A Theory of Large-Amplitude Kelvin Waves[1]

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ABSTRACT

A simple nonlinear model of the generation of Kelvin waves is presented and applied to internal Kelvin waves in Lake Michigan. It is shown that a Kelvin wave which has a wavelength longer than the Rossby radius of deformation steepens. This may explain "warm fronts" in records of nearshore temperature in Lake Michigan.

1. Introduction

This study was motivated by the observations of Mortimer (1963) of large variations in the depth of the thermocline near the shores of Lake Michigan. He found that, although wind-induced vertical motion could explain much of the data, some incidents suggested the propagation of internal Kelvin waves. Csanady (1967, 1968a, b) supported this theory by showing with a simple two-layer model that Kelvin waves of sufficient amplitude would be generated by wind stress. He noted that for many purposes these waves would be indistinguishable from the steady currents which Charney (1955) called "baroclinic coastal jets," since their frequency is small compared to the Coriolis parameter.

Baroclinic edge-waves in nature can be expected to deviate in many ways from linear Kelvin waves in a basin of constant depth. Undoubtedly the nearshore topography is important (Csanady, 1971). It is equally certain that finite-amplitude effects are important since it is not uncommon for the thermocline to intersect the surface. Also, since the observed relative vorticity is comparable to the Coriolis parameter, one can expect the nonlinear acceleration terms to be important. The effects of a continuous vertical density distribution discussed by Csanady (1972) are probably of lesser importance in the Great Lakes but may be important in oceans and lakes with more detailed vertical density structures.

Nonlinear Kelvin waves have been discussed previously by Saylor (1970) and Smith (1972). Saylor noted that nonlinear effects could be expected since Kelvin waves are nondispersive. Smith found that for small-amplitude waves, dispersion due to topography could balance the nonlinearity.

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This work will concentrate on the simplest case—that of a Kelvin wave generated from a state of rest in a single layer of fluid of uniform depth. The results, however, will be used to infer nonlinear effects which may occur in a two-layer system, such as Lake Michigan.

2. The general problem

The water is assumed to obey the shallow water approximation and the longshore component of the current is assumed to be geostrophic. The range of validity of these approximations will be discussed in Section 4. The motion is generated by a longshore acceleration, $F(y,t)$, which has zero curl. The governing equations are:

$$-fv=-g\frac{\partial h}{\partial x}, \tag{2.1}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+fu=-g\frac{\partial h}{\partial y}+F(y,t), \tag{2.2}$$

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(uh)+\frac{\partial}{\partial y}(vh)=0. \tag{2.3}$$

The Coriolis parameter is f (assumed constant), g is gravity, u and v are velocities in the x (onshore) and y (longshore) directions, and h is the water depth. The water occupies the region $x<0$ (see Fig. 1).

At $t=0$, the depth is assumed equal to H, the equilibrium depth, and the geostrophic component of the flow is thus zero:

$$t=0: \quad h=H, \quad v=0. \tag{2.4}$$

At the shore the normal component of the flow is zero:

$$t\geqslant 0, \quad x=0: \quad u=0, \tag{2.5}$$

and far away from the shore, the depth is uniform and

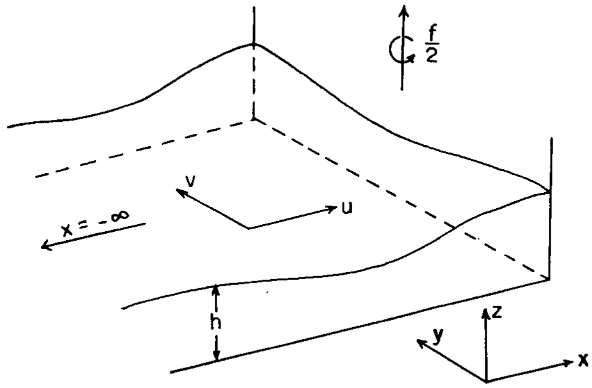

FIG. 1. Perspective sketch of section of shoreline, showing definition of variables used in this study.

the current is simply wind drift:

$$t \geqslant 0, \quad x \to -\infty: \quad v=0, \quad h=H, \quad u=F/f. \quad (2.6)$$

This problem can be reduced by noting that the following potential vorticity law can be derived from (2.1)–(2.3):

