Expected Structure of Extreme Waves in a Gaussian Sea. Part I: Theory and SWADE Buoy Measurements

O. M. PHILLIPS,* DAIFANG GU,* AND MARK DONELAN**

*Department of Earth and Planetary Sciences, The Johns Hopkins University, Baltimore, Maryland
**National Water Research Institute, Canada Centre for Inland Waters, Burlington, Ontario, Canada

(Manuscript received 2 January 1992, in final form 24 July 1992)

ABSTRACT

This paper is concerned with the expected configuration in space and time surrounding extremely high crests in a random wave field, or, equivalently, the mean configuration averaged over realizations of extreme events. A simple, approximate theory is presented that predicts that the mean configuration $\bar{\zeta}(\mathbf{x} + \mathbf{r}, t + \tau)$ surrounding a crest at $(\mathbf{x}, t)$ that is higher than $\gamma\sigma$ (where σ is the overall rms surface displacement and $\gamma \gg 1$), when normalized by $\bar{\zeta}(\mathbf{x},t)$ for $\zeta > \gamma\sigma$, is the space–time autocorrelation function $\rho(\mathbf{r}, t) = \overline{\zeta(\mathbf{x}, t)\zeta(\mathbf{x} + \mathbf{r}, t + \tau)}/\overline{\zeta^2}$ for the entire wave field. This extends and simplifies an earlier result due to Boccotti and is consistent with a precise calculation of the one-dimensional case with $\mathbf{r} = 0$, involving the time history of measurements at a single point. The results are compared with buoy data obtained during the Surface Wave Dynamics Experiment and the agreement is found to be remarkably good.

1. Introduction

Extreme wave events, giant or rogue waves, have been responsible for many marine accidents, some involving loss of life. They occur under storm conditions when the waves are already high, perhaps amplified further by refraction in currents, such as the Agulhas Current off eastern South Africa or the Gulf Stream. The very highest individual waves, the rogue waves, seem to appear with little warning and may be regarded as the statistical extremes in an already rough random sea, occurring sporadically in space and time. Although the *occurrence* of these events may be random, it is of interest to inquire, as does Boccotti (1989), whether in the vicinity of extreme wave crests there is any predictable, expected configuration of the sea surface, any "organized structure" of the surface in space and time with which a mariner and his ship must cope.

Any such "organized structure" in a random sea (or random function in general) is not of course deterministic in the usual sense, but the selection of regions surrounding extreme maxima does extract a regularity from the randomness of the field as a whole. Suppose, for example, we have an extensive record $\zeta(\mathbf{x}, t)$ of surface displacement in a random wind-generated wave field as a function of position and time. From this record, let us pick out the high wave crests, extract instances of wave maxima lying between ζ_m and $\zeta_m + d\zeta_m$, where ζ_m is a large multiple of the rms wave height, and consider the nature of the surface displacements surrounding these maxima. At the maxima, the surface displacements are all in essence the same, ζ_m. Close to the maxima, the surface displacements $\zeta < \zeta_m$. The expected or mean surface displacement is somewhat less than ζ_m while still being very large, and the variance about this mean is small. As the distance in space or time from the maxima increases, the expected value of ζ decreases, and the variance among different realizations increases until ultimately, far from the extreme events, the order is lost, the expected position of the free surface is simply the mean water level, $\bar{\zeta} = 0$, and the variance is that of the overall wave field. This paper is concerned with the question, Given the existence of an extreme wave crest, what is the *expected* surface configuration surrounding this crest or, equivalently, the mean over many realizations of extreme events, and what is the distribution of variance about this mean? The question seems to have been asked first by Boccotti of the Università di Reggio Calabria in a series of papers (Boccotti 1981, 1984, 1988, 1989) that may not have received the attention they deserve.

As long as the wave field can be regarded as a linear superposition of components with random phase, having been generated over a large area in an uncorrelated way, the displacement of the sea surface can be regarded as a random Gaussian process; this description has been used with conspicuous success by Longuet-Higgins (from 1952 to 1984) and by others. However, it is less likely to be accurate in a consideration of extreme crests that may or may not be breaking, and whose profiles in either event are likely to be distorted by nonlinear

Corresponding author address: O. M. Phillips, Department of Earth and Planetary Sciences, The Johns Hopkins University, Olin Hall, 3400 N. Charles Street, Baltimore, MD 21218-2681.

dynamical processes. Longuet-Higgins (1963) showed that these processes overall lead to a Gram–Charlier distribution for the surface elevation at a single location, with a positive skewness proportional to the square of the wave slope, but the increased accuracy of this specification is offset by greatly increased complexity. We therefore suppose here that $\zeta(\mathbf{x}, t)$ is Gaussian and defer the questions of local nonlinear effects for later study.

Even with this assumption, the calculation of the expected surface displacement $\bar{\zeta}(\mathbf{x}, t)$, given that at the origin, say, ζ has a maximum whose value ζ_m is a substantial multiple of the rms value $\sigma = (\overline{\zeta^2})^{1/2}$, is extremely cumbersome and involves the higher moments of the spectrum, which are very sensitive to the small-scale, high-frequency cutoff. This sensitivity reflects the fact that in a broad spectrum, a large dominant wave may have near its crest a number of short gravity or capillary waves with many local maxima. In a consideration of the structure of the highest dominant waves, these local maxima are irrelevant, as Longuet-Higgins (1984) points out in his discussion of group wave statistics, and should be filtered out. The calculation of $\bar{\zeta}(t)$ surrounding high maxima in a simple time series is given below for the case of a narrow spectrum, but the two- and three-dimensional analogues are far more cumbersome and have not been carried through.

