Structure of the Instability Associated with Harmonic Resonance of Short-Crested Waves

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ABSTRACT

Harmonic resonance of a short-crested water wave field, one of the three-dimensional water waves, is due to the multiple-like structure of the solutions. In a linear description, the harmonic resonance is due to the resonance between the fundamental of the wave and one of its harmonics propagating at the same phase speed for particular wave parameter regions depending on the depth of the fluid, the wave steepness, and the degree of three-dimensionality. Former studies showed that (i) an *M*th-order harmonic resonance is associated with a superharmonic instability of class I involving a $(2M + 2)$-mode interaction and (ii) the instability corresponds to one sporadic ''bubble'' of instability. However, these first-step studies dealt with nonbifurcated solutions and thus incomplete solutions. In the present study, the complete set of short-crested wave solutions was computed numerically. It is found that the structure of the solutions is composed of three branches and a turning point that matches two of them. Then, the stability of the bifurcating solutions along their branches was computed. Of interest, another bubble of instability was discovered that is very close to the turning point of the solutions, and the instabilities are no longer sporadic. Harmonic resonances of short-crested water waves are thus associated with two bubbles of instability; the first one is located in a branch in which the turning point is present, and the second one is located in another branch that is continuous in the vicinity of the bifurcation point.

1. Introduction

In a linear description, short-crested wave fields are defined as a superposition of two two-dimensional progressive wave trains of equal wavelengths and intersecting at an angle γ. The description of the geometry of the propagation can be found in Hsu et al. (1979), who defined an angle θ so that $\theta = (\pi - \gamma)/2$. The three-dimensional fields admit two two-dimensional limits: the progressive Stokes wave for $\theta = 90°$ for which the two waves propagate in the same direction and the standing wave for $\theta = 0°$ for which the two waves propagate in opposite directions. Figure 1 shows two wave patterns on deep water: Fig. 1 (left side) represents a short-crested wave field for $\theta = 10°$ that is close to standing waves, exhibiting long crests in the x direction as compared with the other horizontal direction y; Fig. 1 (right side) represents a wave at angle $\theta = 45°$ exhibiting equal wavelengths in both horizontal directions.

The properties of short-crested waves have been discussed in Roberts (1983) for deep water and in Marchant and Roberts (1987) on water of finite depth. In particular, the authors showed how short-crested wave fields may be unsteady through harmonic resonance phenomena. Roberts and Peregrine (1983) calculated low-order analytical solutions for $\theta \to 90°$ and found that harmonic resonances correspond to multiple-like solutions. Okamura (1996) calculated both weakly nonlinear and fully nonlinear short-crested waves in deep water for $\theta \approx 0°$ and found that harmonic resonances correspond to multiple-like solutions. Marchant and Roberts (1987) showed that harmonic resonances occur for short-crested waves in finite depth when harmonic (m, n), that is, $\sin[m\alpha(x - ct)] \cos(n\beta y) \cosh[\kappa_{mn}(z + d)]$, is a solution of the homogeneous differential equation derived from the nonlinear surface conditions, where $\kappa_{mn} = (m^2\alpha^2 + n^2\beta^2)^{1/2}$, $\alpha = \sin\theta$, $\beta = \cos\theta$, c is the propagation velocity of the wave, t is time, z is a vertical coordinate, and d is depth. Such cases occur at critical angles θ_c for which

$$\kappa_{mn} \tanh(\kappa_{mn} d) = m^2 \tanh d. \quad (1)$$

The critical angles for which a harmonic resonance occurs are given in Marchant and Roberts (1987) and in Ioualalen et al. (1996) for different depths. Without loss of generality, in this study we would like to char-

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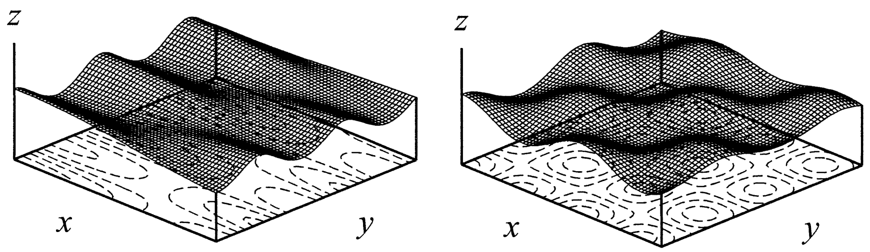

FIG. 1. Short-crested wave fields for angles $\theta = 10°$ (left) and $\theta = 45°$ (right) on deep water.

acterize the harmonic resonance phenomenon through the computation of one of the critical angles. We have chosen the harmonic resonance (2, 6) occurring at depth $d = 1$. In the linear description, the critical angle for which this harmonic resonance occurs is $\theta_c \approx 65.8354°$ (e.g., Ioualalen et al. 1996).