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\left[\frac{\partial v/\partial x+f}{h}\right]=0. \quad (2.7)$$

Therefore, since the potential vorticity is initially uniform, it will be uniform for all time. Setting it equal to the initial potential vorticity, f/H, and using (2.1) gives

$$\frac{gH}{f^2}\frac{\partial^2 h}{\partial x^2}-h=-H. \quad (2.8)$$

The general solution for h which satisfies (2.8) and the boundary condition (2.6) must, therefore, have the form

$$h=H+Q(y,t)\exp[xf/(gH)^{\frac{1}{2}}]. \quad (2.9)$$

From (2.1), v must be of the form

$$v=\sqrt{\frac{g}{H}}Q(y,t)\exp[xf/(gH)^{\frac{1}{2}}]. \quad (2.10)$$

By inserting these expressions into (2.2) and setting $x=0$ (noting that $u=0$ at $x=0$), one finds that Q must satisfy

$$\frac{\partial Q}{\partial t}+\sqrt{\frac{g}{H}}Q\frac{\partial Q}{\partial y}+\sqrt{gH}\frac{\partial Q}{\partial y}=\sqrt{\frac{H}{g}}F(y,t). \quad (2.11)$$

From either (2.2) or (2.3), it follows that the transport normal to the shore is

$$uh=u\{H+Q\exp[xf/(gH)^{\frac{1}{2}}]\}=\frac{HF}{f}\{1-\exp[xf/(gH)^{\frac{1}{2}}]\}$$
$$+\frac{g}{f}Q\frac{\partial Q}{\partial y}\{\exp[xf/(gH)^{\frac{1}{2}}]-\exp[2xf/(gH)^{\frac{1}{2}}]\}. \quad (2.12)$$

The first term, multiplied by the forcing function $F(y,t)$, is zero at the shore and at $x=-\infty$ gives the normal wind drift current there. The second term, the normal component of velocity in a free Kelvin wave, vanishes both at the shore and at $x=-\infty$. It is of second order in the amplitude and is therefore zero in the linear theory. The longshore component of the current is the same as in the linear theory since it is geostrophic.

This separation of variables reduces the original problem given by (2.1)–(2.6) to the problem of solving (2.11) subject to the condition that $Q=0$ at $t=0$. It is straightforward to do this by the method of characteristics. First, define a set of curves $Y_i(t)$ by

$$\frac{dY_i}{dt}=\sqrt{gH}\left[1+\frac{Q}{H}(Y_i,t)\right]. \quad (2.13)$$

On each curve (2.11) is

$$\frac{dQ}{dt}=\sqrt{\frac{H}{g}}F(Y_i,t). \quad (2.14)$$

The initial condition for each pair of equations is

$$t=0: \quad Y_i=Y_0, \quad Q=0. \quad (2.15)$$

3. A simple example

The theory reduces to the linear theory when $Q/H \ll 1$, for then the characteristics are straight lines with a slope of $+(gH)^{\frac{1}{2}}$, indicating Kelvin wave propagation in the positive y direction. To show how nonlinearity alters the solution, it suffices to examine a simple example. We choose

$$F(y,t)=-F_0\sin(y/R). \quad (3.1)$$

This corresponds to a wind independent of time and varying as a sine function in y; it would correspond to the tangential component of a uniform wind over a circular basin of radius R, as in the problem Csanady studied. The angle that the wind comes from is $\theta=y/R=-\pi$. It is natural to form the following nondimensional variables:

$$Q'=Q/H, \quad t'=\frac{t\sqrt{gH}}{R}, \quad \theta=y/R. \quad (3.2)$$

Eqs. (2.13) and (2.14) are then

$$\left.\begin{aligned}\frac{d\theta_i}{dt'}&=1+Q'(\theta_i,t)\\ t'=0:&\quad \theta_i=\theta_0\end{aligned}\right\}, \quad (3.3)$$

$$\left.\begin{aligned}\frac{dQ'}{dt'}&=-\frac{RF_0}{gH}\sin\theta_i\\ t'=0:&\quad Q'=0\end{aligned}\right\}. \quad (3.4)$$