One of Boccotti's principal results, restated somewhat, is the following. Let $\rho(\tau)$ represent the autocorrelation function

$$\rho(\tau) = \overline{\zeta(t)\zeta(t+\tau)}/\overline{\zeta^2}$$

of a Gaussian wave field and let $\tau = \tau'$ be the time delay at which the correlation function attains its first minimum. Suppose that (i) at time $\tau = 0$ the surface displacement $\zeta = \beta$, large compared with $(\overline{\zeta^2})^{1/2}$, and $\dot{\zeta} = 0$ and (ii) at time τ', $\zeta = r\beta$. The expected surface configuration $\bar{\zeta}(\tau)$ is then given by

$$\frac{\bar{\zeta}(\tau)}{\beta} = [1 - (\rho(\tau'))^2]^{-1}\,[\rho(\tau) - \rho(\tau')\rho(\tau - \tau') + \{\rho(\tau - \tau') - \rho(\tau')\rho(\tau)\}r]. \quad (1.1)$$

Although Boccotti's derivation of (1.1) involves no approximations except that of a Gaussian field, the implicit assumption that when β is large, the instant $\tau = 0$ necessarily corresponds to a maximum (with $\ddot{\zeta} < 0$) is in fact equivalent to a narrow spectrum approximation, as will be seen in more detail later. Boccotti extends (1.1) to express the expected space–time configuration of the surface surrounding a high maximum at $\mathbf{x} = 0$, $t = 0$ by replacing $\rho(\tau)$ by the space–time correlation function $\rho(\mathbf{x}, \tau)$, retaining the specifications $\zeta(\mathbf{0}, 0) = \beta$, $\dot{\zeta}(\mathbf{0}, 0) = 0$, and $\zeta(\mathbf{0}, \tau') = r\beta$. In general, these specifications do not define a maximum in ζ at $(\mathbf{0}, 0)$ since in addition, one needs $\nabla\zeta = 0$, $\zeta_{xx}, \zeta_{yy} < 0$ and $\zeta_{xx}\zeta_{yy} - (\zeta_{xy})^2 > 0$. Nevertheless, in the limit where $\beta/(\overline{\zeta^2})^{1/2}$ is very large, one might safely conjecture that points satisfying Boccotti's conditions, which occur rarely, must be close to true maxima. To illustrate his result Boccotti calculates the space–time autocorrelation by inverting the wave spectrum using the JONSWAP spectrum as an example, defines a wave group surrounding extreme events by the extension of (1.1), and gives numerical examples (Boccotti 1988) of the propagation, reflection, and diffraction of such a group.

In this paper, we return to the original question of the expected configuration of extreme wave events and the fluctuations about this configuration. A simple approximate method is given that provides an expression for the configuration in space–time, $\bar{\zeta}(\mathbf{x}, \tau)$ surrounding an extreme maximum at $\mathbf{x} = 0$, $\tau = 0$, which is simpler than (1.1) but in essence equivalent to it. This approximation is confirmed by a detailed calculation of the more restricted one-dimensional problem of the form of $\bar{\zeta}(\tau)$ surrounding extreme maxima in the narrow spectrum limit. Finally, these expressions are compared with buoy data obtained in the Surface Wave Dynamics Experiment (SWADE: Weller et al. 1991) during the storms of October 1990 and are found to be surprisingly accurate.

2. A simple approximation

Rather than seeking the precise points where realizations of $\zeta(\mathbf{x}, t)$ attain maxima, let us consider those regions where $\zeta \geqslant \gamma(\overline{\zeta^2})^{1/2} = \gamma\sigma$, where γ is (formally) a number large compared with unity. For a given γ, at any instant these regions consist of isolated islands, each containing at least one maximum, and as γ increases, the islands shrink, converging towards the maxima and then disappearing. Our interest is in large values of γ, where there are rare, small, isolated islands in which $\zeta \geqslant \gamma\sigma$ and allows us to pose the question thus: Given that at $\mathbf{x}$, t, say, $\zeta > \gamma\sigma$ where γ is large, what is the expected distribution of ζ in the vicinity, and what is the standard deviation about this expected value? The expected distribution in space and time describes the configuration and evolution of the extreme wave events and the standard deviation is the random uncertainty.

Accordingly, let $\zeta_1 = \zeta(\mathbf{x}, t)$ and $\zeta_2 = \zeta(\mathbf{x} + \mathbf{r}, t + \tau)$. From the theorem of conditional probability, the distribution of ζ_2 given that $\zeta_1 \geqslant \gamma\sigma$ is, in the usual notation,

$$p(\zeta_2 | \zeta_1 \geqslant \gamma\sigma) = p(\zeta_2, \zeta_1 \geqslant \gamma\sigma)/p(\zeta_1 \geqslant \gamma\sigma).$$

In a Gaussian process, the two-point probability density function

$$p(\zeta_1, \zeta_2) = [2\pi\sigma^2(1 - \rho^2)^{1/2}]^{-1} \times \exp\left\{-\frac{(\zeta_1^2 - 2\rho\zeta_1\zeta_2 + \zeta_2^2)}{2\sigma^2(1 - \rho^2)}\right\}, \quad (2.1)$$

where $\rho(\mathbf{r}, \tau)$ is the correlation function between ζ_1 and ζ_2. Then

$$p(\zeta_2, \zeta_1 \geqslant \gamma\sigma) = \int_{\gamma\sigma}^{\infty} p(\zeta_1, \zeta_2) d\zeta_1,$$

$$= \frac{\exp(-\zeta_2^2/2\sigma^2)}{2\pi\sigma^2(1-\rho^2)^{1/2}} \int_{\gamma\sigma}^{\infty} \exp\left\{-\frac{(\zeta_1 - \rho\zeta_2)^2}{2\sigma^2(1-\rho^2)}\right\} d\zeta_1,$$

since $\zeta_1^2 - 2\rho\zeta_1\zeta_2 + \zeta_2^2 = (\zeta_1 - \rho\zeta_2)^2 + (1-\rho^2)\zeta_2^2$. The integral can be simplified by a change of variable, and

$$p(\zeta_2, \zeta_1 \geqslant \gamma\sigma) = \frac{\exp(-\zeta_2^2/2\sigma^2)}{\pi\sqrt{2}\sigma} \int_{u_0}^{\infty} e^{-u^2} du, \quad (2.2)$$

where

$$u_0 = \frac{\gamma\sigma - \rho\zeta_2}{\sigma[2(1-\rho^2)]^{1/2}}.$$

Also,

$$p(\zeta_1 \geqslant \gamma\sigma) = \frac{1}{\sqrt{(2\pi)}\,\sigma} \int_{\gamma\sigma}^{\infty} \exp\left(-\frac{\zeta_1^2}{2\sigma^2}\right) d\zeta_1,$$

$$= \frac{1}{\sqrt{(2\pi)}\,\gamma} e^{-\frac{1}{2}\gamma^2} f(\gamma), \quad \text{say}, \quad (2.3)$$

where the function $f(\gamma)$, illustrated in Fig. 1, is about 0.9 for realistically important values of γ. Its asymptotic form (Abramowitz and Stegun 1964, p. 298) is

$$f(\gamma) \sim 1 - \gamma^{-2} + 3\gamma^{-4} - \cdots \quad \text{for } \gamma \gg 1. \quad (2.4)$$