Ioualalen and Kharif (1993) and Ioulalen et al. (1996) computed the stability problem associated with the harmonic resonance phenomenon and found that harmonic resonances are associated with sporadic and weak superharmonic instabilities that have a bubblelike shape in the wave steepness parameter space. This was the first attempt to characterize the stability of resonant short-crested waves. However, the stability problem studied by the authors concerned nonbifurcated solutions and not fully nonlinear solutions.

The aim of the present study is to extend their work to fully nonlinear solutions exhibiting a multiple-like solution behavior as shown by Roberts and Peregrine (1983), who computed weakly nonlinear solutions. It is fair to say that it is not the purpose of the present study to perform an extensive parametrical study of these instabilities (vs the depth of the water column, the wave steepness, or the degree geometry of the flow). Rather, we preferred to exhibit the characteristics of one particular resonance, (2, 6), because it is one of the strongest given the number of mode interactions, and we have fixed an arbitrary depth to avoid any particularities such as deep water.

2. Mathematical formulation of the problem

We consider surface gravity waves on an inviscid, incompressible fluid of finite depth in which the flow is assumed to be irrotational. The governing equations are given in a dimensionless form with respect to the reference length $1/k$ and the reference time $(gk)^{-1/2}$, where g is the gravitational acceleration and k is the wavenumber of the incident wave train.

Let us define a frame of reference $(x^*, y^*, z^*, t^*, \phi^*)$ so that $x^* = x - ct$, $y^* = y$, $z^* = z$, $t^* = t$, and $\phi^* = \phi - cx^*$, where c represents the propagation velocity of the short-crested wave train and is equal to ω/α, with ω being the frequency of the wave, $\alpha = \sin\theta$ being the x-direction wavenumber, and $\beta = \cos\theta$ being the y-direction wavenumber. If we omit the asterisks for sake of simplicity, the governing equations are

$$\Delta\phi = 0, \quad \text{for } -d < z < \eta(x, y, t), \tag{2}$$

$$\phi_z = 0, \quad \text{for } z = -d, \tag{3}$$

$$\phi_t + \eta + (1/2)(\phi_x^2 + \phi_y^2 + \phi_z^2 - c^2) = 0, \quad \text{for } z = \eta(x, y, t), \tag{4}$$

$$\eta_t + \phi_x\eta_x + \phi_y\eta_y - \phi_z = 0, \quad \text{for } z = \eta(x, y, t), \tag{5}$$

where d is the depth of the fluid, $\phi(x, y, z, t)$ is the velocity potential, and $z = \eta(x, y, t)$ is the equation of the free surface. In this new frame of reference propagating at a speed c, the system of Eqs. (2)–(5) admits doubly periodic solutions of permanent form $\overline{\eta}(x, y)$ and $\overline{\phi}(x, y, z)$.

Like Ioualalen and Kharif (1993), we define the following functions to construct a stability problem:

$$\eta(x, y, t) = \overline{\eta}(x, y) + \eta'(x, y, t), \quad \text{and} \tag{6}$$

$$\phi(x, y, z, t) = \overline{\phi}(x, y, z) + \phi'(x, y, z, t), \tag{7}$$

where we assume that the surface elevation and the velocity potential are a superposition of a steady unperturbed wave $(\overline{\eta}, \overline{\phi})$ and infinitesimal unsteady perturbations (η', ϕ') where $\eta' \ll \overline{\eta}$ and $\phi' \ll \overline{\phi}$. After substituting Eqs. (6) and (7) into Eqs. (2)–(5) and linearizing, we obtain the zeroth-order system of equations for which the permanent short-crested wave is a solution and the first-order perturbation equations that represent the stability problem. Both systems of equations will be resolved in the frame of reference moving with the wave.

