NOTES AND CORRESPONDENCE

On Ocean Transport Diagnostics: The Idealized Age Tracer and the Age Spectrum

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ABSTRACT

The idealized age tracer is commonly used to diagnose transport in ocean models and to help interpret ocean measurements. In most studies only the steady-state distribution, the result of many centuries of model integration, has been presented and analyzed. However, in principle the transient solution provides more information about the transport. Here it is shown that this information can be readily interpreted in terms of the ventilation histories of water masses. A simple relationship is derived, valid for stationary transport, between the transient evolution, $\tau_{id}(r, t)$, of the idealized age tracer and the ''age spectrum,'' $G(r, t)$, the distribution of times t since a water mass was last ventilated. Namely, $G(r, t) = -\partial_{tt}\tau_{id}(r, t)$. Implications of the relationship are discussed, and the relationship is illustrated with an idealized model.

1. Introduction

Natural and anthropogenic tracers have been used to estimate the ventilation history of ocean water masses. The ''ages'' constructed from tracers generally reveal different and complementary information about the ventilation. Recent work has made explicit the fact that a water mass must be characterized by a distribution of times since since it last made surface contact, rather than a single ''age'' (Beining and Roether 1996; Delhez et al. 1999; Khatiwala et al. 2001; Haine and Hall 2002), and observable ages represent differently weighted averages over this distribution (Haine and Hall 2002). Many of these ideas build on previous work intepreting stratospheric tracers (Hall and Plumb 1994) and recent more general work on transport timescales in geophysical flows (Holzer and Hall 2000; Beckers et al. 2001; Deleersnijder et al. 2001a).

One age that has meaning independent of any particular ocean tracer is given by the idealized age tracer $\tau_{id}(r, t)$ (e.g., England 1995), which we refer to simply as the ''ideal age.'' It is defined by

$$\frac{\partial \tau_{id}}{\partial t} + L(\tau_{id}) = \Theta(t), \qquad (1)$$

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where L is a general linear transport operator that may include, for example, advection and diffusion, and $\Theta(t)$ is the unit step function [$\Theta(t) = 1$ for $t \geq 0$ and $\Theta(t) = 0$ for $t < 0$]. The boundary condition (BC) is $\tau_{id}(S, t) = 0$, where S is the ocean surface, and the initial condition is $\tau_{id}(r, 0) = 0$. In this note we restrict attention to the case of stationary transport (i.e., the coefficients of L are assumed to be time-independent). Some comments concerning the impact of nonstationarity on our analysis are made in the final section.

In the limit of long elapsed time compared to timescales of the circulation (''steady state''), τ_{id} is a natural diagnostic. The irreducible fluid elements that compose the water mass have had their ''clocks'' increased one time unit per unit time by the source [rhs of (1)] since last boundary contact, where their clocks were reset to zero. Thus, τ_{id} averaged over the elements of the water mass [the ''observable'' quantity, were there such a tracer obeying (1)] is the ''ideal age,'' the average time since the water mass last made surface contact. Due to mixing, the clock times of the individual elements comprising the water mass may vary widely.

The ideal age is a popular and useful diagnostic in ocean models, but generally only the steady-state solution is reported and analyzed. However, the transient approach to steady state contains the transport information provided by the steady-state solution and in principle much more. This transient is a useful diagnostic if it can be interpreted physically in a straightforward manner. Recently, mathematical and physical frameworks have been developed in which passive tracer fields can be expressed in terms of transient evolutions that have interpretations as transit time or age distributions (Holzer and Hall 2000; Beckers et al. 2001; Deleersnijder et al. 2001a). In this note we present an example of this connection between a tracer field and an age-related transient that should be of practical interest to ocean modelers. Namely, we derive a simple and direct relationship between τ_{id} and the distribution of transit times since a water mass made last surface contact. This distribution has been termed the ''age spectrum'' in stratosphere applications (Kida 1983; Hall and Plumb 1994), and we use that nomenclature here.