Thus, from (2.2) and (2.3),

$$p(\zeta_2 | \zeta_1 \geqslant \gamma\sigma) = \frac{1}{\sigma\sqrt{\pi}} \frac{\gamma e^{\frac{1}{2}\gamma^2}}{f(\gamma)} \exp\left(-\frac{\zeta_2^2}{2\sigma^2}\right) \int_{u_0}^{\infty} e^{-u^2} du. \quad (2.5)$$

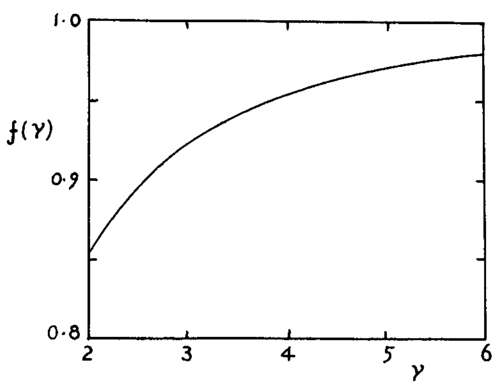

FIG. 1. The function $f(\gamma)$ of Eq. (2.3).

The expected value of $\zeta_2 = \zeta(\mathbf{x} + \mathbf{r}, t + \tau)$, given that $\zeta_1 = \zeta(\mathbf{x}, t) \geqslant \gamma\sigma$, is then

$$\overline{\zeta_2} = \int_{-\infty}^{\infty} \zeta_2 p(\zeta_2 | \zeta_1 \geqslant \gamma\sigma) d\zeta_2$$

$$= \gamma\sigma\rho[f(\gamma)]^{-1} \quad (2.6)$$

after an integration by parts and some reduction. In particular, when $\mathbf{r} = 0$, $t = 0$, the average height of those waves higher than $\gamma\sigma$ is $\gamma\sigma[f(\gamma)]^{-1}$, since the autocorrelation function is unity, so that

$$\frac{\bar{\zeta}(\mathbf{x} + \mathbf{r}, t + \tau)}{\bar{\zeta}(\mathbf{x}, t)_{\zeta > \gamma\sigma}} = \rho(\mathbf{r}, \tau). \quad (2.7)$$

The expected profile of the water surface surrounding extreme events, normalized with respect to the extreme wave height, is simply the overall space–time autocorrelation function of the sea surface; the simplicity and generality of this result is somewhat surprising.

The variance about this mean among different realizations of extreme maxima can also be found simply. From (2.5),

$$\overline{\zeta_2^2} = \int_{-\infty}^{\infty} \zeta_2^2 p(\zeta_2 | \zeta_1 \geqslant \gamma\sigma) d\zeta_2,$$

$$= \frac{1}{\sigma\sqrt{\pi}} \frac{\gamma e^{\frac{1}{2}\gamma^2}}{f(\gamma)} \int_{-\infty}^{\infty} \zeta_2^2 \exp\left(-\frac{\zeta_2^2}{2\sigma^2}\right) \int_{u_0}^{\infty} e^{-u^2} du d\zeta_2,$$

$$= \frac{2\sqrt{2}}{\sqrt{\pi}} \frac{\gamma e^{\frac{1}{2}\gamma^2}}{f(\gamma)} \sigma^2 I(\gamma, \rho), \quad (2.8)$$

where

$$I(\gamma, \rho) = \int_{-\infty}^{\infty} y^2 e^{-y^2} \int_{u_0}^{\infty} e^{-u^2} du dy,$$

and

$$u_0 = \frac{\gamma - \sqrt{2}\rho y}{[2(1-\rho^2)]^{1/2}}.$$

The integral can be evaluated by differentiation with respect to the parameter γ, and it is found that

$$\frac{dI}{d\gamma} = -\frac{\sqrt{\pi}}{2\sqrt{2}} \{\rho^2\gamma^2 + (1-\rho^2)\} e^{-\frac{1}{2}\gamma^2}.$$

Since $I \to 0$ as $\gamma \to \infty$,

$$I = \frac{\sqrt{\pi}}{2\sqrt{2}} e^{-\frac{1}{2}\gamma^2} \{\gamma\rho^2 + \gamma^{-1} f(\gamma)\},$$

and so

$$\overline{\zeta_2^2} = \sigma^2 \left\{\frac{\gamma^2\rho^2}{f(\gamma)} + 1\right\}. \quad (2.9)$$

Hence,

$$\overline{(\zeta_2-\bar{\zeta}_2)^2}=\overline{\zeta_2^2}-\overline{\zeta_2}^2=\sigma^2\left\{1-\frac{\gamma^2\rho^2}{f^2}(1-f)\right\}$$
$$\sim\sigma^2\{1-\rho^2(1-\gamma^{-2})\}\quad\text{for}\quad\gamma\gg1,\quad(2.10)$$

from (2.4).

Near the crests of extreme waves with $\zeta\geqslant\gamma\sigma$, where $\rho\approx1$, the variance about the mean is very small, approximately $\gamma^{-2}\sigma^2$, where σ^2 is the variance or mean square displacement of the wave field as a whole. The crest heights of almost all waves with $\zeta\geqslant\gamma\sigma$, $\gamma\gg1$ are only slightly greater than $\gamma\sigma$ because of the rapid drop-off of the probability distribution. With increasing distance (in space or time) from the crest, the variance increases but remains less than or equal to σ^2 as the envelope of the correlation function decreases. As γ increases, we consider rarer and more extreme wave events, and the random variations from the mean among such events become an even smaller fraction of the maximum wave height.