The zeroth-order system of equations follows:

$$\Delta\overline{\phi} = 0, \quad \text{for } -d < z < \overline{\eta}(x, y), \tag{8}$$

$$\overline{\phi}_z = 0, \quad \text{for } z = -d, \tag{9}$$

$$\overline{\eta} + (1/2)(\overline{\phi}_x^2 + \overline{\phi}_y^2 + \overline{\phi}_z^2 - c^2) = 0, \quad \text{for } z = \overline{\eta}(x, y), \tag{10}$$

$$\overline{\phi}_x\overline{\eta}_x + \overline{\phi}_y\overline{\eta}_y - \overline{\phi}_z = 0, \quad \text{for } z = \overline{\eta}(x, y). \tag{11}$$

The numerical method to solve this problem and obtain short-crested wave solutions of permanent form is an extension of the work by Okamura (1996) to finite depth, so the reader can refer to this work to obtain more details about the method. A concise description is also provided in appendix A.

The first-order system of equations is then

$$\Delta\phi' = 0, \quad \text{for } -d < z < \overline{\eta}(x, y), \tag{12}$$

$$\phi'_z = 0, \quad \text{for } z = -d, \tag{13}$$

$$\begin{aligned}\phi'_t = &-\overline{\phi}_x\phi'_x - \overline{\phi}_y\phi'_y - \overline{\phi}_z\phi'_z \\ &- \eta'(1 + \overline{\phi}_x\overline{\phi}_{xz} + \overline{\phi}_y\overline{\phi}_{yz} + \overline{\phi}_z\overline{\phi}_{zz}), \\ &\text{for } z = \overline{\eta}(x, y),\end{aligned} \tag{14}$$

$$\begin{aligned}\eta'_t = &\,\eta'(\overline{\phi}_{zz} - \overline{\eta}_x\overline{\phi}_{xz} - \overline{\eta}_y\overline{\phi}_{yz}) - \overline{\eta}_x\phi'_x \\ &- \overline{\phi}_x\eta'_x - \overline{\eta}_y\phi'_y - \overline{\phi}_y\eta'_y + \phi'_z, \\ &\text{for } z = \overline{\eta}(x, y),\end{aligned} \tag{15}$$

for which we look for nontrivial solutions of the following superharmonic form:

$$\eta' = \exp(-i\sigma t)\sum_{J=-\infty}^{\infty}\sum_{K=-\infty}^{\infty} a_{JK}\exp[i(J\alpha x + K\beta y)], \tag{16}$$

and

$$\phi' = \exp(-i\sigma t)\sum_{J=-\infty}^{\infty}\sum_{K=-\infty}^{\infty} b_{JK}\exp[i(J\alpha x + K\beta y)] \times \frac{\cosh[\kappa_{JK}(z + d)]}{\cosh(\kappa_{JK}d)}, \tag{17}$$

where $\kappa_{JK} = [(J\alpha)^2 + (K\beta)^2]^{1/2}$.

The resolution of this eigenvalue problem consists in a stability analysis for which we need to determine the set of eigenvalues σ and the coefficients a_{JK} and b_{JK} of their associated eigenvectors. Since the system of Eqs. (12)–(15) represents a Hamiltonian structure, the eigenvalues σ appear in complex conjugate pairs. Thus an instability corresponds to $T(\sigma) \neq 0$. For the wave steepness $h = 0$ in Eq. (A3), the unperturbed wave is given by $\overline{\eta} = 0$ and $\overline{\phi} = -c_0 x$ with $c_0 = \omega_0/\alpha = (\tanh d)^{1/2}/\alpha$. Then the eigenvalues are $\sigma^s_{JK} = -J\alpha c_0 + s[\kappa_{JK} \tanh(\kappa_{JK}d)]^{1/2}$ with $s = \pm 1$, sign$[sT(-i\sigma)]$ being the signature of the perturbation (e.g., MacKay and Saffman 1986). The real-valued set of eigenvalues $\{\sigma^s_{JK}\}$ causes the wave to be neutrally stable for $h = 0$. Instabilities arise as the wave steepness h increases. We use here the method by Ioualalen et al. (1996), who took advantage of the useful work of MacKay and Saffman (1986) on Hamiltonian systems; we apply the necessary condition for instability in terms of collision of eigenvalues of opposite signatures s or at zero frequency. An instability can arise if two modes have the same frequency, that is, $\sigma^s_{J_1K_1} = \sigma^{-s}_{J_2K_2}$ for some wave steepness h. This condition takes the following form for $s = 1$ ($s = -1$ corresponds to an opposite direction of propagation):

$$[\kappa_{J_1K_1}\tanh(\kappa_{J_1K_1}d)]^{1/2} + [\kappa_{J_2K_2}\tanh(\kappa_{J_2K_2}d)]^{1/2} = (J_1 - J_2)\tanh^{1/2}d. \tag{18}$$

The numerical procedure to solve the nontrivial eigenvalue problem is provided in appendix B.

