr>
<td>( c_6 )</td>
<td>7 \times 10^{-2}</td>
<td>present paper/Fernando (1989)</td>
</tr>
<tr>
<td>( c_7 )</td>
<td>0.058</td>
<td>Linden &amp; Shirtcliffe (1978)</td>
</tr>
<tr>
<td>( c_8 )</td>
<td>0.15</td>
<td>Fernando (1989)</td>
</tr>
<tr>
<td>( c_9 )</td>
<td>6.7 \times 10^{-4}</td>
<td>Fernando (1989)</td>
</tr>
<tr>
<td>( c_{10} )</td>
<td>4.5 \times 10^{-3}</td>
<td>Fernando (1989)</td>
</tr>
<tr>
<td>( c_{11} )</td>
<td>0.15 or ( r^{1/2} )</td>
<td>Fernando (1989)</td>
</tr>
<tr>
<td>( c_{12} )</td>
<td>12.5</td>
<td>Present paper</td>
</tr>
<tr>
<td>( c_{13} )</td>
<td>4.7 \times 10^{-4}</td>
<td>Present paper</td>
</tr>
</tbody>
</table>

Using the scaling for convective velocity in turbulent thermal convection (Caughey and Palmer 1979; Hunt 1984),
\[
\overline{w^2} = c_2 (\rho_0 h)^{2/3},
\]
it was possible to show that the thickness of the lower convecting layer in the laboratory staircases should be given by
\[
h = c_3 (\rho_0/N_h)^{1/2}.
\]
The experimental results of Fernando (1987) were found to support (4) over (1). Further, the fact \( h > h_c \) was used to explain why Turner (1968) obtained an anomalously high \( R_s \) value.

3. Laboratory observations on single diffusive interfaces

Although many researchers have studied heat and salt transport through single diffusive interfaces experimentally, there are no generally accepted flux laws for such transports. Turner (1965) generated a diffusive interface by heating a two-layer, salt-stratified fluid from below (Fig. 2). The buoyancy fluxes due to heat \( q_h \) and salt \( q_s \) were measured and the results indicated that for the range of stability ratios \( 2 < R_s < 7 \), the flux law can be given by
\[
R_F = \frac{q_s}{q_h} = c_4,
\]
where \( c_4 \approx 0.15, R_s = g\beta\Delta S/g\alpha\Delta T \), and \( g\beta\Delta S \) and \( g\alpha\Delta T \) are the buoyancy jumps across the interface due to salinity and temperature, respectively. For \( R_s < 2 \), the flux ratio \( R_F \) showed an abrupt increase. Later experiments, however, revealed that (5) is valid only for the experimental parameter range used by Turner (1965). Crapper (1975) demonstrated that the critical value of \( R_s \), below which an abrupt increase of \( R_F \) occurs, is not a constant but varies from one experiment to another (also see Taylor 1988).

In addition to heating from below, Marmorino and Caldwell (1976) used cooling from above to achieve a quasi-stationary state in the buoyancy transfer process. They showed that \( R_F \) is a function of \( R_s \) and \( q_0 \). The
buoyancy (heat) flux was found to be given by the empirical relation

\[ q_h \approx c_5 \left( \frac{k_h}{\nu} \right)^{1/3} (g \alpha \Delta T)^{4/3} \times \exp\left\{4.6 \exp[-0.54(R_p - 1)]\right\}, \]  

where \( c_5 \approx 8.58 \times 10^{-3} \). Outside the range, \( 2 < R_p < 5 \), Eq. (6) was noted to give significantly different heat fluxes than that predicted by the equation of Huppert (1971), which is based on Turner’s (1965) data, viz.,

\[ q_h = c_6 \left( \frac{k_h}{\nu} \right) (g \alpha \Delta T)^{4/3} R_p^{-2}, \]  

where \( c_6 \approx 0.323 \). Based on data obtained from sugar-salt double-diffusive systems, Shirtcliffe (1973) suggested that (7) may not be applicable to situations other than heat-salt systems. Shirtcliffe also proposed that the general flux law, for high stability ratios, can be written as \( R_F = c_1 = \tau^{1/2} \), where \( \tau = k_c/k_h \) is the Lewis number, and \( k_c \) is the molecular diffusivity of salt. The experiments of Takao and Narusawa (1980), however, did not support this postulate; instead they found \( c_4 \approx 0.039\tau^{-1/2} \). Using a Turner (1965)-type experiment, Newell (1984) investigated the heat and salt transports at high \( R_p \). The resulting flux law took the form

\[ R_F \approx \tau R_p, \quad \text{for} \quad R_p > 7. \]  