Boccotti's (1989) results are couched in terms of the asymptotic limit $\gamma\to\infty$, and he writes of "the quasi-determinism of the highest waves" in a random sea. His expression (1.1) is considerably more complicated than our (2.7), which reduces to his (1.1) on the recognition that, at a given position with $\mathbf{r}=0$, since ζ is Gaussian, $\bar{\zeta}(\tau)$, like $\rho(\tau)$, is an even function of τ. Thus,

$$\frac{\bar{\zeta}(\tau)}{\beta}=\frac{\bar{\zeta}(-\tau)}{\beta}$$
$$=[1-(\rho(\tau'))^2]^{-1}[\rho(\tau)-\rho(\tau')\rho(\tau-\tau')+\{\rho(\tau-\tau')-\rho(\tau')\rho(\tau)\}r],$$

and by comparison with (1.1) it is found that

$$\{r-\rho(\tau')\}\{\rho(\tau-\tau')-\rho(-\tau-\tau')\}=0$$

for all τ. The second factor is not zero for all τ so that $r=\rho(\tau')$. Substitution into (1.1) gives

$$\frac{\bar{\zeta}(\tau)}{\beta}=\rho(\tau),$$

which is the limit of (2.6) or (2.7) as $\gamma\to\infty$ and $f(\gamma)\to1$. Despite the somewhat different formulations of the question, the asymptotic results are the same—as our intuition might have suggested.

3. A more precise calculation for $\bar{\zeta}(\tau)$

The principal limitation in the approximate method of the previous section is that the locations of extreme maxima are defined only as being in the "islands" for which $\zeta\geqslant\gamma\sigma$. When γ is large, the islands are very rare and very small so that there is little uncertainty in the definition of the point where $\mathbf{r}=0$; $\tau=0$; but when γ is only moderate, the "islands" are larger, and the expressions found for nominal values of $\mathbf{r}$ and τ, in fact, represent averages in the space or time domain over intervals equal to the size of each island. A more precise calculation is the direct one: given that at time t_m, say, $\zeta(t)$ has a local maximum with $\zeta(t_m)\geqslant\gamma\sigma$; then what is the expected value of $\zeta(t_m+\tau)$? This calculation, even for a single variable, is a good deal more cumbersome than the approximate method for estimating $\bar{\zeta}$ as a function of both space and time, but it does serve as a check on the asymptotic results for $\gamma\gg1$, as well as a comparison benchmark for the approximation at the moderate values of γ where its interpretation begins to become fuzzy.

Let $\zeta_1=\zeta(t+\tau)$, $\zeta_2=\zeta(t)$, $\zeta_3=\dot{\zeta}(t)$, and $\zeta_4=\ddot{\zeta}(t)$. The probability density function of $\zeta(t_m+\tau)$, assuming that $\zeta(t_m)$ is a maximum with $\zeta(t_m)\geqslant\gamma\sigma$, is

$$p(\zeta(t_m+\tau))=\frac{\int_{\gamma\sigma}^{\infty}d\zeta_2\int_{-\infty}^{0}p(\zeta_1,\zeta_2,\zeta_3=0,\zeta_4)|\zeta_4|\,d\zeta_4}{\int_{\gamma\sigma}^{\infty}d\zeta_2\int_{-\infty}^{0}p(\zeta_2,\zeta_3=0,\zeta_4)|\zeta_4|\,d\zeta_4},\quad(3.1)$$

where $p(\zeta_1,\zeta_2,\zeta_3,\zeta_4)$ and $p(\zeta_2,\zeta_3,\zeta_4)$ are the joint probability density functions for the variables named. In a Gaussian wave field,

$$p(\zeta_2,\zeta_3=0,\zeta_4)=(2\pi)^{-3/2}\{m_2(m_0m_4-m_2^2)\}^{-1/2}\times\exp\left\{-\frac{m_4\zeta_2^2+2m_2\zeta_2\zeta_4+m_0\zeta_4^2}{2(m_0m_4-m_2^2)}\right\},\quad(3.2)$$

where m_i represents the ith moment of the spectrum. Also,

$$p(\zeta_1,\zeta_2,\zeta_3=0,\zeta_4)=(2\pi)^{-2}|\mathbf{M}_{ij}|^{-1/2}\exp\left\{-\frac{1}{2}\sum_{i,j=1,2,4}M_{ij}\zeta_i\zeta_j\right\},\quad(3.3)$$

where $\mathbf{M}_{ij}$ is the matrix inverse of

$$\mathbf{D}_A=\begin{bmatrix} m_0 & \psi(\tau) & -\dfrac{\partial\psi(\tau)}{\partial\tau} & \dfrac{\partial^2\psi(\tau)}{\partial\tau^2}\\ \psi(\tau) & m_0 & 0 & -m_2\\ -\dfrac{\partial\psi(\tau)}{\partial\tau} & 0 & m_2 & 0\\ \dfrac{\partial^2\psi(\tau)}{\partial\tau^2} & -m_2 & 0 & m_4\end{bmatrix},\quad(3.4)$$

with $\psi(\tau)=\overline{\zeta(t)\zeta(t+\tau)}=\overline{\zeta^2}\rho(\tau)$. The expected value of $\zeta(t_m+\tau)$, given that $\zeta(t_m)$ is a maximum larger than $\gamma\sigma$, is therefore

$$\overline{\zeta(t_m+\tau)}=\int_{-\infty}^{\infty}\zeta_1p\{\zeta(t_m+\tau)\}d\zeta_1.\quad(3.5)$$

Evaluation of the integrals in closed form does not seem to be possible in general, but when the spectrum is narrow in the sense that $\Delta/\gamma^2 \ll 1$ or $\gamma^2/\Delta \gg 1$, where $\Delta = (m_0 m_4/m_2^2) - 1$, it is found that the integral of the numerator of (3.1) involved in (3.5) is

$$\frac{\gamma\sqrt{m_2}}{2\pi m_0}\psi(\tau)e^{-\gamma^2/2} - (2\pi^2 m_2)^{-1/2}\frac{\partial^2\psi}{\partial\tau^2}\int_{\gamma/\sqrt{2}}^{\infty} e^{-u^2}du, \quad (3.6)$$

with a relative error of order $(\sqrt{\Delta}/\gamma)\exp(-\gamma^2/2\Delta)$. The details of the integrations can be obtained from the authors by request. The numerator can likewise be evaluated as

$$\frac{\sqrt{m_2}}{2\pi\sqrt{m_0}}e^{-\gamma^2/2}\{1 + O((\sqrt{\Delta}/\gamma)\exp-\gamma^2/2\Delta)\}. \quad (3.7)$$