2. Idealized age tracer and age spectrum

Our goal is to derive a relationship between ideal age and the age spectrum, G. Because the relationship of G to Green's functions has been developed by Holzer and Hall (2000), a natural place to start is by expressing τ_{id} in terms of Green's functions. Given the unit uniform source of (1) one has

$$\tau_{id}(r, t) = \int_0^t dt' \int_D d^3r' \rho(r') G(r, t, r', t'), \qquad (2)$$

where D is the physical domain (i.e., the ocean), ρ the fluid density, and $G(r, t, r', t')$ the Green's function, the response at (r, t) to a point source at (r', t'). The function G obeys a BC compatible with τ_{id}; namely, $G(r_S, t, r', t') = 0$ for r_S on S. For stationary transport G depends only on elapsed time $\xi = t - t'$, and (2) can be rewritten

$$\tau_{id}(r, t) = \int_0^t d\xi \int_D d^3r' \rho(r') G(r, r', \xi). \qquad (3)$$

Expression (3) is the response at a location r to a spatially distributed source. We replace this with the response integrated over the domain to a point source at r by exploiting the reciprocity condition for Green's functions: $G(r, r', \xi) = G^\dagger(r', r, -\xi)$, where $G^\dagger$ is the Green's function for the adjoint flow (e.g., Morse and Feshbach 1953). The reciprocity condition says that the response at r to a point source at r' in the time-forward flow is the same as the response at r' to a source at r in the time-reversed adjoint flow (TRAF). Thus,

$$\tau_{id}(r, t) = \int_0^t d\xi \int_D d^3r' \rho(r') G^\dagger(r', r, -\xi)$$
$$= \int_0^t d\xi M^\dagger(r, -\xi), \qquad (4)$$

where $M^\dagger \equiv \int_D d^3r \rho G^\dagger$ is the total tracer mass in D at elapsed time $-\xi$ ''after'' the unit tracer release from r in the TRAF.

We now consider the budget of $M^\dagger(r, -\xi)$ as ξ increases ($-\xi$ decreases). Initially in the TRAF the unit tracer mass is localized near the release point r, and $M^\dagger = 1$. Subsequently, tracer begins to make contact with the boundary where it is lost, and $M^\dagger$ declines from unity. The rate of change $\partial_\xi M^\dagger(r, -\xi)$ (subscripts indicate differentiation) must equal the flux of tracer mass out through the boundary. That is,

$$\frac{\partial}{\partial \xi} M^\dagger(r, -\xi)$$
$$= \int_S d^2r_S \rho(r_S) \kappa(r_S) \mathbf{n} \cdot \boldsymbol{\nabla}_{r_S} G^\dagger(r_S, r, -\xi), \qquad (5)$$

where $\mathbf{n}$ is the unit normal vector on S directed into D, κ is the diffusivity, and the gradient is evaluated on S. [Note that the flux into S is purely diffusive; $G^\dagger(r_S, r, -\xi) = 0$ by the boundary condition, so that the advective flux $\mathbf{v}G^\dagger$ vanishes.] As discussed by Holzer and Hall (2000), a general Green's function solution for a tracer with arbitrary sources and boundary conditions reveals that the rhs of (5) is the kernel in a convolution with the tracer's time-dependent BC on S. The rhs of (5) acts as a propagator, $\mathcal{G}(r, \xi)$, of BCs on S in the time-forward flow. Thus, (5) can be written $\partial_\xi M^\dagger(r, -\xi) = \mathcal{G}(r, \xi)$, or

$$M^\dagger(r, -\xi) = 1 - \int_0^\xi d\xi' \mathcal{G}(r, \xi'). \qquad (6)$$

Therefore, from (4), the ideal age is

$$\tau_{id}(r, t) = t - \int_0^t d\xi \int_0^\xi d\xi' \mathcal{G}(r, \xi') \qquad (7)$$

or, upon rearranging,

$$\mathcal{G}(r, t) = -\frac{\partial^2}{\partial t^2} \tau_{id}(r, t). \qquad (8)$$

