Comments on "Reflections over Neap to Spring Tide Ratios and Spring Tide Retardment in Co-oscillating Basins with Reference to Observations from the North Sea"

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1. Introduction

The age of the tide has been a classical problem in the theory of tides. A brief historical review can be found in Garrett and Munk (1971). Recently Gade (1998) published a paper, in which a functional relationship (hereafter abbreviated as G relationship) between the age of the tide and the neap to spring tide ratio (hereafter we will call them the T age and the N/S ratio, respectively) was proposed. However, the theory posed by Gade might be applicable to the response of the temperature in a shallow reservoir to the solar radiation but is in conflict with the tidal dynamics. In fact, the variations of T age and N/S ratio in shelf seas are related to the dimensions of the sea, the nonlinearity of bottom friction, and the time required in wave propagation. To show this, we first give an energy equation (in section 2), which governs the Gade's theory, and prove that this equation does yield the G relationship. By comparison with the well-established tidal energy equation, it is pointed out that the equation governing Gade's theory has an essential difference from the energy equation derived from the hydrodynamic equations. Section 3 discusses the mechanisms responsible for the changes in T age and N/S ratio in shelf seas. Two universal mechanisms are illustrated in details. In section 4, we show that the spring–neap modulation of the area-integrated tidal energy does satisfy the G relationship, but this property does not reflect the senses of the T age and N/S ratio referred in the literature. Finally, some further comments are made in section 5 to indicate some important, though not crucial, problems in relation to Gade's paper.

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2. A comparison of the energy equation derived from tidal dynamics with that governing Gade's theory

The energy equation derived from tidal dynamics was first given by Taylor (1919), and then improved by Garrett (1975). The equation reads

$$\frac{\partial E}{\partial t} = gh\langle \mathbf{u}\boldsymbol{\nabla}\zeta_e\rangle - \boldsymbol{\nabla}\cdot(gh\langle \mathbf{u}\zeta\rangle) - \langle \mathbf{f}\cdot\mathbf{u}\rangle, \quad (2.1)$$

where $E = \frac{1}{2}h\langle \mathbf{u}^2\rangle + \frac{1}{2}g\langle \zeta^2\rangle$ is the energy density, $\mathbf{u}$ the tidal current vector, ζ the tidal elevation, ζ_e the height of the equilibrium tide, g the acceleration due to gravity, h the water depth, t the time, $\boldsymbol{\nabla} = \mathbf{i}\partial/\partial x + \mathbf{j}\partial/\partial y$, $\mathbf{f}$ is the bottom friction vector, and the angle brackets denote averaging over a cycle. Here the terms associated with advection have been ignored for their insignificance. A more complete energy equation can be found in Fang et al. (1999).

In coastal seas such as the North Sea, the first term on the right-hand side (rhs) of (2.1) can be ignored, yielding

$$\frac{\partial E}{\partial t} = -\boldsymbol{\nabla}\cdot\mathbf{F} - \langle \mathbf{f}\cdot\mathbf{u}\rangle, \quad (2.2)$$

in which $\mathbf{F}$ represents the energy flux density:

$$\mathbf{F} = gh\langle \mathbf{u}\zeta\rangle. \quad (2.3)$$

Gade (1998) deduced his theory with a schematic diagram without giving governing equation explicitly. However, the description of Gade (1998) can be translated into the following equation:

$$\frac{\partial E}{\partial t} = R - S, \quad (2.4)$$

where R is the energy input and S the energy loss. Gade (1998) introduced two models, that is, the "linear energy flux model" and "nonlinear energy flux model." These two models produced rather close results, and the result of the linear model was cited in the abstract of the paper. In the following we will mainly concentrate on his linear model. In this model, the amplitude of tide was used to represent the energy of tide. This is not a good approach as will be shown in section 5. But for the moment, we will not distinguish between the energy and the amplitude in the discussion of this section. In Gade (1998), the energy input R was assumed to be proportional to the amplitude of ''external forcing'' P, and S proportional to the amplitude of tide Q. Thus (2.4) becomes

$$\frac{\partial Q}{\partial t} = k_1 P - k_2 Q, \tag{2.5}$$

where P and Q were expressed in the forms

$$P(t) = P_0 + P_1 \cos\omega t, \tag{2.6}$$

$$Q(\mathbf{x}, t) = Q_0(\mathbf{x}) + Q_1(\mathbf{x}) \cos[\omega t - \phi(\mathbf{x})], \tag{2.7}$$

respectively, where ω is the difference in angular speeds between the tidal constituents M_2 and S_2, or the angular speed of neap–spring modulation. Inserting (2.6) and (2.7) into (2.5) yields

$$\left.\begin{aligned} Q_0/P_0 &= k_1/k_2 \\ k_2 Q_1 \cos\phi + \omega Q_1 \sin\phi &= k_1 P_1 \\ -\omega \cos\phi + k_2 \sin\phi &= 0 \end{aligned}\right\}. \tag{2.8}$$