Turner et al. (1970) investigated the buoyancy transport through a multicomponent diffusive interface and proposed that the flux law be given by

\[ R_F \approx \tau^{1/2} R_p. \]  

According to Turner (1979, p. 278), Eq. (9) may have applicability to two-component systems.

Fernando (1989) assumed that the salt and heat interfacial-layer thicknesses within a diffusive interface are governed by a balance between the thickening of the interface due to diffusion and the entrainment of the thickened layer by the convective turbulence. It was shown that for \( R_p < \tau^{-1/2} \), \( q_h \) should be given by

\[ q_h = c_7 \left[ \frac{k_h^3 (g \alpha \Delta T)^{6}}{h_u^2} \right]^{1/5} \left(1 - \tau^{1/2} R_p\right)^{1/5}, \]  

\[ \approx c_7 \left[ \frac{k_h^3 (g \alpha \Delta T)^{6}}{h_u^2} \right]^{1/5}, \quad \text{for} \quad \tau^{1/2} R_p \ll 1; \]  

where \( h_u \) is the thickness of the upper convecting layer. The corresponding flux law is the same as (9). Fernando (1989) further carried out a Turner (1965)-type experiment to investigate the validity of the assumptions underlying (10a); they were found to be reasonably accurate. Since that paper does not contain a comparison between (10b) and the experimental data, we have presented Fig. 3, which provides support for (10b). Further, we find \( c_7 \approx 7 \times 10^{-2} \).

![Graph showing the relationship between heat flux and stability number](image)

**Note:**

The models of Linden and Shirtcliffe (1978) and Fernando (1989) employ different physics. In the former, it is assumed that the thermals rising from the instabilities at the interface govern the convection in the upper layer, whereas the latter assumes that, once the convection is established, thermals are no longer important and turbulence dynamics govern the buoyancy transfer process.

Using measurements of r.m.s. buoyancy fluctuations and flow visualization studies, Fernando (1989) inferred that the sudden increase of heat flux, as observed by Turner (1965), occurs due to a change in the buoyancy transfer mechanism. Accordingly, at high interfacial stabilities, the turbulent eddies tend to flatten at the density interface, as demonstrated by Hannoun et al. (1988). At low stabilities, the eddies can penetrate into the interfacial layer, thus facilitating physical contact between the eddies of the two layers and increasing the surface area available for the buoyancy transfer. The criterion for the onset of “low stability” transport was shown to be
\[ R_p (g \alpha \Delta T)^{4/5} \left[ \frac{k_h^2 h_p^4}{k_h} \right]^{1/10} \left[ 1 - R_p^{-1} \right]^{1/2} \left[ 1 - R_p^{1/2} \right]^{1/5} \]

\[ \approx c_9 (q_0 h_t)^{2/3}, \quad (13) \]

where \( c_9 \approx 0.15 \) and \( h_t \) is the depth of the lower convecting layer. The left-hand side of (13), \( \mathcal{P} \), represents the potential energy increase associated with the distortion of the interface due to impinging eddies while the right-hand side represents the kinetic energy of the eddies of the lower layer, according to (3). If \( \mathcal{P} > c_9 (q_0 h_t)^{2/3} \), the buoyancy transport process is considered to be diffusive; if not, it belongs to the “low stability” regime.

The flux transport expressions for the “low stability regime” were found to be

\[ q_s \approx c_{10} (g \beta \Delta S) \left( \frac{w}{h} \right)^{1/2}, \quad (14) \]

\[ q_h \approx c_{11} (g \alpha \Delta T) \left( \frac{w}{h} \right)^{1/2}, \quad (15) \]

where \( c_{10} \approx 6.7 \times 10^{-4}, c_{11} \approx 4.5 \times 10^{-3} \). The flux law becomes \( R_p \approx 0.15 R_p \).

