On “A Consistent Theory for Linear Waves of the Shallow-Water Equations on a Rotating Plane in Midlatitudes”

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ABSTRACT

Recently, Paldor et al. provided a consistent and unified theory for Kelvin, Poincaré (inertial–gravity), and Rossby waves in the rotating shallow-water equations (SWE). Unfortunately, the article has some errors, and the effort is made to correct them in this note. Also, the eigenvalue problem is rewritten in a dimensional form and then nondimensionalized in terms of more traditional nondimensional parameters and compared to the dispersion relations of the old and new theories. The errors in Paldor et al. are only quantitative in nature and do not alter their major results: Rossby waves can have larger phase speeds than what is predicted from the classical theory, and Rossby and Poincaré waves can be trapped near the equatorward boundary.

1. Introduction

In most textbooks on geophysical fluid dynamics (Cushman-Roisin 1994; Holton 1992; Gill 1982; Pedlosky 1987; Salmon 1998) the shallow-water equations (SWE) are studied in their linearized form to derive the dispersion relations and the velocity and free-surface height profiles for Kelvin, Poincaré (inertial–gravity), and Rossby waves; however, the waves are derived in two slightly different contexts. The high-frequency Kelvin and Poincaré waves are derived on an $f$ plane, whereas the low-frequency Rossby waves are obtained using a $\beta$ plane, where the variation of the Coriolis parameter with latitude due to the sphericity of the earth is essential for wave propagation. In other contexts, Rossby waves can be generated whenever there is a background vorticity gradient; two primary mechanisms are topography and background shear flows (Poulin and Flierl 2003a, b). To establish a unified theory that contains all three different wave motions, Paldor et al. (2007, hereafter PRM) study the linearized SWE at midlatitudes on a $\beta$ plane and obtain dispersion relations for the waves by solving the boundary value problem numerically. Some of the statements in PRM are not entirely clear and several of the plots appear to have errors in them.

The outline of this note is as follows: first, we derive the same eigenvalue problem as in PRM but in a dimensional form that parallels the treatment of many textbooks. Then we nondimensionalize and define the nondimensional parameters of PRM in terms of more traditional parameters. By studying the dispersion relation in dimensional form, we show how the classical theory necessarily describes the high meridional-wave-number waves but can produce significant differences with some of the low meridional–wavenumber waves. Finally, we reproduce the calculations presented in PRM using a spectral eigenvalue solver (Trefethen 2000) and present corrected versions of their plots that contain errors.

2. Derivation of the model

We begin by stating the linearized SWE in dimensional form as stated in Eq. (2.2) of PRM:

\[
\frac{\partial u}{\partial t} - (f_0 + \beta y)u = -g \frac{\partial \eta}{\partial x},
\]

\[
\frac{\partial v}{\partial t} - (f_0 + \beta y)u = -g \frac{\partial \eta}{\partial y},
\]

\[
\frac{\partial \eta}{\partial t} + H \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0.
\]

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As is usually done, we look for normal-mode solutions in time and in the zonal direction $x$,
\[
(u, v, \eta) = [\hat{u}(y), -ik\hat{V}(y), \hat{\eta}(y)]e^{i(kx-Ct)} + c.c,
\]
where $c.c$ denotes complex conjugate, and we have introduced hats to differentiate between the separated functions and the total fields. Note that the factor of $-ik$ in the meridional velocity implies that $\hat{V}$ does not have dimensions of velocity.

When we substitute the normal-mode decomposition into Eqs. (1), (2), and (3), we obtain
\[
\frac{\partial \hat{V}}{\partial y} = \frac{f_0 + \beta y}{C} \hat{V} + g \frac{\hat{\eta}}{C}, \quad (4)
\]
\[
\frac{\partial \hat{\eta}}{\partial y} = \frac{k^2 C^2 - (f_0 + \beta y)^2}{g C} \hat{V} - \frac{f_0 + \beta y}{C} \hat{\eta}. \quad (5)
\]