The second and third equations of (2.8) give

$$\cos\phi = \frac{k_1 k_2 P_1}{(\omega^2 + k_2^2) Q_1}, \tag{2.9}$$

and the third equation alone yields

$$\cos^2\phi = \frac{k_2^2}{\omega^2 + k_2^2}. \tag{2.10}$$

Thus we further have

$$\cos\phi = \frac{k_2 Q_1}{k_1 P_1} = \frac{P_0}{P_1} \cdot \frac{Q_1}{Q_0}. \tag{2.11}$$

This is exactly the same as the G relationship (Gade 1998):

$$\cos\phi = \left(\frac{1 + \gamma}{1 - \gamma}\right)\left(\frac{1 - \eta}{1 + \eta}\right). \tag{2.12}$$

To show the difference between Eqs. (2.2) and (2.4) we first assume that $\mathbf{f}$ is proportional to $\mathbf{u}$, that is,

$$\langle \mathbf{f} \cdot \mathbf{u} \rangle = \rho \langle \mathbf{u}^2 \rangle, \tag{2.13}$$

where ρ is a frictional coefficient. Then, (2.2) reduces to

$$\frac{\partial E}{\partial t} = -\nabla \cdot \mathbf{F} - \alpha E, \tag{2.14}$$

where α is a constant. The form of Eq. (2.5) has certain similarity to (2.14). However, the difference in the first term on the rhs between these two equations is of substantial importance. Equation (2.5) might be applicable to water temperature in a shallow reservoir, if we assume that the input energy directly depends on the external forcing (the solar radiation) and the back radiation is proportional to the energy (temperature) itself in the reservoir. (The back radiation is proportional to the fourth power of the absolute temperature. However, we can use a linear relation to approximate the fourth-power relation within the variation range.) On the contrary, Eq. (2.14) shows that the tidal energy in a water column incorporating a concerned specific location cannot directly feel the external tidal forcing from the open ocean. Rather, it can only receive the tidal energy flux from and radiate out its energy towards the water columns in the immediate vicinity of the concerned water column.

3. Two universal factors causing changes in the N/S ratio and T age

One of the major factors determining the N/S ratio and T age in a coastal sea is the resonance property of the sea. If the period of free oscillation is closer to that of M_2, the N/S ratio in this region will be greater than the value in the outer ocean. The North Sea is such an example. If it is closer to that of S_2, the N/S ratio will be smaller. The Coral Sea is a typical example where S_2 can even be greater than M_2, and the T age can be negative. This factor is dependent on the specific dimensions of the sea area and will not be further studied here.

There are two universal factors that can cause changes in the T age and N/S ratio, respectively.

One factor resulting in T age was pointed out long ago by Whewell (Garret and Munk 1971). As we know, the simplest progressive long wave has the form

$$\zeta = H \cos(\sigma t - kx), \tag{3.1}$$

where

$$k = \sigma/\sqrt{gh}. \tag{3.2}$$

From (3.1) and (3.2) we see that the phase lag kx increases more rapidly for the tidal wave with greater frequency. Consequently, the T age increases with the propagation of tidal waves.

A factor responsible for the increase of the N/S ratio is the nonlinear property of tidal friction. It is widely accepted that the frictional force is approximately proportional to the velocity squared. Suppose the N/S ratio in the open ocean is 0.5, say, the ratio of the frictional forces during the neap and spring tide periods should be 0.25 approximately. Hence the attenuation rate of the tide wave during the neap period is lower than that during the spring period. This causes an increase in the N/S ratio.

Explicit analytical solutions for tidal waves propagating in a uniform canal subject to linear and nonlinear friction can be found in Fang and Wang (1966) and Fang (1987), respectively. These solutions can also be derived from the energy equation (2.2) as follows. In shelf seas the friction term in the momentum equation is generally much smaller than the major terms. For progressive tidal waves propagating in a semi-infinite uniform canal, the following relation between the tidal current and tidal elevation holds approximately:

$$\zeta \approx (h/g)^{1/2}u. \tag{3.3}$$

Thus

$$E = \frac{1}{2}h\langle u^2\rangle + \frac{1}{2}g\langle\zeta^2\rangle \approx h\langle u^2\rangle, \quad \text{and}$$