4. Extension of the Fernando (1987) model to oceanic staircases

One cannot expect exact similarity between oceanic and laboratory thermohaline staircases for several reasons. For instance, the salinity stratification in the ocean is often nonlinear and the positions of the interfaces are fixed rather than migrating since the divergence of salt and heat flux across each layer is small (Hoare 1968). Moreover, changes in the buoyancy transport process across the interfaces are so slow that it is possible to treat the oceanic layered convection as a quasi-stationary process for relatively short lengths of time. It is possible, however, to extend the model proposed by Fernando (1987), for the thickness of the bottom layer of a thermohaline staircase structure (section 2), and the results of Fernando (1989) for single diffusive interfaces (Section 3) to oceanic situations.

If each individual layer of the staircase results from an initially smooth salinity profile (Fig. 4), then the observed salinity jump \( \Delta S \) across the layers can be used to obtain “smoothed” initial local buoyancy gradient due to salinity

\[ N_s^2 = -g \beta (dS/dz) \approx g \beta \Delta S / h_t, \quad (16) \]

where \( h_t \) is the convective-layer thickness. The net destabilizing buoyancy flux \( q_T \) across the diffusive interface, which drives the convection, can be written as

\[ q_T = q_h - q_s = q_h [1 - c_{12} R_p], \quad (17a) \]

or

\[ q_T \approx q_h \quad \text{when} \quad c_{12} R_p \ll 1, \quad (17b) \]

where \( c_{12} \approx 0.15 \) for the “low stability” transport, from (14) and (15), and \( c_{12} = c^{1/2} \) for the “diffusive” transport, from (12). Also the buoyancy jump across the convective layer becomes

\[ \Delta b = g \beta \Delta S (1 - R_p^{-1}) \approx N_s^2 h_t (1 - R_p^{-1}). \quad (18) \]

Using (2), (3), (17a), and (18), it follows that

\[ h_t = c_{13} \left[ \frac{q_h}{N_s^2} \right]^{1/2} \left( \frac{1 - c_{12} R_p}{1 - R_p^{-1}} \right)^{1/2}, \quad (19a) \]

\[ h_t = c_{13} \left[ \frac{1}{N_s^3} \right]^{1/2} \left( \frac{1}{1 - R_p^{-1}} \right)^{1/4}, \quad (19b) \]

where \( c_{13} = (c_2/c_1)^{3/4} \), and we have assumed that the convecting-layer thicknesses are determined by a balance of kinetic and potential energies of the turbulent eddies and the layers are in a quasi-stationary state. Note that all of the quantities in (19) are local for a

---

1 The Prandtl number for both laboratory and oceanic cases is of the order 10. Hence the critical Rayleigh number, at which the flow become fully turbulent, is \( \text{Ra}_c = g \alpha \Delta T h_t^3 / k \nu_p \approx 10^7 \), where \( h_t \) is the convective-layer thickness (Turner 1979, p. 220). The fact that the laboratory \( (\text{Ra} \approx 10^7) \) and oceanic \( (\text{Ra} \approx 10^{10}) \) convective layers are fully turbulent provides a justification for the extrapolation of laboratory data to oceanic cases.

2 Here the curvature of the density profile is neglected for motions with length-scales of the order of the layer thicknesses.

3 Because of the assumption that, for scales of order \( h_t \), the curvature of the buoyancy profiles is negligible, it is possible to approximate the temperature and salinity jumps across the interface as \( \Delta T \) and \( \Delta S \).
particular layer. Further we obtain $c_{13} \approx 12.5$. Figure 5 shows a comparison between the geophysical data collected by previous workers and the theoretical result (19b). Note that for all cases depicted, the buoyancy (heat) fluxes have been evaluated using methods that do not employ laboratory flux laws. Although there is scatter, the data clearly follow the trend predicted by 19(b). The constant $c_{13}$ can be evaluated, using Figure 8, as $c_{13} \approx 14$, which is close to the predicted value.

One of the problems that arises in using (19) is the uncertainty of the buoyancy (heat) flux $q_h$. Often $q_h$ is not measured but is inferred from various methods, such as the laboratory flux laws for single diffusive interfaces (Kelley 1984). However, as pointed out below, for some cases, the selection of the flux law may require a priori knowledge of $q_h$ through the interfaces.