The dimensional parameters $f_0 = 2\Omega \sin \phi_0$, $\beta = 2\Omega/R \cos \phi_0$, $g$, and $H$ are the Coriolis parameter, beta parameter, gravitational acceleration, and the typical depth scale of the homogeneous fluid, respectively. As is typical, $\Omega$ is the frequency of rotation, $\phi_0$ is the latitude, and $R$ is the radius of the earth. If we substitute $g = \alpha$, $f_0 = \sin \phi_0$, $\beta = \cos \phi_0$, $H = 1$, we obtain the nondimensional Eqs. (2.4), (2.5b), and (2.5b) of PRM.

a. Scaling in comparison to QG

The first difficulty that we have observed in PRM is the scaling argument that follows directly after Eq. (2.5b). In the asymptotic limit of small-phase speeds, they estimate the orders of magnitude of $\hat{\eta}/\hat{V}$ and $\hat{u}/\hat{V}$ to be $O(1)$ and $O(1/C)$, respectively. A more systematic analysis shows that they are both order-one quantities.

To demonstrate this, observe that in the limit of $C \ll 1$, Eq. (5) yields the following asymptotic relationship:
\[
\frac{\hat{\eta}}{\hat{V}} = -\frac{f_0 + \beta y}{g} + \frac{C \frac{\partial \hat{V}}{\partial y}}{g \hat{V}} + O(1^3) = -\frac{f_0 + \beta y}{g} + O(1). \quad (7)
\]

This equation states that in the limit of slow-phase speeds, or nearly steady motions (as is appropriate for Rossby waves), the meridional velocity is in geostrophic balance with the free-surface height. We thus confirm the first result of PRM that $\hat{\eta}/\hat{V}$ is order one with respect to the phase speed.

To compare the orders of the two velocity fields, substitute Eq. (7) into Eq. (4) and obtain
\[
\frac{\partial \hat{V}}{\partial y} = -\frac{f_0 + \beta y}{C} + g \frac{\hat{\eta}}{\hat{V}} = \frac{d\hat{V}}{dt} + O(C), \quad (8)
\]
which states that the ratio of the two velocities is an order-one quantity with respect to $C$. PRM only considered the orders of magnitude of the leading-order terms and consequently overlooked the fact that the terms of order $1/C$ cancel exactly. We can then conclude that both quasigeostrophy and the SWE, in the parameter regime considered in PRM, have $\hat{\eta}/\hat{V}$ and $\hat{u}/\hat{V}$ as order one in the limit of small phase speeds. In one of the cases studied in PRM, the wave is trapped because the meridional velocity vanishes in the northern region of the channel. In this region, we cannot apply this scaling analysis because the denominator in both Eqs. (7) and (8) vanishes.

b. Kelvin wave speed

In the paragraph above Eq. (2.6) in PRM, they state correctly that for a Kelvin wave in nondimensional form, $C = \alpha$, where $\alpha$ is their nondimensional parameter defined in our Eq. (11), which implies that
\[
C = \pm \sqrt{\frac{gH}{2\Omega R}}.
\]
Alternatively, if $C$ is a dimensional phase speed, then $C = \sqrt{gH}$. In PRM, they first introduce $C$ as a nondimensional parameter, and then in the paragraph following Eq. (2.6), they must take it to be dimensional because they state that $C = \pm \sqrt{gH}$. This ambiguity occurs because they are using the same symbol $C$ for both the dimensional and nondimensional forms.

3. The eigenvalue problem

Equations (4)–(6) can be combined to form one scalar, second-order boundary value problem,
\[
\frac{d^2 \hat{V}}{dy^2} + \left[ \frac{k^2 C^2}{gH} - \frac{\beta}{C} \frac{(f_0 + \beta y)^2}{gH} \right] \hat{V} = 0, \quad (9)
\]
with the boundary condition that $\hat{V}$ vanishes on the two boundaries; that is,
\[
\hat{V}(y = \pm R\delta\phi) = 0. \quad (10)
\]
Equation (9) is a dimensional version of Eq. (2.7) in PRM and appears as Eq. (15.18) of LeBlond and Mysak (1982) and as Eq. (11.6.1) of Gill (1982) for the special case of equatorial dynamics $f_0 = 0$.