From (3.4), it follows that, for a large-amplitude wave to be formed, F_0 must be comparable to gH/R. If F_0 is set equal to gH/R, and Q' is eliminated, the characteristics can be computed from

$$\frac{d^2\theta_i}{dt'^2}+\sin\theta_i=0,$$

$$\left.\begin{aligned}\theta_i&=\theta_0\\ \frac{d\theta_i}{dt'}&=1\end{aligned}\right\},\ t'=0. \tag{3.6}$$

This "pendulum" problem was solved with the aid of a table of elliptic functions. The total depth at the shore $(Q'+1)$ is shown in Fig. 2 for $t'=1.0$, along with the linear solution. Since there is little difference, it can be concluded that the linear theory describes the initial growth quite well even up to the point where the depth approaches zero. Since the depth becomes negative soon afterward, the forcing was set to zero at $t'\geqslant 1$. The solution for $1+Q'$ at $t'=1.5$ is shown in Fig. 3. Since the characteristics are straight lines of differing slope, they must eventually cross no matter what the amplitude is. For the wave studied here, this "breaking" occurs in a time of approximately a tenth of the period of the linear wave. In order to understand this phenomenon it is necessary to discuss the validity of the approximations.

4. The validity of the solution

The major approximation is the assumption that the longshore component of the flow is geostrophic. This will be valid as long as the term du/dt, missing from (2.1), is small compared to fv. The normal component u, as given by (2.12), is composed of two terms. One depends on the forcing function which in nature has a wide range of time and space scales. The only part of relevance here is that which has a time scale larger than f^{-1}. The same criterion applys to the acceleration term for the onshore component of the wind. Thus, this theory excludes one of the most interesting cases—that of a strong wind of short duration. The other term which does not vanish for $F=0$ is

$$uh=\frac{g}{f}Q\frac{\partial Q}{\partial y}\{\exp[xf/(gH)^{\frac{1}{2}}]-\exp[2xf/(gH)^{\frac{1}{2}}]\}.$$

Using this expression and (2.10), one can estimate the ratio

$$\frac{|du/dt|}{|fv|}\approx\frac{gH}{f^2R^2}\delta=\lambda^2\delta,$$

where δ is $|Q|/H$, the amplitude of the wave, and λ is the ratio of the Rossby radius of deformation $[(gH)^{\frac{1}{2}}/f]$ to the longshore length scale of the wave. Thus, the longshore component is geostrophic for long Kelvin waves or for small-amplitude waves. The nonlinear terms in (2) and (3), however, have a magnitude of δ. Thus, it is consistent to include these terms but drop the du/dt term only if $\lambda^2\ll 1$.

We conclude that a freely propagating Kelvin wave which has a wavelength larger than the Rossby radius of deformation ultimately steepens. Our theory describes the steepening up to the point where the width of the "front" is comparable to the radius of deformation. The steepening is accompanied by a large growth of the normal component of the current. This suggests that if the missing terms were included, the motion would no longer be confined to the coastal region; the adjustment might involve exchange of the coastal and interior water.

One cannot say whether the steepening continues until the hydrostatic theory is no longer valid. The work of Houghton (1969) implies that there is a critical amplitude above which the formation of a hydraulic jump occurs. For a Kelvin wave this critical amplitude is probably larger than for the inertio-gravity wave studied by Houghton, since in his problem

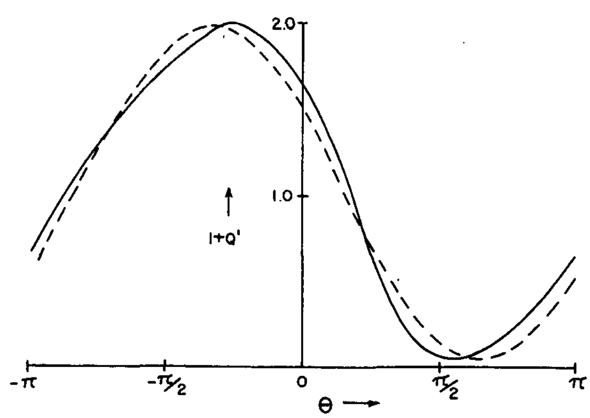

FIG. 2. Nondimensional depth of water at the shore of a circular basin at $t'=1.0$. The solid line is the nonlinear solution, and the dashed line the linear solution.