3. Description of the multiple-like solutions of permanent form

In this work we are interested in the harmonic coefficient ϕ_{26} whose mode (2, 6) is responsible for a harmonic resonance at angle $\theta_c \approx 65.8354°$ for depth $d = 1$ in the linear approximation. Figure 2 exhibits the multiple-like structure of the coefficient ϕ_{26} as a function of the coefficient ϕ_{11} of the fundamental mode for the wave parameters $d = 1$ and $\theta = 66°$ near the critical angle θ_c. A high-density computation has been performed in the vicinity of the turning point for which a bifurcation of the wave occurs near the value $\phi_{11} \approx 0.167\,091$.

We have obtained all solutions of the short-crested waves: branch 1; branch 2, including the critical point; and branch 3. With their numerical perturbation method, Ioualalen et al. (1996) obtained branch 1 and the part of branch 3 on the right side of the turning point and failed to find branch 2. Then they used the Shanks transform to match artificially branches 1 and 3 in the vicinity of the turning point that we obtain here. They computed the stability of the solution corresponding to a nonbifurcated solution. Because their solutions are much different from ours near the critical point, we are now interested in studying the stability of the multiple-like solutions that represent the fully nonlinear wave field

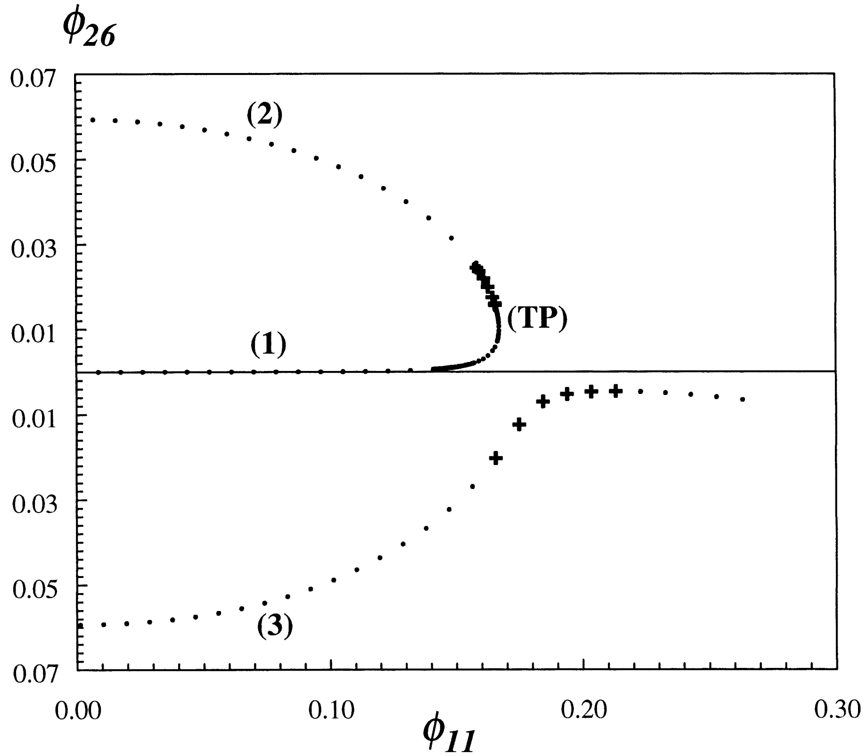

FIG. 2. Coefficient ϕ_{26} vs coefficient ϕ_{11} for depth $d = 1$ and angle $\theta = 66°$. The different branches of the solutions are numbered 1–3 and TP indicates a turning point. The plus signs denote the unstable solutions; see Fig. 3.

along all the three branches and around the turning point to characterize definitely the stability behavior of harmonic resonances.

4. Resonant interactions: Superharmonic instabilities associated with the harmonic resonance (2, 6)

In this section, the superharmonic instability of short-crested waves subject to a harmonic resonance (2, 6) is computed for depth $d = 1$ and angle $\theta = 66°$. We conjecture that one example is sufficient to assess the stability of the bifurcating solutions of short-crested waves because all harmonic resonances are composed of the same multiple-like structure. We have performed extensive computations of harmonic resonances for other purposes, and they do not need to be shown here because their behavior is recurrent.