Since the integral in (3.6) can be expressed as

$$\frac{1}{\gamma\sqrt{2}}e^{-\gamma^2/2}f(\gamma),$$

we obtain from (3.5), (3.6), and (3.7) that

$$\overline{\zeta(t_m + \tau)} = \gamma\sigma\rho(\tau) - \frac{f(\gamma)\sigma^3}{\gamma m_2}\frac{\partial^2\rho(\tau)}{\partial\tau^2}. \quad (3.8)$$

Now,

$$m_2 = -\sigma^2[d^2\rho/d\tau^2]_0,$$

so that the mean height of the maxima higher than $\gamma\sigma$, obtained by putting $\tau \to 0$ in (3.8), is

$$\bar{\zeta}(t_m) = \gamma\sigma\left\{1 + \frac{f(\gamma)}{\gamma^2}\right\},$$

and

$$\frac{\overline{\zeta(t_m + \tau)}}{\bar{\zeta}(t_m)} = \left\{\rho(\tau) + \frac{f(\gamma)}{\gamma^2}\frac{\ddot{\rho}(\tau)}{\ddot{\rho}(0)}\right\}\left\{1 + \frac{f(\gamma)}{\gamma^2}\right\}^{-1}, \quad (3.9)$$

which agrees with the approximation (2.7) (for the restricted case $\mathbf{r} = 0$) to the lowest order in γ^{-2}.

The variations of $\zeta(t_m + \tau)$ about the mean among different realizations of extreme maxima can likewise be calculated when $\gamma^2/\Delta \gg 1$:

$$\overline{\zeta^2(t_m + \tau)} = \int_{-\infty}^{\infty}\zeta_1^2 p\{\zeta(t_m + \tau)\}d\zeta_1,$$

and

$$\begin{aligned}&\overline{[\zeta(t_m + \tau) - \bar{\zeta}(t_m + \tau)]^2} \\ &= \overline{\zeta^2(t_m + \tau)} - \overline{\zeta(t_m + \tau)}^2 \\ &= \sigma^2[1 - \rho^2(\tau)] - \frac{\sigma^4}{m_2}\times \\ &\quad\times\left\{\left(\frac{d\rho}{d\tau}\right)^2 + 2\rho\frac{d^2\rho}{d\tau^2}(1 - f(\lambda)) + \frac{1}{\gamma^2}\frac{\sigma^2}{m_2}\left(\frac{d^2\rho}{d\tau^2}\right)^2 f^2(\gamma)\right\}. \quad (3.10)\end{aligned}$$

The approximate theory of section 2 does not, of course, provide the terms in (3.10) involving time derivatives of ρ, of which the first is the most important when γ is large. In either case, $\overline{(\zeta - \bar{\zeta})^2}$ is of order γ^{-2} when $\tau = 0$ and approaches σ^2 as the time interval increases and $\rho(\tau) \to 0$. As γ increases the rms variations about the mean become a decreasing fraction of the maximum elevation itself.

4. Comparison with buoy data from SWADE

The results of the previous two sections have been compared with buoy measurements obtained during the Surface Wave Dynamics Experiment at the location Discus-North (38.37°N, 73.65°W, NDBC station 44001) during a time interval of 6 hours beginning at 1916 (UTC) on 26 October 1990. The wind speed averaged 19.6 m s^{-1}. The buoy was a discus of 3-m diameter (Steele et al. 1992), modified as described by Weller et al. (1991). The vertical acceleration of the buoy was digitized at 1 Hz and recorded on board. The data were later filtered numerically to exclude periods

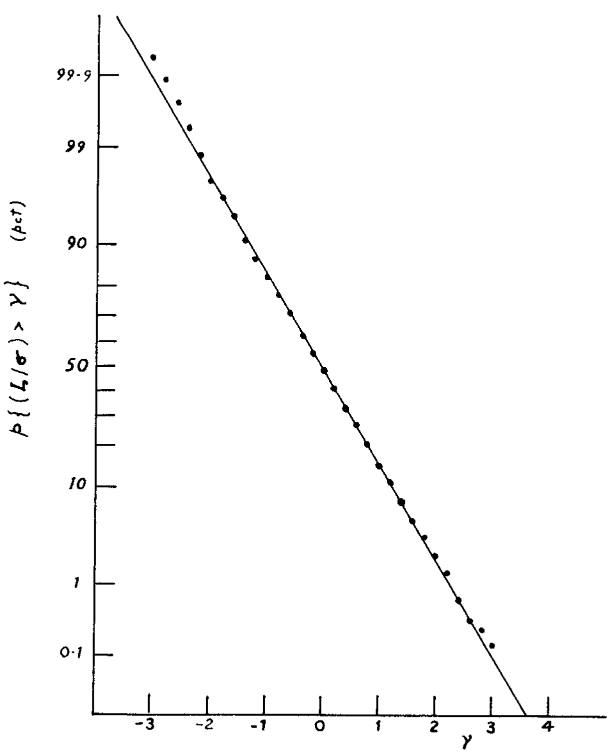

FIG. 2. Cumulative probability density function of the surface displacement.