5. Evaluation of some individual cases

Extensive field measurements have been reported in lakes and oceans in which diffusive interfaces have been found. Possible applications of the present experimental findings to interpret such geophysical observations are discussed in this section. In the process of comparison, equation (13) will be used, with $h_s$ and $h_i$ replaced by $h_s$ and $q_0$ replaced by $q_h$, to investigate the nature of interfacial buoyancy transport. The value of $q_h$ will then be checked a posteriori using (10b) if the interface belongs to the diffusive regime, or using (15), viz.

$$q_h \approx c_{14}(g\alpha\Delta T)^{3/2}h_s^{1/2},$$

$$c_{14} \approx 4.7 \times 10^{-4},$$

if the interface belongs to the "low stability" regime.

Before proceeding further, it is instructive to restate the limits of applicability of the proposed models. It is assumed that, in the oceanic regions concerned, the turbulence is mainly due to the thermal convection and the Rayleigh number is sufficiently high to maintain the convection in a fully developed state; all other turbulence sources, such as velocity shear, is assumed to play an insignificant role. The predictions for heat and salt transports are valid only under certain conditions and the corresponding restrictions are tabulated in Table 2. Further we have assumed that the model constants of Fernando (1989), based on unsteady laboratory experiments, are valid for quasi-steady oceanic cases. Another important assumption is the stationarity of the interface. According to Fernando (1989), interfacial migrations due to turbulent entrainment (which give rise to additional buoyancy fluxes) are frequent under "low stability" conditions and in calculating $c_{10}$ and $c_{11}$ (and thus $c_{14}$), care has been taken to exclude the experimental cases with significant interfacial movement. Further, it was noted that the mechanisms of interfacial migration are different for different Richardson number ($Ri_s = \Delta h_s/w^2_A$) regimes. When $Ri_s < 15$, the interfaces migrate rapidly due to direct engulfment of interfacial-layer fluid by the turbulent eddies (as in a nonstratified fluid) whereas at $Ri_s > 15$, the stratification becomes important and the entrainment takes place due to splashing of interfacial-layer fluid into the mixed layer by the impinging eddies, as in Fernando (1987). For this case the entrainment coefficient is a decreasing function of $Ri_s$. Further it was found that the effects due to entrainment become vanishingly small when $Ri_s > 240$.

One of the most widely investigated cases is the layered convection of polar oceans and lakes. For instance, Lake Vanda in Antarctica has been studied by Hoare (1966, 1968), Shirkcliffe and Calhaem (1968), and Fedorov (1970). The upward heat flux in this region, expressed in terms of a buoyancy flux, has been estimated to be $1.2 \times 10^{-8} \text{ m}^2 \text{ s}^{-3} (42 \text{ W m}^{-2})$ and the average thickness $h_i$ of the layers is about 1.5 m (Fedorov 1970). Other relevant data are $\alpha \approx 1.2 \times 10^{-4} \text{ K}^{-1}, g\alpha\Delta T \approx 5.8 \times 10^{-4} \text{ m s}^{-2}$, and $R_p \approx 1.25$ (Hoare 1966). The diffusive buoyancy flux due to heat through the interfaces can be calculated using (10b) as $5 \times 10^{-10} \text{ m}^2 \text{ s}^{-3} (1.75 \text{ W m}^{-2})$, which is much lower than the quoted value. The right-hand side of (13) is $10^{-6} \text{ m}^2 \text{ s}^{-2}$ while the left-hand side is $6 \times 10^{-7} \text{ m}^2 \text{ s}^{-2}$, indicating "low stability" transport through the interface.

![Figure 5](image_url)

**Fig. 5.** The variation of the average convection-layer thickness $h_i$ with local $(q_h/N_s)^{1/2}(1 - R_p)^{-3/4}$, as evaluated using oceanic observations. $\Delta$—Lake Vanda (Hoare 1966, 1968); $\square$—in Red Sea (Swallow and Crease 1965; Fedorov 1970); $\diamond$—Mediterranean waters in Atlantic Ocean (Siedler 1968); $\nabla$—Lake Kivu (Newman 1976); $\star$—large steps in the Weddell Sea [step size was taken from Muench et al. (1989) and the heat flux was taken from Gordon (1981)]; $\blacksquare$—Lake Vanda (Shirkcliffe and Calhaem 1968); and $\square$—Western Canadian Arctic (Melling et al. 1984).
### Table 2. Summary of salient parameterizations.