Following Ioualalen et al. (1996), a superharmonic instability associated with a harmonic resonance ($\pm m$, n) may arise only if the two eigenvalues $\sigma^{s}_{mn} = \sigma^{-s}_{-mn}$ of opposite signatures are equal at a given wave steepness h. The collision of the two eigenmodes ($\pm m$, n) is then interpreted as Ioualalen and Kharif's (1993) class Ia (m, n) instability, and it can be interpreted as a resonance between the two eigenmodes ($\pm m$, n) and the $2m$ modes (1, ± 1) of the basic short-crested wave; that is,

$$\Omega_1 = -\Omega_2 + m\Omega_{01} + m\Omega_{02}, \quad \text{and} \tag{19}$$

$$\mathbf{k}_1 = \mathbf{k}_2 + m\mathbf{k}_{01} + m\mathbf{k}_{02}, \tag{20}$$

where $\Omega_i = [|\mathbf{k}_i| \tanh(\kappa_{mn} d)]^{1/2}$; $\Omega_{0i} = (\tanh d)^{1/2}$, for $i = 1, 2$; $\mathbf{k}_1 = (\alpha m, \beta n)$; $\mathbf{k}_2 = (-\alpha m, \beta n)$; $\mathbf{k}_{01} = (\alpha, \beta)$; and $\mathbf{k}_{02} = (\alpha, -\beta)$.

In Fig. 3 are plotted the frequencies of the eigenvalues $\sigma_{\pm 26}$ along all the branches of the short-crested wave solutions shown in Fig. 2. The stability of branch 3 shows that frequencies of the modes (± 2, 6) coalesce between $\phi_{11} \approx 0.160$ and $\phi_{11} \approx 0.214$. The two modes are neutrally stable with a nonzero frequency for infinitesimal wave steepness h, that is, $\phi_{11} \to 0$, and then give rise to a bubble of instability in that range of ϕ_{11} with zero frequency. The bubble of instability is physically associated with a resonant interaction; the coalescence of the two eigenmodes with zero frequency simply means that the harmonics (± 2, 6) propagate at the same phase speed as the basic wave, bearing in mind that the stability problem has been computed in the frame of reference moving with the basic wave. This phase-locking of the resonant modes with the basic wave is what we have expected.

Most interesting, the superharmonic instability is no longer sporadic, as mentioned by Ioualalen et al. (1996). The authors found that the instability occurs in a range of h of order h^4; ours occurs in the range of order h^3.