The nondimensionalization of the SWE in PRM is done in the following way: the time scale is the inverse of the earth’s rotation frequency, $(2\Omega)^{-1}$, the length
scale is the radius of the earth, $R$, and the velocity scale combines these two, $2\Omega R$. Due to the linear dynamics, the only nondimensional parameter that appears explicitly in their Eq. (2.7) is 
\[
\alpha = \frac{gH}{(2\Omega R)^2}.
\]

The above parameter is the square of the ratio of the external radius of deformation at the poles normalized by the radius of the earth; thus, it can be seen as the square of the inverse of a Froude number.

Perhaps a more natural nondimensionalization of Eq. (9) is as follows: use the half-width of the channel $L = R \delta \phi$ for the length scale; the inverse of $f_0$ as a typical time scale; and $L f_0$ as a velocity scale. We define the external Rossby deformation radius as $L_R = gH/f_0$, and we arrive at the following nondimensional equation and parameters [Eq. (2.9) of PRM]:
\[
\varepsilon^2 \frac{d^2 \hat{V}}{dz^2} + [E_n - (1 + b z^2)] = 0 \quad \text{and} \quad V(z = \pm 1) = 0, \quad (12)
\]

with $z = y/(R \delta \phi)$ and the following nondimensional parameters of 
\[
\varepsilon = \left( \frac{L_R}{L} \right)^2, \quad (13)
\]
\[
b = \frac{\beta L}{f_0}, \quad (14)
\]
\[
E_n = \frac{k^2 C_n^2}{f_0^2} - L_R^2 \left( k^2 - \frac{\beta}{C_n} \right). \quad (15)
\]

We have added a subscript $n$ to the eigenvalue and the phase speed to emphasize that there are different eigenvalues for $n$ ranging over the nonnegative integers. It is clear from Eqs. (13) and (14) that the $\varepsilon$ is the square of the inverse of the Froude number and that $b$ measures the relative importance of the $\beta$ parameter to the reference Coriolis parameter $f_0$. The definitions of $\varepsilon$ and $b$ are equivalent to Eqs. (2.10a) and (2.10b) of PRM. The definition for the eigenvalue $E_n$ in Eq. (15) varies slightly from PRM because we have chosen our length scale based on the channel width $L = R \delta \phi$ instead of simply $R$. Also, in the limit of large horizontal length scales, $\varepsilon^2 \ll 1$, it is apparent that $E_n$ is proportional to the square of the nondimensional meridional wavenumber.

a. Dispersion relations

If we consider the dimensional eigenvalue problem, Eq. (9), we observe that in the limit where $b \ll 1$ (either very long length scales or near-equatorial dynamics) we can solve the problem exactly. The solution is
\[
\hat{V} = \sin \left[ \frac{(n + 1) \pi y}{2L} \right] \quad \text{with} \quad E_n = 1 + \left[ \frac{(n + 1) \pi \varepsilon}{2} \right] \quad \text{for} \quad n = 0, 1, 2, \ldots, \quad (16)
\]

where $E_n$ contains the leading-order effect of $\beta$. The dispersion relation is a cubic equation that relates the eigenvalue $E_n$ with the phase speed, which in dimensional form is
\[
\frac{k^2}{f_0^2} C_n^4 = \frac{E_n + L_R^2 k^2}{E_n - L_R^2} C_n - L_R^2 \beta = 0. \quad (17)
\]

The solutions for each eigenvalue correspond to one Rossby and two Poincaré waves.