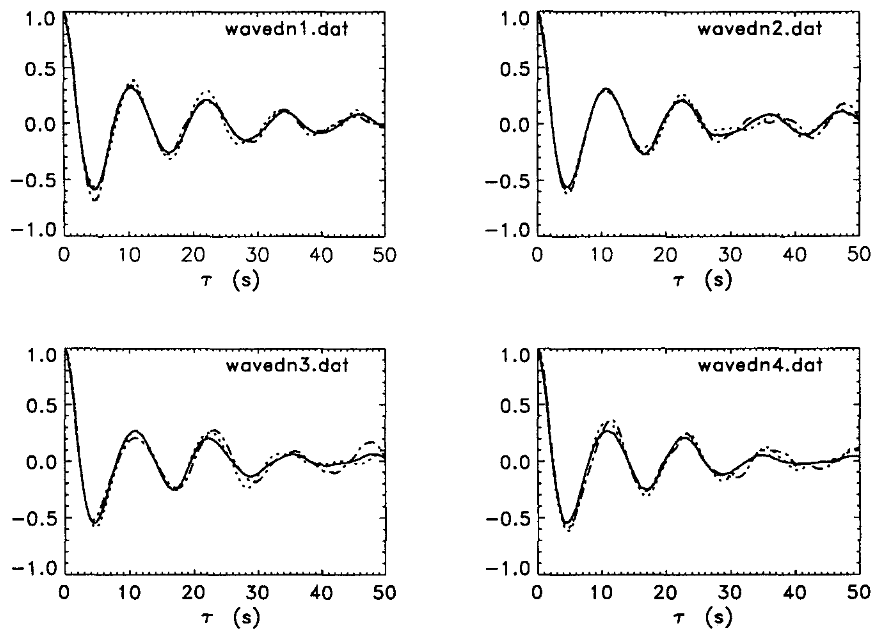

FIG. 3. Mean profiles of surface displacement around wave crests higher than 2.0σ in the four datasets—the dotted curves being for times following the crest and the dot–dash lines preceding them. The solid curve is the expression (3.9) calculated from the overall autocorrelation function for surface displacement.

longer than 25 sec and then integrated twice to provide $\zeta(t)$ at 1 Hz. Four sequences of data, each ninety minutes long, were chosen to represent reasonably steady wind and wave conditions for evaluation of the theoretical results. The surface elevations measured by the buoy were closely Gaussian. Figure 2 shows a representative cumulative pdf in which the departures from a Gaussian distribution are most evident as a slight deficiency of very low values (deep troughs). Each of the four data segments, labeled wavedyn 1 to 4, contained about 750 wave crests, of which about 10% were higher than 2σ and 1% higher than 3σ. This is consistent

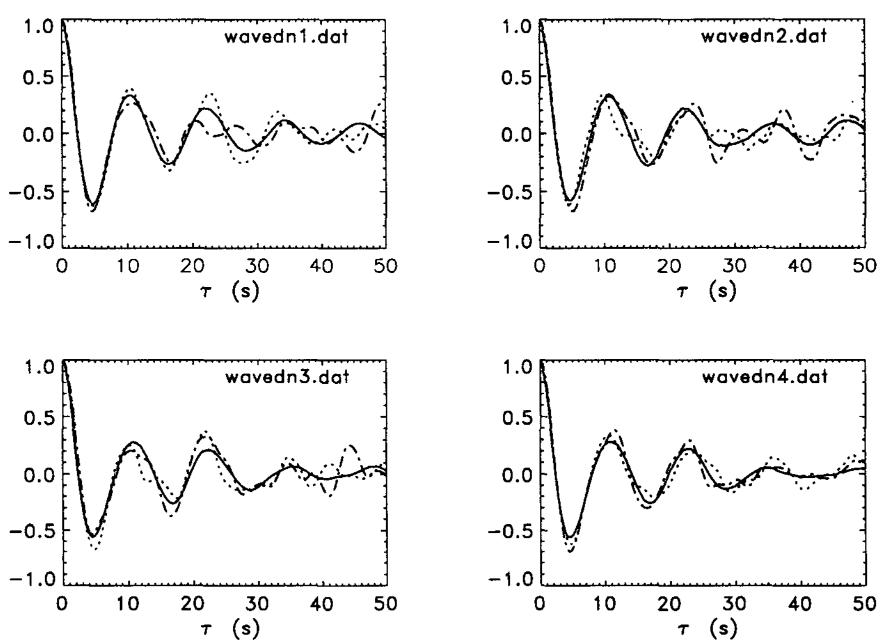

FIG. 4. As for Fig. 3 but for wave crests higher than 2.5σ.

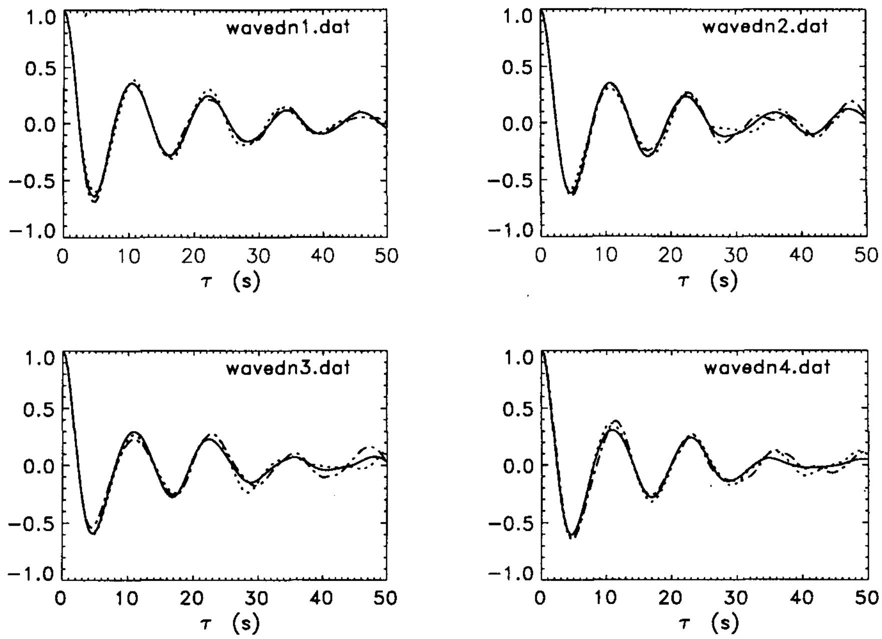

FIG. 5. Mean profiles of surface displacement at time τ before and after points where $\zeta(t) > 2\sigma$. The continuous curve represents the approximation (2.7).

with a value of the spectral width parameter ϵ of Cartwright and Longuet-Higgins (1956) of about 0.6 and of our Δ of 1.0. The relative errors in the expressions (3.6) and (3.7) are thus of order 5%.