<table>
<thead>
<tr>
<th>Predicted parameter(s)</th>
<th>Equation</th>
<th>Restrictions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buoyancy fluxes in ‘diffusive’ transport regime (Fernando 1989)</td>
<td>( q_b \approx c (k_b \alpha^{3/4} \Delta T^{1/2} / \rho_s)^{1/3} )</td>
<td>( R_s &lt; c_0 )</td>
</tr>
<tr>
<td></td>
<td>( \frac{q_b}{\rho_s} = -1^{1/2} R_s )</td>
<td>( P &gt; c_0 (\rho_s h_b)^{2/3} )</td>
</tr>
<tr>
<td>Buoyancy fluxes in ‘low stability’ transport regime (Fernando 1989)</td>
<td>( q_b \approx c_1 (k_b \alpha^{3/4} \Delta T^{1/2} h_b)^{1/2} )</td>
<td>( P &lt; c_0 (\rho_s h_b)^{2/3} )</td>
</tr>
<tr>
<td></td>
<td>( \frac{q_b}{\rho_s} = 0.15 R_s )</td>
<td></td>
</tr>
<tr>
<td>Thickness of the convecting layers in stationary thermohaline staircases (present work)</td>
<td>( h_s \approx c_3 \left( \frac{q_b}{N_s^2} \right)^{1/2} \frac{(1 - c_3 R_s)^{1/2}}{(1 - R_s^{-1})^{1/4}} )</td>
<td>( R_s &lt; c_1^2 )</td>
</tr>
<tr>
<td></td>
<td>( h_s \approx c_3 \left( \frac{q_b}{N_s^2} \right)^{1/2} \frac{1}{(1 - R_s^{-1})^{1/4}} )</td>
<td>( R_s &lt; c_1^2 )</td>
</tr>
</tbody>
</table>

Hence, the buoyancy flux due to heat can be evaluated, using (20), as \( 8.5 \times 10^{-9} \text{ m}^2 \text{s}^{-3} \) (30 W m\(^{-2} \)), which is in fair agreement with the quoted value.

Another example is the deep-water, near-bottom geothermal inversion (Lubimova et al. 1965; Federov 1970). The typical quoted values are \( h_b \approx 10 \text{ m}, q_b \approx 1.9 \times 10^{-11} \text{ m}^2 \text{s}^{-3} \) (0.09 W m\(^{-2} \)), \( \alpha = 0.9 \times 10^{-4} \text{ K}^{-1} \), \( g \alpha \Delta T \approx 2.6 \times 10^{-6} \text{ m s}^{-2} \), and \( R_s \approx 1.20 \). The diffusive flux evaluated using (10b) is \( q_b \approx 3.6 \times 10^{-13} \text{ m}^2 \text{s}^{-3} \) (1.7 \times 10^{-3} W m\(^{-2} \)), which is two orders of magnitude smaller than the observed value. However, the left and right hand sides of (13) become \( 1.5 \times 10^{-8} \text{ m}^2 \text{s}^{-2} \) and \( 5.0 \times 10^{-8} \text{ m}^2 \text{s}^{-2} \), respectively, and hence, the “low stability” transport expression (20) can be used to calculate \( q_b \). The calculated value \( 6.5 \times 10^{-12} \text{ m}^2 \text{s}^{-3} \) (0.03 W m\(^{-2} \)) is not far from the quoted value.