The dispersion relation for Rossby waves corresponds to the solution with slow phase speeds, meaning we can neglect the cubic term and hence obtain
\[
C_{\text{Rossby}} \approx \frac{-\beta}{k^2 + E_n - L_R^2}. \quad (18)
\]

In the case of relatively small $b$, we obtain using Eq. (16) that the leading-order expansion of the frequency is
\[
\omega_{\text{Rossby}} \approx \frac{-\beta k}{k^2 + \left[ \frac{(n + 1) \pi \varepsilon}{2L} \right]^2 + L_R^2}, \quad (19)
\]

which is the dispersion relation for Rossby waves in a channel according to the classical theory. For the faster Poincaré waves, we assume that $C \gg 1$ and neglect the constant term in Eq. (17):
\[
C_{\text{Poincare}}^2 \approx g H + E_n \left( \frac{f_0^2}{k^2} \right). \quad (20)
\]

In the limit of small $b$, we get that the leading-order equation is
\[
\omega_{\text{Poincare}}^2 \approx f_0^2 + g H \left[ 1 + \left[ \frac{(n + 1) \pi \varepsilon}{2L} \right]^2 \right], \quad (21)
\]

which is the dispersion relation for Poincaré waves in a channel as determined from the classical theory. If we focus on the first mode, as is done in PRM, we set $n = 0$ in Eq. (16) and obtain that $E_0 = 1 + (\pi \varepsilon/2)^2$. When $E_0$ is not equal to this critical value, differences occur between the two theories but only for the low meridional-wavenumber modes. This is because the relative error between the exact and approximate eigenvalues tends to 0 quadratically with increasing $n$. 

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b. Wave trapping

In Eq. (9) we denote the quantity in the square brackets as \( Q(y) \). The eigenvalue problem can have either oscillatory or exponential solutions depending on whether the function \( Q(y) \) is positive or negative, respectively. In the classical theory, the coefficient is always nonnegative, which is why the modes are necessarily oscillatory. Alternatively, if \( Q(y) \) is sufficiently large, or conversely if \( E_n \) is sufficiently small, \( Q(y) \) is negative for low meridional wavenumbers and the solution decays away from one boundary and is thus trapped. Because \( Q(t) \) is a decreasing function of latitude, trapping must necessarily occur at the southern, or more generally equatorward, boundary. The criterion for trapping is that for a particular mode \( b > \sqrt{E_n - 1} \).

If \( Q(y) \) is strictly positive, then we can use Wentzel–Kramers–Brillouin (WKB) theory (Nayfeh 2000) to approximate the solution for any modes where \( E_n - 1 > b \) and \( \varepsilon \ll 1 \). To ensure satisfying the boundary condition at the southern wall, we can write the solution as

\[
V = Q^{1/4}(y) \sin[e^{-1}\int_{-L}^{y} \sqrt{Q(x)} \, dx] , \tag{22}
\]

where the eigenvalues are determined by the following integral equation:

\[
\int_{-1}^{1} \sqrt{E_n - (1 + bx)^2} \, dx = 0 . \tag{23}
\]

Even though this integral can be evaluated exactly, it yields a highly nonlinear algebraic equation, which is why we prefer to leave it in this compact form. This equation gives a more accurate approximation of \( E_n \) than does Eq. (16).

In the case where \( Q(y) \) changes sign, we have a turning point problem (Nayfeh 2000). Approximate solutions can be found in the two regions using WKB theory in each of the regions where \( Q(y) \) is of a particular sign; at the point where \( Q(y) = 0 \), a turning point, we can approximate it using an Airy function. The mathematics do not give rise to any simple equations analogous to Eq. (23), which is why we do not present the details.

An alternative approach in this parameter regime is to approximate \((f_0 + b \gamma)^2 \approx (f_0^2 + 2b)\gamma\) thereby only neglecting the term of order \( b^2 \) rather than order \( b \), as we did before. The resulting equation is an Airy equation and the solutions can be written in terms of Airy functions, though this does not yield an easy means to approximate the eigenvalues \( E_{\alpha} \).

c. Problems with the analytical solutions

Section four of PRM is dedicated to finding the eigenvalues of the boundary value problem. In this section, PRM comment on how the only analytical solutions exist for the special cases of \( b = 0 \) and \( b = 1 \). In the former case, they present phase speeds for the Rossby and Poincaré waves; however, these are somewhat misleading. If we reinterpret \( b = 0 \) in Eq. (14) in terms of our dimensional parameters, we see that this requires that either \( \beta \) or \( L \) vanish. The latter option is not possible because the channel width cannot be zero, and thus this must correspond to the \( f \)-plane limit. In this case, Rossby waves cannot propagate because they have a zero phase speed. Equation (4.2) suggests that the Rossby wave phase speed is nonzero; however, when the right-hand side is evaluated with \( b = 0 = \cos\theta_0 \), we observe that the phase speed is in fact 0. It is certainly possible to find an asymptotic solution for the case of \( 0 < b \ll 1 \), but this is beyond the scope of this note.