The autocorrelation function $\bar{\rho}(\tau) = \overline{\zeta(t)\zeta(t+\tau)}/\overline{\zeta^2}$ was calculated for each dataset. The records were then searched for maxima higher than 2σ, the time of occurrence being denoted as t_m, and the mean surface displacement in the vicinity of these maxima $\bar{\zeta}(t_m + \tau)$ was found as a fraction of $\bar{\zeta}(t_m)$ by averaging over this ensemble. The resulting average configurations are shown in Fig. 3, the dotted line indicating $\tau > 0$ and the dot–dash line $\tau > 0$, preceding the maxima. The expected configuration calculated from $\rho(\tau)$ using the

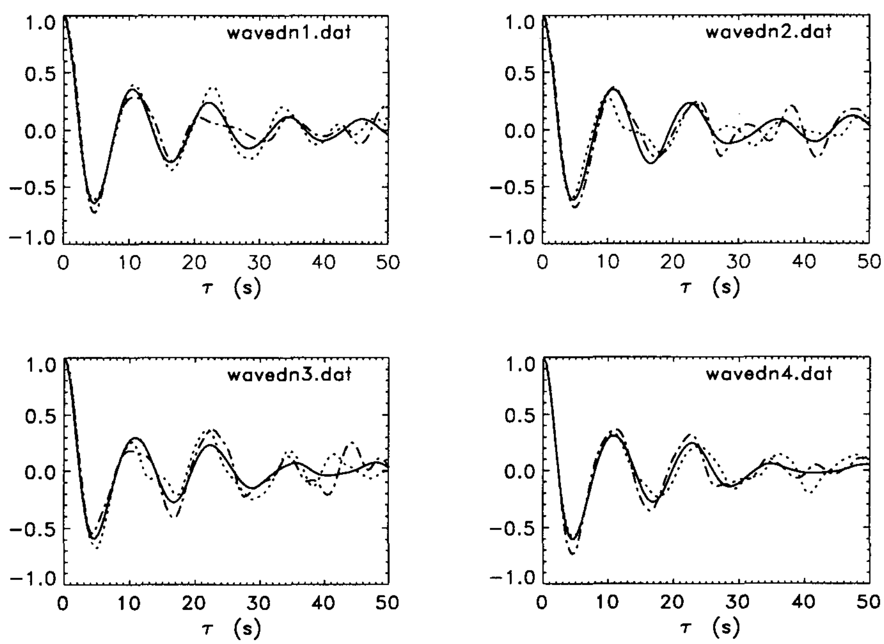

FIG. 6. As for Fig. 4 but for $\zeta(t) > 2.5\sigma$.

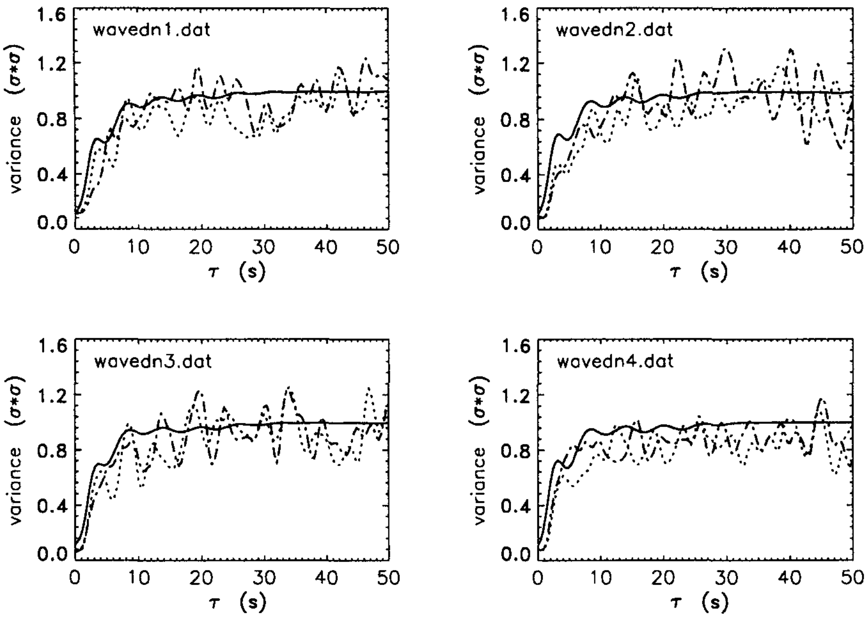

FIG. 7. Predicted and measured variances about the mean profiles surrounding wave crests higher than 2σ.

narrow spectrum approximation (3.9) is shown as the solid line; the agreement is quite surprising in view of the relative smallness of γ (i.e., 2). As γ increases, the theoretical expressions are expected to become more accurate, but the statistics deteriorate since fewer such maxima occur in records of finite length. Corresponding curves for $\gamma = 2.5$, shown in Fig. 4, exhibit more scatter since now the mean profiles are averaged over only 15–20 events, and the standard deviation in the estimate of the mean is larger.

Figures 5 and 6 compare the simple approximate expression (2.7), (with $\mathbf{r} = 0$) with averaged values of $\overline{\zeta(t+\tau)}$ given that $\zeta(t) > \gamma\sigma$, and the results are found to be almost indistinguishable from those of Figs. 3

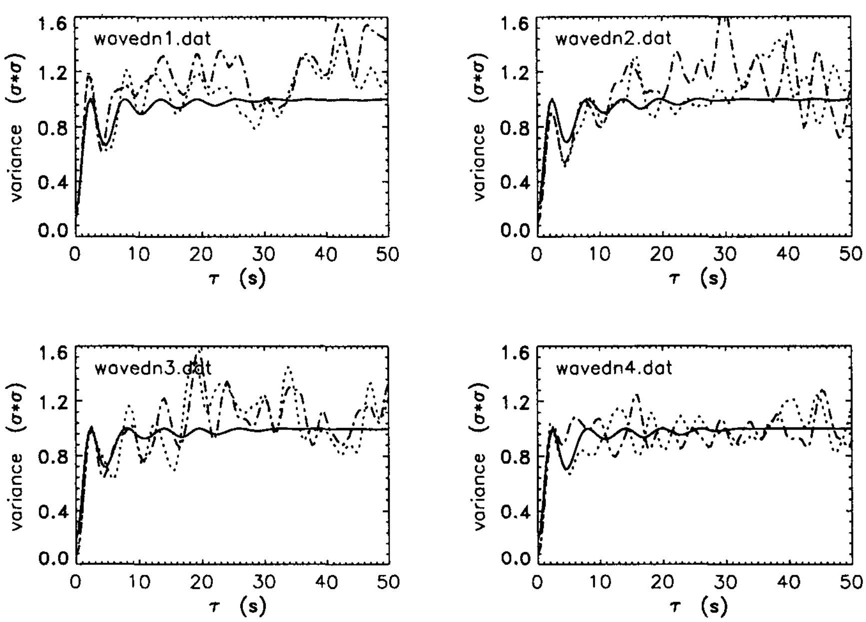

FIG. 8. Measured variances around regions where $\zeta(t) > 2.0\sigma$ compared with the approximate expression §2.