Newman (1976) has performed a detailed investigation on heat and salt transport in Lake Kivu. The quoted parameters (at station D1) are \( h_b \approx 1.4 \text{ m}, q_b \approx 4.0 \times 10^{-10} \text{ m}^2 \text{s}^{-3} \) (0.71 W m\(^{-2} \)), \( \alpha = 1.2 \times 10^{-10} \text{ m}^2 \text{s}^{-3} \), \( g \alpha \Delta T \approx 7.1 \times 10^{-5} \text{ m s}^{-2} \), and \( R_s \approx 2 \) whereas \( \alpha \) can be evaluated as \( 2.4 \times 10^{-4} \text{ K}^{-1} \). In this case, the diffusive buoyancy (heat) flux calculated using (10b), \( 4.0 \times 10^{-11} \text{ m}^2 \text{s}^{-3} \) (0.071 W m\(^{-2} \)), is much smaller than the quoted value. The left and right hand sides of (13) are \( 1.5 \times 10^{-7} \) and \( 1.1 \times 10^{-7} \text{ m}^2 \text{s}^{-2} \), respectively, suggesting the existence of marginal conditions for “low stability” buoyancy transport, and the possibility of using (14) and (15) to evaluate the buoyancy fluxes. The calculated values of buoyancy fluxes due to heat and salt, \( 3.4 \times 10^{-10} \text{ m}^2 \text{s}^{-3} \) (0.61 W m\(^{-2} \)) and \( 1.0 \times 10^{-10} \text{ m}^2 \text{s}^{-3} \), respectively, agree well with Newman’s (1976) estimations.

The role of diffusive interfaces in maintaining the marginal ice edge of the Arctic Ocean has been discussed by Stegan et al. (1985). Their measurements indicated that, under the ice edge, the ocean is stratified into two layers in such a way that overlying cold relatively fresh water is separated from the bottom warm salty water by a diffusive interface. Since the gradient Richardson number in the interfacial region is about 4, Stegan et al. (1985) concluded that there cannot be substantial turbulent mixing due to internal wave (shear) instabilities (but see Kantha 1986). The total estimated buoyancy flux due to heat required to maintain the ice edge, \( 2.6 \times 10^{-8} \text{ m}^2 \text{s}^{-2} \) (180 W m\(^{-2} \)), calculated on the basis of the ice melting rate, therefore, has to come from other sources, such as lateral advection and double diffusion. The typical values obtained, from Figure 2 of Stegan et al. (1985), are \( g \alpha \Delta T \approx 5.9 \times 10^{-4} \text{ m s}^{-2} \), \( R_s \approx 4 \), \( h_b \approx 35 \text{ m}, h_s \approx 20 \text{ m}, \text{ and } \alpha \approx 6 \times 10^{-5} \text{ K}^{-1} \). The left and right hand sides of (13) become \( 6.7 \times 10^{-6} \text{ m}^2 \text{s}^{-2} \) and \( 10^{-5} \text{ m}^2 \text{s}^{-2} \), respectively, suggesting “low stability” transport through the interface. Equations (14) and (15) can be used to evaluate heat and salt fluxes through the interface as \( q_b \approx 3 \times 10^{-8} \text{ m}^2 \text{s}^{-3} \) (207 W m\(^{-2} \)) and \( q_s \approx 1.9 \times 10^{-8} \text{ m}^2 \text{s}^{-3} \) (1.9 \times 10^{-10} kg m\(^{-2} \text{s}^{-1} \)). It appears that double diffusive transport through the interface can alone account for the heat flux responsible for maintaining the ice edge.

It is interesting to compare the heat-flux predictions based on Fernando (1989) with those based on previous measurements and theories. Such a comparison is given in Table 3, for the cases discussed above. It is possible to conclude that, in general, the flux-laws based on Fernando (1989) give better predictions for oceanic heat fluxes. Also it is evident that in some cases the predictions made on the basis of the present work differ from the quoted value by a factor as much as three. This discrepancy can be attributed to the low Richardson numbers \( R_i \) at the interface which permit interfacial migrations due to turbulent entrainment: \( R_i \) values for the cases given in Table 3 and the corresponding ratio between the quoted and predicted fluxes are given in Table 4. The increase in the variance be-
between the predicted and quoted values with decreasing $R_i^*$ corroborates the notion that additional fluxes resulting from the turbulent entrainment may be responsible for the observed disparities at low $R_i^*$.

In all of the above comparisons with Fernando (1989), a “quoted” value of $q_h$ was used a priori to evaluate the “regime” of interfacial buoyancy transfer. However, as pointed out by a referee, the “quoted” values are subject to a variety of uncertainties and hence must be critically judged. In the previous sections two equations, (10) and (20), to estimate the heat flux have been introduced, but to determine which equation is valid, using (13), a knowledge of $q_h$ is required. Nevertheless, under certain circumstances, it is possible to use (10b), (20), and (13) to estimate $q_h$ through the interface.