For the case of \( b = 1 \), if we define \( x = (1 + z)/\sqrt{\varepsilon} \), Eq. (12) can be transformed into the following equation:

\[
\frac{d^2 V}{dx^2} + \left( \frac{E}{\varepsilon} - x^2 \right) \tilde{V} = 0 \quad \text{and} \quad V(x = 0, 2/\sqrt{\varepsilon}) = 0.
\]

If \( x \) is defined on an infinite domain, it is well known (Lifshitz and Landau 1981) that the eigenvalues are \( E = (2n + 1)\varepsilon \). If, furthermore, we impose the condition that at \( x = 0 \) the solution is 0, then we only retain the odd eigenfunctions and thus \( E = (4m + 3)\varepsilon \) for any positive integer \( m \), as verified by the power series expansion of section 4(b) of PRM. Our domain is bounded, however, so these eigenvalues are only approximate; indeed, the smaller the \( \varepsilon \), the more eigenvalues are accurately described by this approximate relationship.

PRM focus on wave dynamics at the midlatitudes, which is why they restrict their attention to values of \( b \) smaller than one. In the subtropics, \( b \) is larger than unity; at the equator, \( b \) is unbounded. It is for this reason that on global scales, there is a wide range of values of \( b > 1 \), which are of great interest to study. At the equator, exact solutions for the phase speed and structure of the normal-mode solutions can be found in classical textbooks such as section 11.6 of Gill (1982). Using the methods described in the next section, we have verified numerically that solutions to Eq. (12) for increasing values of \( b \) approach those of the analytical solutions on an equatorial \( \beta \) plane.
4. Numerical solutions

Following the works of Poulin and Flierl (2003a,b); Le Sommer et al. (2006); Scherer and Zeitlin (2008); and Gula et al. (2008, manuscript submitted to J. Fluid Mech.), we solve Eq. (12) using a spectral collocation method on a Chebyshev grid as explained in Trefethen (2000). We use 120 grid points but found that there was excellent convergence with only 50 points in the Chebyshev grid. We have posted our code online (http://www.math.uwaterloo.ca/~fpoulin/LinearSWE) for public accessibility. We have reproduced the plots presented in PRM and have found several discrepancies.

In our Figs. 1, 2, we compare the dispersion relations between the classical and new theories for a wider range of wavenumbers for the Rossby and Poincaré waves, respectively. The eigenvalues for these two cases are $E_0/\alpha = 0.288$ and $E_0/\alpha = 0.753$, which yield significant differences between the two theories. As stated in PRM, the new theory in the first subplot of Fig. 1 yields Rossby wave phase speeds that are over 2 times larger than the classical theory. This is an important finding because the Ocean Topography Experiment (TOPEX/Poseidon) data seem to have found Rossby waves that travel faster than previously expected. PRM suggest their theory as a possible mechanism to explain that discrepancy. Importantly, the third plot in Fig. 3 of PRM has the uppermost contour mislabeled; it is the 0.95 contour, not 1.00 as indicated in their figure label.