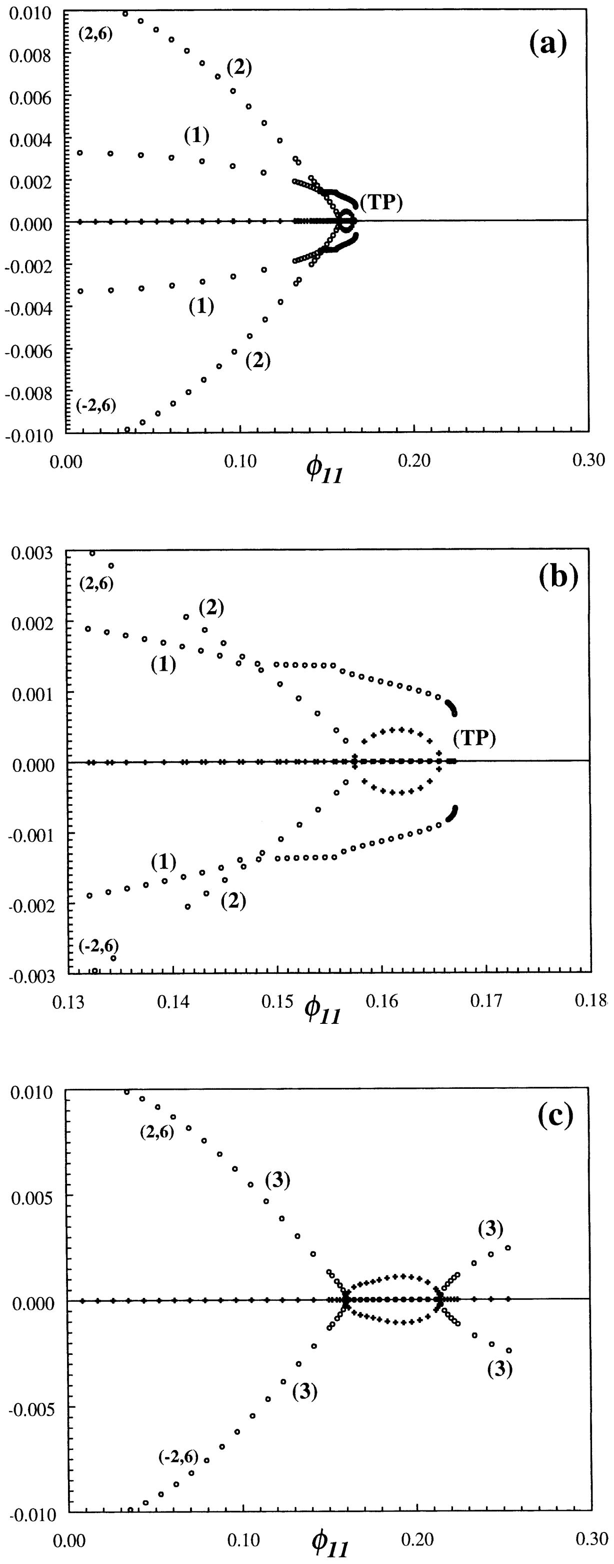

FIG. 3. Frequency $[-\Re(\sigma_{\pm 26})]$ (open circles) and growth rate $[-T(\sigma_{\pm 26})]$ (plus signs) as a function of coefficient ϕ_{11} for depth $d = 1$ and angle $\theta = 66°$: (a) branches 1 and 2 including the TP area, (b) a magnification of (a), and (c) branch 3. See Fig. 2 for definition of the branches.

The difference comes out of the short-crested wave solutions. Our short-crested wave, branch 3, is computed numerically through a fully nonlinear method, whereas theirs is obtained by matching artificially the solutions by using the Shanks transform. Thus they obtained a solution that is not strictly similar to our branch 3.

The stability of branches 1 and 2 of our solutions shows that no instability appears on branch 1 while a bubble of instability occurs for $0.1571 < \phi_{11} < 0.1656$ on branch 2, the region that is very close to the turning point ($\phi_{11} \approx 0.167\,091$). The superharmonic instability is more sporadic and of lower intensity (larger timescale) than that of branch 3. Note that the inflexion of the frequencies appearing around $\phi_{11} \approx 0.146$ and $\phi_{11} \approx 0.155$ on branch 1 is simply due to the presence of a four-mode Benjamin–Feir instability; as in the superharmonic instability with which we are concerned, the modes $(1, \pm 1)$ of the unperturbed wave of permanent form are used in the interaction process of the instability. At the end, the frequencies of the modes $(\pm 2, 6)$ are perturbed by the competition with the Benjamin–Feir instability. For clarity we have not reported this instability in the figure.

5. Conclusions

In this work, we have characterized the potential evolution of multiple-like solutions of short-crested water waves subject to the harmonic resonance phenomenon. We have studied their superharmonic stability and given the values of the growth rates of their associated instability. The growth rates represent rough predictions of their potential timescale evolution. For that purpose we have revisited the work of Ioualalen et al. (1996) and have numerically computed fully nonlinear solutions of short-crested wave fields near a critical region in which a harmonic resonance occurs. Our solutions differ from theirs because they found only one unique solution with a perturbation method. When finding accurate solutions near the critical region where a small divisor occurs, they matched the two solutions artificially by using the Shanks transform and finally obtained one unique solution. In the present work, we computed multiple-like solutions through a fully nonlinear method. At the end, we obtained three branches; the first one is nearly similar to theirs except that ours is more relevant in the region in which the harmonic resonance occurs, and we also obtained two other branches connected by a turning point in the vicinity of the harmonic resonance critical wave parameter.

As a result, like Ioualalen et al. (1996), we recovered a bubble of instability that is physically associated with a resonant interaction for the common branch we found. The growth rate differs, but, most interesting, the superharmonic instability is no longer sporadic like the one they computed. We then discovered another superharmonic instability that is very close to the bifurcation point to which Ioualalen et al. (1996) had no access with their asymptotic method. The unstable solutions are summarized in Fig. 2, and their behavior is thus characterized through this study.