The final step in out development is to note that, in addition to being a BC propagator, $\mathcal{G}$ has the interpretation as the distribution of transit times ξ since fluid at r last made contact with S (Hall and Plumb 1994; Holzer and Hall 2000). Indeed, it was the connection of $\mathcal{G}$ to $M^\dagger$ that led Holzer and Hall (2000) to this interpretation. Physically, $M^\dagger(r, -\xi)$ is the fraction of tracer released from r in the TRAF that has not made boundary contact in the elapsed time $-\xi$. Therefore, $\delta\xi \partial_\xi M^\dagger(r, -\xi)$ is the fraction that makes first boundary contact in the elapsed time interval $-\xi \to -\xi - \delta\xi$; that is, $\partial_\xi M^\dagger$, which equals $\mathcal{G}$, has the interpretation as the distribution of times to first boundary contact (the first arrival time pdf) in the time-reversed adjoint flow. This is equivalent to the distribution of times since last boundary contact in the time-forward flow.

Expression (8), together with the transit time distribution interpretation, is the key result of this work. The transient solution to the ideal age equation and the age spectrum are related in a simple fashion, a result not previously noted, but which follows naturally from the more general Green's function frameworks of Holzer and Hall (2000) and Beckers et al. (2001). The transient solution of the ideal age contains valuable transport information—far more, in fact, than the steady-state ideal age alone—and this information is readily interpretable in terms of the ventilation history of the water mass. Relationship (8) can be physically understood in a straightforward fashion. At early enough times the fluid at an interior point r has not yet felt the influence of the boundary, so that the unit source [rhs of Eq. (1)] causes a linear increase in time of τ_{id}. From (8), the boundary propagator G is zero at this time, precisely what is meant by not yet having felt the influence of the boundary. Once the zero BC starts to make itself felt, then τ_{id} starts to increase more slowly than linear. Eventually, when τ_{id} reaches steady state ($\partial/\partial t = 0$), the boundary has made its full influence felt, and there is no further boundary information to propagate, whence G returns to zero. Note that relationship (8) implies that the curvature of ideal age is always negative.

3. Ideal age convergence

Khatiwala et al. (2001) simulated in an ocean GCM the ideal age and the "cumulative mean age," defined as $\Gamma(r, t) \equiv \int_0^t d\xi \xi G(r, \xi)$, where $\Gamma(r, t)$ is the mean transit time since last boundary contact of the tracer that has accumulated at r by time t. As $t \to \infty$ all fluid elements of the water mass have made boundary contact at some past time, and Γ becomes the mean transit time since the entire water mass was last at the boundary (the "mean age"), equal to the ideal age. Khatiwala et al. (2001) made the numerical observation that although τ_{id} and Γ tend to the same value, τ_{id} converges more rapidly. This is readily verified from the general analysis here. Taking the partial first moment ($\int_0^t d\xi \xi$) of (8) one finds

$$\Gamma(r, t) = \tau_{id}(r, t) - t\frac{\partial}{\partial t}\tau_{id}(r, t). \tag{9}$$

The ideal age increases monotonically from zero, approaching its equilibrium value asymptotically so that the second term on the rhs of (9) is positive. At long times $\partial/\partial t \sim 0$ and $\Gamma \sim \tau_{id}$. However, at intermediate times $\Gamma < \tau_{id}$. Therefore, Γ converges more slowly than τ_{id}. This has practical consequences for diagnostics of numerical models. If it is the sole intention of a modeler to obtain the ideal age (mean age) then the approach of Eq. (1) is more efficient. (However, it should be recognized that the full age spectrum contains far more information than just its mean.)

4. Illustration

The expressions relating τ_{id}, G, and Γ derived above are true for any stationary transport operator L, whether that of the time-averaged observed circulation, a 3D numerical model, or an idealized analytic model. It is instructive, however, to illustrate with an example. We choose 1D advection–diffusion with constant coefficients because of its familiarity and common usage.