Figure 4 in PRM compares the dispersion relation for $\alpha = 10^{-5}$ and $\alpha = 10^{-3}$ between the classical and new theories. The $x$ axis is $0 \leq k \leq 20$, which only focuses on a narrow range of wavenumbers and completely neglects the region where the frequency increases with $k$. Figure 5 in PRM depicts the structure of the first mode for the following parameters: $\epsilon = 2$, $b = 0.1$, $k = 1$, $\phi_0 = 1$, and $E_0 = 10.871$. The figure shows the following fields: $\hat{V}$; $\hat{\eta}$, $\hat{u}$ for the Rossby wave; $\check{h}$, $\check{u}$ for the Poincaré wave; the exponents of the Kelvin wave in both the classical $f$-plane theory and the new theory normalized such that they are both one at $z = -1$. We reproduce the calculation in Fig. 3 and find that the first and fourth plots are identical; however, the magnitude and structure of the $u$ fields for the Rossby wave is incorrect. First, they seem to have plotted the Rossby wave $\check{u}$ field in the Poincaré plot, and vice versa; in addition, the zonal velocity fields have incorrect magnitudes. Our figure indicates that the magnitude of $\check{u}/\hat{V}$ is $O(1)$, not $O(1/C)$ as suggested in PRM. Notice that
we have scaled the velocities by factors of 100 and 10 so that we can see the structure of the $\vec{u}$ field. In this case, the approximately analytical solution in Eq. (16), which is precisely what the classical theory predicts, is extremely accurate and predicted both the eigenvalue, $E_0 = 10.869$, and the eigenvector.

Figure 6 of PRM is organized in the same fashion as the previous figure, except that the parameters are $\epsilon = 0.055$, $b = 0.15$, $k = 1$, and $\phi_0 = 1$. Our correction of their figure is presented in Fig. 4. Again, PRM plotted the zonal velocities in the wrong subplots and their magnitudes differ from those in our corrected figure. Moreover, the magnitudes of $\eta$ differ as well. The meridional velocity and the exponents of the Kelvin wave appear to be identical. The eigenfunction in the first subplot is trapped near the southern wall, as previously stated in PRM. This occurs because $Q(t)$ is positive near the south wall and changes sign at the turning point $z = -0.47$, which is where the solution begins to decay exponentially.

PRM plotted the horizontal divergence $\delta$ and the vertical component of vorticity $\zeta$ for $\epsilon = 2$, $b = 0.1$, $k = 1$, and $\phi_0 = 1$. We have a corrected version of their Fig. 7 in our Fig. 5. The vorticity and divergence fields for the vorticity are essentially the same; however, the divergence and vorticity of the Poincaré waves appear to differ by approximately a factor of 2. Figure 8 of PRM is like the previous figure, except it has the parameters $\epsilon = 0.055$, $b = 0.55$, $k = 1$, and $\phi_0 = 1$. We produce a corrected version in our Fig. 6. Again, both the divergence and vorticity fields for the Rossby wave are correct. The form of the vorticity and divergence of the Poincaré wave are similar but differ in magnitude. These figures both illustrate, as stated in PRM, that the Rossby waves have negligible divergence in comparison to their vorticity. In contrast, the Poincaré waves have a divergence and a vorticity that are of the same order.

5. Conclusions

We have derived a dimensional version of the eigenvalue problem in PRM on a $\beta$ plane that contains Kelvin, Poincaré, and Rossby waves. If the beta effect is relatively small, $b \ll 1$, the dispersion relation for the later two waves reduces to the classical theory if
Because this is not necessarily the case in general, as is illustrated in Fig. 2 of PRM, this new theory is able to describe linear dynamics more accurately than the classical theory. The difference only occurs for low wavenumber waves, but these are usually the waves of greatest interest, which is one reason why this is a significant finding.

By solving the boundary value problem numerically we are able to obtain solutions for $E_n$ and the corresponding eigenfunctions. We have used our code to plot corrections to the figures of PRM, which contain errors. We have restricted our attention to the first mode because that is where the differences are most significant between the two theories. The two major findings of PRM that we have verified are that the phase speed for Rossby waves can be larger than the classical theory, which perhaps explains in part the recent findings on TOPEX/Posiedon data. As well, Rossby and Poincaré waves can be trapped near the equatorward wall.

PRM focused on wave dynamics at the midlatitudes and solved for typical wave structures in this region. The model is more universal than the classical theory because it can be applied at any latitude. At the equator, exact solutions are known, as can be found in Gill (1982). It would be of interest to study the wave behavior in the subtropics because that encompasses a rather large region of the earth’s surface. Also, it should be easy to generalize this approach to the study of stratified flows.

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