It is possible to recast these three equations as

$$Nu \approx c_7 R_h^{1/5} Pr^{1/5}, \quad \text{("diffusive" regime)}, \quad (21)$$

$$Nu \approx c_{14} R_h^{1/2} Pr^{1/2}, \quad \text{("low stability" regime)}, \quad (22)$$

and

$$R_p R_h^{2/15} Pr^{2/5} \tau^{1/2} (1 - R_i^{-1})^{1/2} (1 - R_i^{2/15”)} \gamma \approx c_9 Nu^{2/3} \quad \text{(transition)}, \quad (23)$$

**TABLE 3.** $q_h$ calculated m$^2$ s$^{-3}$ (W m$^{-2}$)

<table>
<thead>
<tr>
<th>Location</th>
<th>$q_h$ (m$^2$ s$^{-3}$ (W m$^{-2}$) quoted</th>
<th>Huppert [Eq. (7)]</th>
<th>Marmorino and Caldwell [Eq. (6)]</th>
<th>Linden and Shirtscliffe [Eq. (11)]</th>
<th>Fernando (see text)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lake Vanda</td>
<td>$1.2 \times 10^{-8}$ (42)</td>
<td>$2.1 \times 10^{-8}$ (73)</td>
<td>$4.8 \times 10^{-8}$ (168)</td>
<td>$5.1 \times 10^{-9}$ (18)</td>
<td>$8.5 \times 10^{-9}$ (30)</td>
</tr>
<tr>
<td>Deep water near bottom geothermal inversions</td>
<td>$1.9 \times 10^{-11}$ (0.09)</td>
<td>$1.7 \times 10^{-11}$ (0.08)</td>
<td>$4.0 \times 10^{-11}$ (0.19)</td>
<td>$3.8 \times 10^{-12}$ (0.018)</td>
<td>$6.5 \times 10^{-12}$ (0.03)</td>
</tr>
<tr>
<td>Lake Kivu</td>
<td>$4 \times 10^{-10}$ (0.71)</td>
<td>$5 \times 10^{-10}$ (0.88)</td>
<td>$7.7 \times 10^{-10}$ (1.37)</td>
<td>$2.7 \times 10^{-10}$ (0.47)</td>
<td>$3.4 \times 10^{-10}$ (0.61)</td>
</tr>
<tr>
<td>Marginal ice zone</td>
<td>$2.6 \times 10^{-8}$ (180)</td>
<td>$2.1 \times 10^{-9}$ (15)</td>
<td>$2.3 \times 10^{-9}$ (16)</td>
<td>$3.2 \times 10^{-9}$ (22)</td>
<td>$3 \times 10^{-8}$ (207)</td>
</tr>
</tbody>
</table>

**TABLE 4.** Dependency of heat-flux estimation errors on $R_i^*$.

<table>
<thead>
<tr>
<th>Location</th>
<th>$(q_h)<em>{\text{quoted}}/(q_h)</em>{\text{calculated}}$</th>
<th>$R_i^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lake Vanda</td>
<td>1.4</td>
<td>32</td>
</tr>
<tr>
<td>Deep water near bottom geothermal inversions</td>
<td>3.0</td>
<td>16</td>
</tr>
<tr>
<td>Lake Kivu</td>
<td>1.15</td>
<td>146</td>
</tr>
<tr>
<td>Marginal ice zone</td>
<td>0.85</td>
<td>547</td>
</tr>
</tbody>
</table>

**FIG. 6.** A $R_i^*-R_p$ diagram depicting the regimes of buoyancy transport through diffusive interfaces. The diagram shows that only the diffusive transport regime can be identified in a straightforward manner.
where $\text{Nu} = q_h/(k_h g \alpha \Delta T/h)$ is the Nusselt number, \( \text{Pr} = \nu/k_h \) is the Prandtl number, and $R_{a_h} = g \alpha \Delta T h^3/k_h \nu^2$ is the Rayleigh number. The following cases must be considered:

i) From (21) and (23), it appears that the condition for "diffusive" transport is satisfied when

$$Y > c_9 (c_7 \text{Ra}_h^{1/5} \text{Pr}^{1/15})^{2/3},$$

or

$$R_p (1 - \tau^{1/2} R_p^{-1}) > c_9 c_7^{2/3} \tau^{-1/2}. \quad (24)$$

For $\tau^{1/2} \approx 0.1$, condition (24) translates to $R_p > 0.35$, which is satisfied by all the statically stable diffusive interfaces with $R_p > 1$; and

ii) From (22) and (23), the condition for 'low stability' transport can be written as

$$Y < c_9 (c_4 \text{Ra}_h^{1/2})^{2/3},$$

or

$$R_p (1 - \tau^{1/2} R_p^{-1}) < [c_9 c_4^{2/3} \text{Pr}^{1/15} \tau^{-1/2}] \text{Ra}^{1/5}. \quad (25)$$