$$F = gh\langle u\zeta\rangle \approx cE,$$

with $c = (gh)^{1/2}$ representing the wave celerity. The energy equation (2.14) now further reduces to

$$\frac{\partial E}{\partial t} = -c\frac{\partial E}{\partial x} - \alpha E \tag{3.4}$$

when friction is linearly proportional to velocity. Providing that the energy density at the open boundary has the form

$$E(0, t) \equiv A(t) = A_0 + A_1 \cos\omega t, \tag{3.5}$$

the solution of E is then

$$E = [A_0 + A_1 \cos\omega(t - x/c)]e^{-\beta x}, \tag{3.6}$$

where $\beta = \alpha/c$ is a constant. This solution indicates that the N/S ratio at any arbitrary location x is

$$\frac{E_{\text{neap}}}{E_{\text{spring}}} = \frac{(A_0 - A_1)e^{-\beta x}}{(A_0 + A_1)e^{-\beta x}} = \frac{A_0 - A_1}{A_0 + A_1}, \tag{3.7}$$

which does not vary with location, while the T age equal to x/c [see (3.6)] increases with x.

The above analysis shows that the T age and N/S ratio can be independent of each other.

If friction is taken to be proportional to the velocity squared, the energy equation (2.2) can be written in the form

$$\frac{\partial E}{\partial t} = -c\frac{\partial E}{\partial x} - \alpha' E^{3/2}, \tag{3.8}$$

where α' is a constant. This is a nonlinear equation. The solution for the corresponding primitive equations was studied by Fang (1987). Here we will not give the details. For simplicity, we only consider the solution in the area where $x \ll 2(A/A_1)(c/\omega)$. It is not difficult to prove that the following function

$$E = [A_0 + A_1 \cos\omega(t - x/c)]\left(1 + \frac{1}{2}\lambda A^{1/2}x\right)^{-2} \tag{3.9}$$

satisfies (3.8) approximately. Here $\lambda = \alpha'/c$. In contrast to the linear case, now the N/S ratio is

$$\frac{E_{\text{neap}}}{E_{\text{spring}}} = \frac{A_0 - A_1}{A_0 + A_1}\left[\frac{1 + \frac{1}{2}\lambda(A_0 + A_1)^{1/2}x}{1 + \frac{1}{2}\lambda(A_0 - A_1)^{1/2}x}\right]^2, \tag{3.10}$$

which is $\geq(A_0 - A_1)/(A_0 + A_1)$ and increases with x.

4. Spring–neap modulation of the area-integrated tidal energy

In this section we will show that under certain circumstance the spring–neap modulation of the integrated tidal energy does possess a property satisfying the G relationship. We will also show that this property is different from the T age and N/S ratio referred to in the literature. The aim is to give a better understanding of the inapplicability of the G relationship.

First, we deal with the energy equation (2.2). Integrating (2.2) over an area Ω we obtain the following integrated tidal energy equation

$$\frac{\partial E^*}{\partial t} = -\int_B \mathbf{F}\cdot\mathbf{n}\, dl - \iint_\Omega \langle\mathbf{f}\cdot\mathbf{u}\rangle\, dS, \tag{4.1}$$

where E^* is the total energy in Ω:

$$E^* = \iint_\Omega E\, dS, \tag{4.2}$$

and B is the boundary curve of Ω, $\mathbf{n}$ is the outward pointing unit vector normal to the element dl. The boundary B can be divided into three parts: B_1 and B_2 represent the parts where the energy flux enters into and leaves from Ω, respectively; B_3 the closed boundary, where no energy exchange occurs. Thus,

$$\frac{\partial E^*}{\partial t} = -\int_{B1} \mathbf{F}\cdot\mathbf{n}\, dl - \int_{B2} \mathbf{F}\cdot\mathbf{n}\, dl - \iint_\Omega \langle\mathbf{f}\cdot\mathbf{u}\rangle\, dS. \tag{4.3}$$

In this equation the first term on the rhs appears as the external forcing. From this equation it can be seen that the radiation (the second term on the rhs) depends on the wave properties on B_2, while the dissipation depends on those in Ω. Therefore, they cannot be combined into and expressed by a single term as in Gade (1998). If Ω has only one open boundary B_1, then the second term on the rhs of (4.3) vanishes and the integrated energy equation becomes

$$\frac{\partial E^*}{\partial t} = -\int_{B1} \mathbf{F}\cdot\mathbf{n}\, dl - \iint_\Omega \langle\mathbf{f}\cdot\mathbf{u}\rangle\, dS. \tag{4.4}$$

Furthermore, if the friction is linearly proportional to the velocity, the equation may have the following form:

$$\frac{\partial E^*}{\partial t} = k_1 A - k_2 E^*. \tag{4.5}$$

This form is the same as (2.5). Thus we can imagine that in this case the T age and N/S ratio of E^* satisfies the G relationship. However, the T age and N/S ratio of E^* are different from those at individual points in Ω. To illustrate this subtle but important difference let us examine the tidal waves propagating in a semi-infinite uniform canal under the action of linear friction.