The evolution equation for ideal age is

$$\frac{\partial \tau_{id}}{\partial t} + u\frac{\partial \tau_{id}}{\partial x} - k\frac{\partial^2 \tau_{id}}{\partial x^2} = \Theta(t), \tag{10}$$

where u is the velocity, k the diffusivity, and $\Theta(t)$ the unit step function. The BCs are $\tau_{id}(0, t) = 0$ and that $\tau_{id}(x, t)$ should not grow exponentially with x, and the initial condition is $\tau_{id}(r, 0) = 0$. One method of solution is to take the Laplace transform, consider the sum of homogeneous and particular solutions, and use the BCs to constrain general constants. This yields the Laplace space solution

$$\tilde{\tau}_{id} = \frac{1}{s^2} - \frac{1}{s^2}\exp\left(\frac{ux}{2k} - x\sqrt{\frac{s}{k} + \frac{u^2}{4k^2}}\right), \tag{11}$$

where $\tilde{\tau}_{id}$ is the Laplace transform of τ_{id} and s is the transform variable. We were not able to obtain a closed-form solution for the inverse transform. An integral expression of the solution is

$$\tau_{id} = t - \frac{x}{\sqrt{4\pi k}}\int_0^t \frac{(t - t')}{t'^{3/2}} e^{-(x-ut')^2/4kt'}\, dt', \tag{12}$$

similar to solutions for closely related problems obtained by Deleersnijder et al. (2001b).

By comparison, the evolution equation for the age spectrum is

$$\frac{\partial G}{\partial t} + u\frac{\partial G}{\partial x} - k\frac{\partial^2 G}{\partial x^2} = 0 \tag{13}$$

with the BCs $G(0, t) = \delta(t)$ and no exponential growth with x. The solution can be found in several studies and is

$$G(x, t) = \frac{x}{\sqrt{4\pi k t^3}} e^{-(x-ut)^2/4kt}. \tag{14}$$

To obtain the relationship with τ_{id} we also note that the Laplace space solution to (13) is

$$\tilde{G}(x, t) = \exp\left(\frac{ux}{2k} - x\sqrt{\frac{s}{k} + \frac{u^2}{4k^2}}\right). \tag{15}$$

Inspection of (11) and (15) reveals

$$\tilde{G}(x, s) = 1 - s^2\tilde{\tau}_{id}(x, s). \tag{16}$$

Using relationships between the derivatives of transforms and their inverses (e.g., Abramowitz and Stegun 1972), one finds that the Laplace transform of equality (16) is

$$G(x, t) = -\frac{\partial^2}{\partial t^2}\tau_{id}(x, t). \tag{17}$$

General relationship (8) is verified. [Relationship (17)