The criteria (24) and (25) are depicted in Fig. 6, for the case $\text{Pr} \approx 10$ and $\tau \approx 0.1$. Note that, in practical situations with $R_p > 1$, it is possible to identify the "diffusive" regime, if (25) is not satisfied. If (25) is satisfied, then the interfacial transport can be either "diffusive" or "low stability" and an idea of $q_h$ is necessary to determine the regime. Assuming quasi-stationarity of the staircase structure, it is possible to use (19) to estimate the buoyancy (heat) flux and then to check the calculated value using (13), and (10) or (20). It should be borne in mind that in such cases (19b) and (20) give rise to the condition $R_p \approx 1.2$ for the steps whereas (19b) and (10b) yield $h_s = 14 (k_h^2/\nu g \alpha \Delta T)^{1/3} (R_p - 1)^{-5/3}$.

Acknowledgments. The author wishes to thank Ted Foster, Dan Kelley, and Laurie Padman for critically reviewing an earlier version of this paper, anonymous referees for their valuable comments, and D. F. Janowski and G. Oth for their help in numerous ways. This research was supported by the Fluid Dynamics Program of the National Science Foundation (Grant MSM 8504909 and MSM 8657378), the Office of Naval Research (Contracts N-00014-87-K-0423 and N-00014-88-K-0250), and the Arizona State University Faculty Grant-in-Aid Program.

APPENDIX

Nomenclature

$c_p$, $c_{1,2}$ . . .

specific heat

constants (see Table 1)

$g$

gravitational acceleration

$g \beta \Delta S$

buoyancy jump across the interface due to salinity

$g \alpha \Delta T$

buoyancy jump across the interface due to temperature

$h$

thickness of the bottom mixed layer

$h_c$

thickness of the bottom layer at the breakdown of the thermal boundary layer

$h_s$

thickness of the convecting layer in a thermohaline staircase

$h_u$

thickness of the upper convecting layer

$h_l$

thickness of the lower convecting layer

$k_x$, $k_h$

diffusivities of salt and heat

$N_s$

buoyancy frequency based on salinity stratification

$Nu$ $q_h/(k_h g \alpha \Delta T/h)$ Nusselt number

$\text{Pr} = \nu/k_h$

Prandtl number

$q_h$ buoyancy flux (due to heat) through the interface

$q_s$ buoyancy flux (due to salt) through the interface

$q_T$

net destabilizing buoyancy flux

$(g \alpha Q/\rho_0 c_p)$ buoyancy flux at the heating surface

$Q$

heat flux at the heating surface

$R_{a_h}$

$g \alpha \Delta T h^3/k_h \nu^2$ thermal Rayleigh number

$R_{a_s}$

$g \alpha \Delta T h^3/k_h \nu^2$ Rayleigh number based on the convecting layer thickness

$R_c$

Critical thermal Rayleigh number

$R_{F}$

$= (q_h/q_s)$ flux ratio

$R_{\beta}$

$(g \beta \Delta S/g \alpha \Delta T)$ stability ratio

$S$

mean salinity

$T$

mean temperature

$w$

rms velocity of the convective layers

$Y$

vertical coordinate

$\alpha$

coefficient of volume expansion

$\beta$

salinity contraction coefficient

$\delta$

thickness of the thermal boundary layer

$\Delta S$

salinity jump across the interface

$\Delta T$

temperature jump across the interface

$\nu$

kinematic viscosity

$\rho_0$

reference density

$\Delta b$

$(g \beta \Delta S - g \alpha \Delta T)$ buoyancy jump across the interface

$\tau$

$= k_h/k_h$ Lewis number

REFERENCES