The solution of E is given by (3.6), and we have shown in (3.7) that the N/S ratio does not change with x, hence, the area-mean value of the N/S ratio is also not changed. However, the properties of the integrated energy are different. From equation (3.6) we have

$$\begin{aligned} E^* &= \int_0^\infty E\,dx \\ &= A_0 \int_0^\infty e^{-\beta x}\,dx + A_1 \int_0^\infty e^{-\beta x} \cos\omega(t - x/c)\,dx \\ &= B_0 + B_1 \cos(\omega t - \theta), \end{aligned} \tag{4.6}$$

where

$$B_0 = \beta^{-1} A_0, \qquad B_1 = [\beta^2 + (\omega/c)^2]^{-1/2} A_1,$$
$$\tan\theta = \omega/(c\beta). \tag{4.7}$$

From (4.7) we further have

$$\cos\theta = \beta/[\beta^2 + (\omega/c)^2]^{1/2} = \frac{A_0}{A_1} \cdot \frac{B_1}{B_0}. \tag{4.8}$$

This relationship is the same as the G relationship [cf. (2.11) and (2.12)].

In the tidal literature, the T age, in terms of Eq. (3.6), is x/c, which is space dependent. The neap and spring tides appear at the times $t_{\rm neap} = x/c + \pi/\omega$ and $t_{\rm spring} = x/c$, which are also space dependent. The N/S ratio is equal to a constant $(A_0 - A_1)/(A_0 + A_1)$, as shown in (3.7). It is clear that the T age and N/S ratio of E does not satisfy the G relationship. The validity of the G relationship for E^* is, in fact, a result of the operation of integration. This operation adds up the energy values at all points on the x axis, whereas these values are taken at the same time but have different phases. To be specific, the spring tide for E^* appears at the time $t^*_{\rm spring} = \theta/\omega$, but this time is not the time of spring tide for almost all points on the x axis.

5. Further comments

There are some further comments on the paper of Gade (1998).

1) The functions $P(t) = P_0 + P_1 \cos\omega t$ and $Q(t) = Q_0 + Q_1 \cos(\omega t - \phi)$ are not good expressions for amplitudes. Rather, they are suitable expressions for energy. As given in Gade [1998, p. 751, Eq. (4)], the squared amplitude for the sum of the constituents M_2 and S_2 is $H^2 = M^2 + S^2 + 2MS\cos\omega t'$, which has the same form as $P(t)$ or $Q(t)$. When E is expressed in the form $E = E_0 + E_1 \cos(\omega t - \phi')$, the N/S ratio of the energy is $(E_0 - E_1)/(E_0 + E_1)$ and that of the amplitude should be $[(E_0 - E_1)/(E_0 + E_1)]^{1/2}$. In addition, the "nonlinear energy flux modeling" of Gade (1998) was not appropriately formulated, it is not really a nonlinear model applicable to tidal dynamics. For the tidal energy balance, the most important nonlinearity is present in the dissipation; see (3.8) for example.
2) In the coastal seas, the ratio $\eta = (M - S)/(M + S)$ is a rather rough estimate for the N/S ratio. The constituents $2MS_2$ (contained in μ_2) and $2SM_2$ have significant contribution to the N/S ratio. These constituents are mainly due to frictional nonlinearity (eg., Fang 1987).
3) The ϕ–η curve of Gade (1998) passed the point $(\phi, r) = (0, 0.36)$. There are two problems we should consider. First, the energy flux to the North Sea is mainly from the North Atlantic Ocean, where ϕ is much greater than zero; the direct energy input from the tidal potential is small. Second, for the World Ocean, the energy input can be expressed by (Garrett 1975)

$$G^* = \iint_\Omega g\langle \zeta_e(\partial\zeta/\partial t)\rangle\,dS. \tag{5.1}$$

Because the maximum amplitude of $\partial\zeta/\partial t$ does not occur at the time of new or full moon due to existence of the T age, the maximum value of G^* will also not occur at new or full moon. If the mean T age in the deep ocean is ϕ_d/ω, the maximum G^* should occur roughly at the time of $\frac{1}{2}\phi_d/\omega$.

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