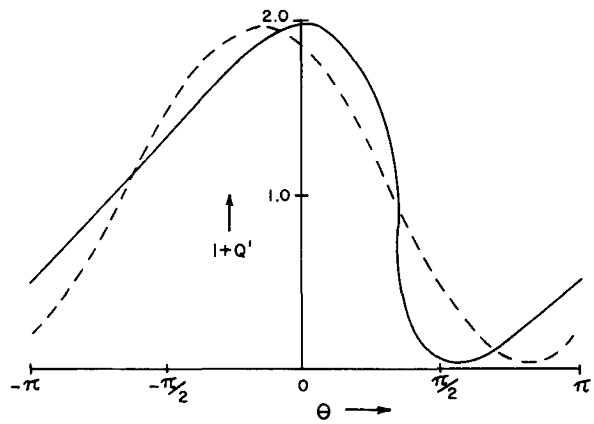

FIG. 3. Same as Fig. 2, but at $t'=1.5$ after the wind is stopped at $t'=1.0$.

the solution was constrained to be independent of one direction.

5. Application to Lake Michigan

It is empirically true that in most baroclinic coastal currents in the Great Lakes the velocity falls off rapidly with depth. Thus, in the stratified season a lake may behave qualitatively like an "equivalent one-layer" system, i.e., one obeying (2.1)–(2.3) where h is the depth of the thermocline and g is a reduced gravity, $(\Delta\rho/\rho)g$. If the longshore acceleration F is set equal to $\tau_y(y,t)/(\rho_0 H)$, where τ_y is the longshore wind stress, and the Kelvin wave velocity $c=[gH(\Delta\rho/\rho)]^{\frac{1}{2}}$ is defined, then (2.11) can be written

$$\frac{\partial Q}{\partial t}=\frac{\tau_y}{\rho_0 c}-c\frac{\partial Q}{\partial y}-c\frac{Q}{H}\frac{\partial Q}{\partial y}. \qquad (5.1)$$

This equation states that the rise and fall of the thermocline at the shore is due to three mechanisms. The first is the effect of a longshore wind which produces an Ekman drift of the surface water toward or away from the shore. The second is the propagation of linear Kelvin waves. The third is advection by the geostrophic longshore current. Charney (1955) considered only the first term, and Csanady (1968b) considered the first two. It is interesting to note that the Coriolis parameter does not appear in this equation.

For Lake Michigan in summer, the following values are typical:

$$\tau_y=\tfrac{1}{3}\ \text{dyn cm}^{-2}$$

$$H=1500\ \text{cm}$$

$$f=10^{-4}\ \text{sec}^{-1}$$

$$\frac{\Delta\rho}{\rho}=1.5\times10^{-3}$$

$$c=\left(g\frac{\Delta\rho}{\rho}H\right)^{\frac{1}{2}}=48\ \text{cm sec}^{-1}$$

$$R=100\ \text{km}.$$

Thus, the wind stress term in (5.1) is approximately 6 m day^{-1} and the time scale, R/c, for which the thermocline intersects the surface is $2\frac{1}{2}$ days. The ratio $RF_0/(gH)$ in (3.4) is equal to unity as in the example of Section 3. The steepening would occur in a time of $3\frac{1}{2}$ days from the imposition of this wind stress. In the lake the wind stress is probably larger but all the other factors not considered here (topography, friction, etc.) probably serve to inhibit the nonlinear steepening.

Mortimer (1963) has published a preliminary analysis of 20 years of temperature records from municipal water intakes on Lake Michigan. He found that a common occurrence was the propagation of a steep rise in temperature eastward around the southern end, and northward as far as Ludington, the most northerly intake. This usually appeared after a wind shift from the southwest to the northeast. The first response of the lake is strong upwelling on the eastern shore with corresponding downwelling on the western shore. If this is followed by several days of light winds, the rise in temperature propagates around the southern end at approximately the speed of an internal Kelvin wave in a constant-depth model. Prof. Mortimer (personal communication) reports that although this "warm front" behavior occurred on several occasions in most (but not all) summers, he saw no examples of a sharp fall of temperature propagating around the lake. This is what would be expected from the theory presented here. The records await a more systematic examination, but except in isolated incidents it will probably prove very difficult to separate the propagation of baroclinic edge-waves from wind induced upwelling and nonlinear effects.

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