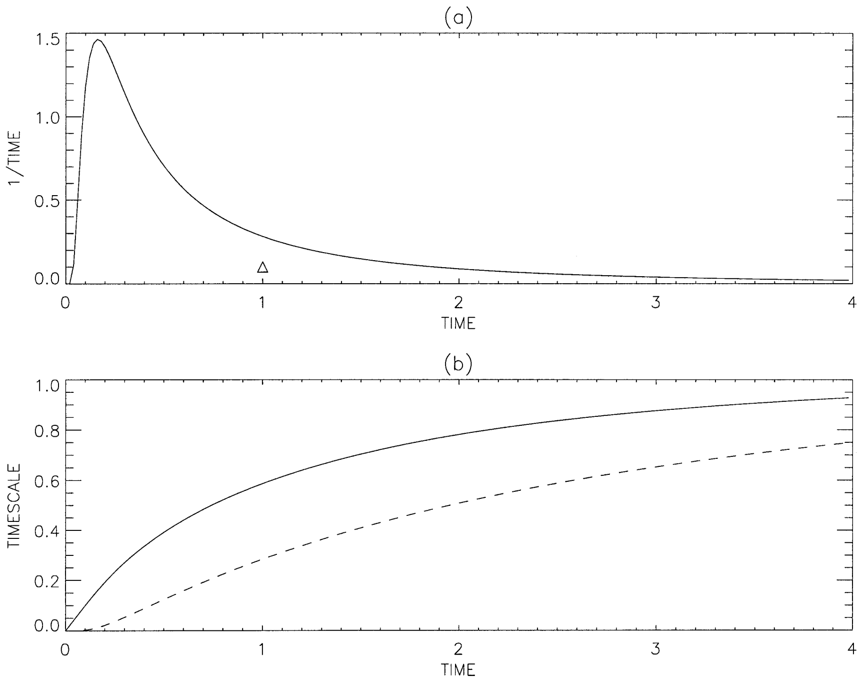

FIG. 1. (a) Age spectrum for the 1D advection–diffusion model with constant coefficients. The spatial evaluation point is $x = k/u$. The time axis is in units of k/u^2, and the mean age is indicated by the symbol. (b) The transient ideal age (solid) and cumulative mean age (dash) for this model, also in units of k/u^2.

could also be obtained by direct differentiation of expression (12).]

The age spectrum for this 1D model is plotted in Fig. 1a, with the mean age indicated by the symbol. Note that the mean age for this idealized model is simply x/u. Despite its simplicity, the age spectrum for this model is qualitatively similar to the spectrum computed in the Atlantic sector GCM of Khatiwala et al. (2001). Figure 1b shows $\tau_{\rm id}$, obtained from relationship (8), and Γ. Note that $\tau_{\rm id}$ and Γ both converge to the mean age, but $\tau_{\rm id}$ does so more rapidly.

5. Discussion and summary

We have derived a simple relationship, valid for any type of stationary transport, between the transient ideal age $\tau_{\rm id}$ of ocean water masses and the age spectrum, G, the distribution of times since the water mass was last ventilated; namely, $G(r, t) = -\partial_{tt}\tau_{\rm id}(r, t)$. This encompasses, but is more general than, the steady-state relationship that the ideal age equals the mean age (the mean over the age spectrum) (e.g., Boering et al. 1996; Khatiwala et al. 2001). The relationship follows naturally from the general frameworks developed by Holzer and Hall (2000), Beckers et al. (2001), and Deleersnijder et al. (2001a) that provide connections between various tracers using the machinary of Green's functions. The relationship derived here implies that the transient ideal age has diagnostic value—much more, in fact, than the equilibrium ideal age itself. No single timescale can completely summarize both the bulk advection and mixing that connect the surface to subsurface regions. By contrast, a tracer's transient solution displays explicitly its sensitivity to the full distribution of timescales, though this information may be convolved with the tracer's particular boundary condition and source distribution. Green's functions, exploited here, are powerful tools to relate tracers to each other and to extract tracer-independent transport information.

The relationships between $\tau_{\rm id}$, G, and Γ concern transport diagnostics that are not generally directly observable, although they may possibly be inferred from combinations of tracers. For ocean model studies, however, these relationships have direct practical utility. First, if one's goal were simply to obtain a model's equilibrium mean age (ideal age) distribution, the ideal age approach is better, since it converges faster. However, it must be recognized that no single timescale can fully characterize the transport, and the age spectrum contains more information than just its mean. Second, to the extent that transport is stationary, it is redundant to simulate both the ideal age and some measure of the boundary propagator. For example, one can find in the ocean modeling literature studies in which both the ideal age and the response to a step function surface boundary condition are simulated (e.g., England 1995). According to the analysis here this is equivalent to simulating both $t - \int_0^t d\xi \int_0^\xi d\xi' G(r, \xi')$ and $\int_0^t d\xi G(r, \xi)$. One could simulate G alone and afterwards reconstruct the other diagnostics. In fact, it may be of interest to modelers who have archived transient data from ideal age simulations to compute *G*, thereby providing additional insight to transport in their models.

Finally, we note that the relationships derived here are strictly valid only for stationary transport. This is clear from relationship (8). Here *G* is always positive, but if there are temporal cycles (e.g., seasonal) in the circulation than τ_{id} will exhibit ''wiggles,'' and its first and second time derivatives will at times be negative. Thus, relationship (8) cannot hold. Nonetheless, if the circulation cycles have periods either much longer than or much shorter than the mean age, then (8) may still hold approximately. If the cycle periods are very long, then the circulation appears approximately stationary over the timescales of interest. If the cycle periods are very short (e.g., seasonal), then suitable filtering will result in an approximate relationship.

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