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APPENDIX A

Computation of the Multiple-like Solutions of Permanent Form

We look for nonlinear short-crested waves of permanent form that are solutions of the system of Eqs. (8)–(10) and

$$\begin{aligned} &c^2\overline{\phi}_{xx} + (\overline{\phi}_x + c)[(\overline{\phi}_x - c)\overline{\phi}_{xx} + \overline{\phi}_y\overline{\phi}_{xy} + \overline{\phi}_z\overline{\phi}_{zx}] \\ &\quad + \overline{\phi}_y[(\overline{\phi}_x - c)\overline{\phi}_{xy} + \overline{\phi}_y\overline{\phi}_{yy} + \overline{\phi}_z\overline{\phi}_{zy}] \\ &\quad\quad + \overline{\phi}_z[(\overline{\phi}_x - c)\overline{\phi}_{xz} + \overline{\phi}_y\overline{\phi}_{yz} + \overline{\phi}_z\overline{\phi}_{zz} + 1] \\ &\quad\quad\quad = 0, \end{aligned} \tag{A1}$$

where the condition in Eq. (A1) is exactly the same as the kinematic boundary condition in Eq. (11) and expresses that the pressure on a free surface is constant following the motion of the fluid. Though Eq. (A1) is much more complicated than the usual kinematic condition [Eq. (11)], the lack of a space derivative of the surface elevation in the expression helps us to obtain short-crested waves with our method.

The velocity potential $\overline{\phi}$ is expressed as follows:

$$\overline{\phi} = -cx + \sum_{k=0}^{N} \sum_{j=2-(k\,\mathrm{mod}2)}^{N} \phi_{jk} \sin(j\alpha x)\cos(k\beta y) \times \frac{\cosh[\kappa_{jk}(z + d)]}{\cosh(\kappa_{jk} d)}, \tag{A2}$$

where N is the maximum order of expansion and is chosen to be odd. All calculations are carried out using $N = 19$ in this paper. Further, we use the following three conditions to obtain a short-crested wave field: the condition for periodicity and symmetry,

$$\begin{aligned} \overline{\phi}(x, y, z) &= \overline{\phi}(x, y + 2\pi/\beta, z), \\ \overline{\phi}(x, y, z) &= \overline{\phi}(x, -y, z), \\ \overline{\phi}(x, y, z) &= -\overline{\phi}(-x, y, z), \\ \overline{\phi}(x, y, z) &= \overline{\phi}(x + 2\pi/\alpha, y, z), \\ \overline{\phi}(x, y, z) &= -\overline{\phi}(\pi/\alpha - x, \pi/\beta - y, z); \end{aligned}$$

the wave steepness condition,

$$h = \frac{\overline{\eta}(0, 0) - \overline{\eta}(\pi/\alpha, 0)}{2}, \tag{A3}$$

which shows one-half of the nondimensional peak-to-trough height for nonresonant waves; and the zero mean height condition,

$$\int_{x=0}^{\pi/\alpha} \int_{y=0}^{\pi/\beta} \overline{\eta}(x, y)\, dx\, dy = 0. \tag{A4}$$

The above symmetry condition leads to

$$\phi_{jk} = 0, \quad \text{when } j + k \text{ is odd.} \tag{A5}$$

Note that the relation between wave steepness h and the coefficient ϕ_{11} of the fundamental mode (1,1) in Eq. (A2) is

$$h \rightarrow (\tanh d)^{1/2}\phi_{11}, \quad \text{when } \phi_{11} \rightarrow 0$$

for nonresonant waves.

We use the collocation method to obtain short-crested water waves. Substituting collocation points into Eqs. (10) and (A1), we obtain independent equations. The collocation points are selected as

$$\alpha x_j = \frac{j - 1}{N - 1}\pi, \qquad j = 1, 2, \ldots, \frac{N - 1}{2}, \tag{A6}$$

and

$$\beta y_i = \frac{i - 1}{N}\pi, \qquad i = 1, 2, \ldots, N + 1. \tag{A7}$$

Note that the symmetry condition gives

$$\beta y_i = \frac{i - 1}{N}\pi, \qquad i = 1, 2, \ldots, \frac{N + 1}{2}, \qquad \text{for } x = \frac{\pi}{2}. \tag{A8}$$

We use the x derivative of the kinematic condition in Eq. (A1) instead of only the kinematic condition for $x = 0$ because it becomes trivial at $x = 0$. The number of the collocation points is $N(N + 1)/2$. We substitute Eq. (A2) into Eqs. (10) and (A1) and the above collocation points into x, y in them to obtain $N(N + 1)$ independent equations. We have two further independent equations: the wave amplitude condition [Eq. (A3)] and the zero height condition [Eq. (A4)].