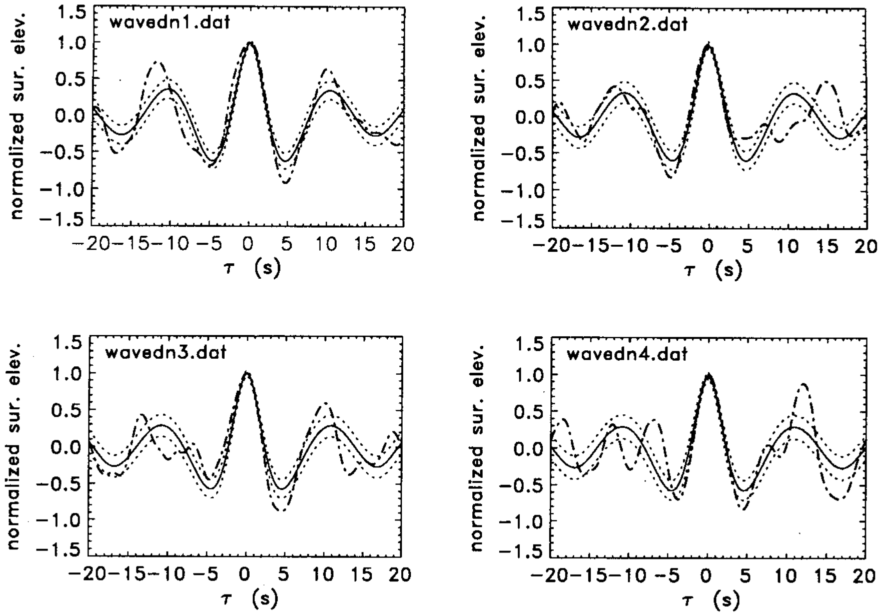

FIG. 9. Profiles surrounding the highest individual wave crests in each of the datasets. In wavedyn 1 on the upper left, the crest height was 4.66 m above mean water level, corresponding to $\gamma = 3.74$. On the upper right, $\zeta_m = 3.86$ m, $\gamma = 3.26$; lower left $\zeta_m = 3.88$ m, $\zeta = 3.31$; and lower right $\zeta_m = 3.58$ m, $\gamma = 3.05$. The expected profile is the solid curve, with $\pm\sigma$ the dotted curves; the individual measured profiles are the dot–dash curves.

and 4. The expected variances about the mean profiles, calculated from Eq. (3.10) are shown in Fig. 7 for $\gamma = 2.0$, together with those measured at points before and after the maxima. The measured variances approach those of the wave field itself somewhat more slowly than the theory would suggest; the scatter is a good deal larger when $\gamma = 2.5$ because of the smaller sample size. Figure 8 shows corresponding variances at time $t + \tau$ given $\zeta(t) > \gamma\sigma$ (rather than being a precise maximum); these seem to be generally somewhat larger than in the previous case as one might possibly have anticipated.

As we examine ever higher crests, the analysis predicts that the expected root-mean-square variation about the expected value in the vicinity of the crests becomes an ever smaller fraction of the crest height. To examine this, we selected the largest individual wave crests in each record; these are shown as the dot–dash lines in Fig. 9. The solid line represents the expected profile calculated from the correlation function, and the dotted lines give the predicted standard deviation. In each case, the shape of the highest wave itself is predicted with good fidelity, though neighboring waves seem to be underpredicted. On the other hand, the agreement between these individual realizations and the theory is best for the very highest wave with $\gamma = 3.74$ and less persuasive for the highest wave in wavedyn 4 for which $\gamma = 3.05$.

Acknowledgments. It is a pleasure to acknowledge the support of the Office of Naval Research under Grant N00014-90-J-1623 for the research of OMP and DG, and Grant N00014-88-J-1028 for the research of MD.

REFERENCES

Abramowitz, M., and I. A. Stegun, 1964: *Handbook of Mathematical Functions.* Nat. Bur. Stand. Appl. Math. Ser. 55, 1045 pp.

Boccotti, P., 1981: On the highest waves in a stationary Gaussian process. *Atti. Acc. Lig.*, **38**, 45–73.

——, 1984: Sea waves and quasi-determinism of rare events in random processes. *Atti. Acc. Naz. Lincei.*, **76**(2), 132–157.

——, 1988: Refraction, reflection, and diffraction of irregular gravity waves. *Excerpta*, **3**, 47–89.

——, 1989: On mechanics of irregular gravity waves. *Atti Acc. Naz. Lincei Mem.*, **19**, 109–170.

Cartwright, D. E., and M. S. Longuet-Higgins, 1956: The statistical distribution of the maxima of a random function. *Proc. Roy. Soc., A* **237**, 212–232.

Longuet-Higgins, M. S., 1952: The statistical distribution of the heights of sea waves. *J. Mar. Res.*, **11**, 245–266.

——, 1963: The effects of nonlinearities on statistical distributions in the theory of sea waves. *J. Fluid. Mech.*, **17**, 459–480.

——, 1984: Statistical properties of wave groups in a random sea state. *Phil. Trans. Roy. Soc. London A***312**, 219–250.

Steele, K. E., C.-C. Teng, and D. W. C. Wang, 1992: Wave direction measurements using pitch–roll buoys. *Ocean Eng.*, **19**, 349–375.

Weller, R. A., M. A. Donelan, M. G. Briscoe, and N. E. Huang, 1991: Riding the crest: A tale of two wave experiments. *Bull. Amer. Meteor. Soc.*, **72**, 163–183.