We solve the $N(N + 1) + 2$ nonlinear simultaneous equations by the Newton method. A third-order short-crested wave is chosen as an initial solution of the iteration. We stop the iteration if the maximum of the differences between unknown quantities before an iteration and those after the iteration is smaller than 10^{-11}.

APPENDIX B

Resolution of the Stability Problem

Once the series in Eqs. (16) and (17) are truncated up to order M and the short-crested wave of permanent form is computed, then both expressions are substituted into the surface conditions in Eqs. (14) and (15). The perturbation equations lead to a generalized eigenvalue problem of the form: $\mathbf{Au} = i\sigma\mathbf{Bu}$, where σ is the set of eigenvalues to be computed with the corresponding eigenvectors $\mathbf{u} = (a_{jk}, b_{jk})^{\mathrm{T}}$. Here, $\mathbf{A}$ and $\mathbf{B}$ are complex matrices, functions of the basic flow. A spectral method of Galerkin type is used to solve the eigenvalue problem. For that purpose, Eqs. (14) and (15) are numerically integrated over one space period in the two horizontal coordinates using Fourier transforms over a set of $\nu \times \mu$ points whose coordinates are $\alpha x_u = 2\pi u/\nu$, $u = 0, \ldots, \nu - 1$ and $\beta y_\upsilon = 2\pi\upsilon/\mu$, $\upsilon = 0, \ldots, \mu - 1$. The following eigenvalue problem is obtained at $z = \overline{\eta}(x, y)$:

$$\sum_{J=-M}^{M}\sum_{K=-M}^{M} F_{J-l,K-r}\{E_{JK}^{(1)}\}a_{JK} + \sum_{J=-M}^{M}\sum_{K=-M}^{M} F_{J-l,K-r}\{G_{JK}^{(1)}\}b_{JK} = i\sigma a_{lr},$$

$$\sum_{J=-M}^{M}\sum_{K=-M}^{M} F_{J-l,K-r}\{E_{JK}^{(2)}\}a_{JK} + \sum_{J=-M}^{M}\sum_{K=-M}^{M} F_{J-l,K-r}\{G_{JK}^{(2)}\}b_{JK} = i\sigma \sum_{J=-M}^{M}\sum_{K=-M}^{M} F_{J-l,K-r}\{H_{JK}\}b_{JK},$$

where

$$E_{JK}^{(1)} = -\overline{\phi}_{zz} + i\alpha J\overline{\phi}_x + i\beta K\overline{\phi}_y + \overline{\eta}_x\overline{\phi}_{xz} + \overline{\eta}_y\overline{\phi}_{yz},$$

$$G_{JK}^{(1)} = \{(i\alpha J\overline{\eta}_x + i\beta K\overline{\eta}_y)\cosh[\kappa_{JK}(\overline{\eta} + d)] - \kappa_{JK}\sinh[\kappa_{JK}(\overline{\eta} + d)]\}\frac{1}{\cosh(\kappa_{JK}d)},$$

$$E_{JK}^{(2)} = 1 + \overline{\phi}_x\overline{\phi}_{xz} + \overline{\phi}_y\overline{\phi}_{yz} + \overline{\phi}_z\overline{\phi}_{zz},$$

$$G_{JK}^{(2)} = \{(i\alpha J\overline{\phi}_x + i\beta K\overline{\phi}_y)\cosh[\kappa_{JK}(\overline{\eta} + d)] + \kappa_{JK}\sinh[\kappa_{JK}(\overline{\eta} + d)]\}\frac{1}{\cosh(\kappa_{JK}d)},$$

$$H_{JK} = \frac{\cosh[\kappa_{JK}(\overline{\eta} + d)]}{\cosh(\kappa_{JK}d)}.$$

The functions

$$F_{J-l,K-r}\{f_{JK}\} = \sum_{u=0}^{\nu-1}\sum_{\upsilon=0}^{\mu-1} f_{JK}\exp[i\alpha(J - l)x_u]\exp[i\alpha(K - r)y_\upsilon],$$

where $l = -M, \ldots, M$ and $r = -M, \ldots, M$, are computed using two-dimensional FFT. Convergence of the eigenvalues σ is obtained by increasing M. All instability calculations are carried out using $M = 15$.

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