Hybrid Rossby-Shelf Modes in a Laboratory Ocean

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(Manuscript received 16 July 2008, in final form 1 April 2009)

ABSTRACT

Idealized laboratory experiments reveal the existence of forced–dissipative hybrid Rossby-shelf modes. The laboratory ocean consists of a deeper ocean (accommodating basin-scale Rossby modes) and a coastal step shelf (accommodating trapped shelf modes). Planetary Rossby modes are mimicked in the laboratory via a uniform topographic slope in the north–south direction. Hybrid modes are found as linear modes in numerical calculations, and similar streamfunction patterns exist in streak photography of the rotating tank experiments. These numerical calculations are based on depth-averaged potential vorticity dynamics with Ekman forcing and damping. Preliminary nonlinear calculations explore the deficiencies observed between reality and the linear solutions. The aim of the work is twofold: to show that idealized hybrid Rossby-shelf modes exist in laboratory experiments and to contribute in a general sense to the discussion on the coupling and energy exchange associated with hybrid modes between shallow coastal seas and deep-ocean basins.

1. Introduction

There is evidence that planetary-scale Rossby waves have been generated off the western coast of the United States, either by unstable coastal boundary currents or by coastal waves matching in frequency and scale (cf. Kelly et al. 1998). Bokhove and Johnson (1999), therefore, investigated the matching of planetary Rossby modes with coastal shelf modes in a cylindrical basin. Otherwise said, linear free modes were calculated with so-called semi-analytical “mode matching” techniques, as well as linear forced–dissipative finite-element methods, to find resonances. Two parameter regimes were considered: an ocean one and a laboratory analog. These laboratory-scale hybrid Rossby-shelf modes had been considered with validating laboratory rotating tank experiments in mind.

Such an experimental validation is the topic of the present paper. Planetary barotropic Rossby modes have been shown before in the laboratory using the analogy between planetary β–plane Rossby modes and topographic shelf modes for a uniform basin-scale north–south background topography—for example, in the classic book of Greenspan (1968). Rotating tank experiments geared toward enforcing resonant hybrid coastal and planetary modes appear to be (relatively) new. In preparation of the rotating tank experiments, linear forced–dissipative finite-element calculations of barotropic potential vorticity dynamics have revealed the resonant frequencies of two primary hybrid Rossby-shelf modes. These primary forcing frequencies were then used to drive the harmonic “wind” forcing provided via the Ekman pumping and suction due to an oscillating rigid lid. Various forcing strengths have been imposed in which a match with the linear calculations is best suited by weak forcing, whereas better visualization requires a larger signal-to-noise ratio and, consequently, stronger forcing. The latter promotes, however, the emergence of nonlinear effects. We, therefore, also compare the experimental streamfunction fields with some nonlinear simulations of barotropic potential vorticity dynamics in an attempt to explain the differences between linear theory and the experimental results under stronger forcing.

In ocean general circulation models (OGCMs), the coastal regions are often underresolved; furthermore, it is hypothesized that significant energy exchange takes place between the deeper ocean and the shallower coastal zones, for example, Wunsch (2004). Vertical walls placed in the shallow seas are generally used in coastal zones as lateral boundaries in OGCMs, whereas in reality, mass, momentum, and energy are transferred across these...
virtual vertical walls. As a consequence, such an exchange would not be resolved or modeled properly in the OGCMs. Suitable parameterizations of these unresolved processes would therefore be required in the coastal zones of OGCMs. The laboratory experiments and finite element calculations we present aim to serve as an idealized barotropic system to investigate this modal coupling between basin- and coastal-scale dynamics.

The outline of this paper is as follows: barotropic potential vorticity dynamics is introduced in section 2, and linear finite-element calculations are presented to find the relevant forcing frequencies. These forcing frequencies are a building block in section 3, where the experimental setup and results are presented. Preliminary nonlinear simulations in section 4 indicate the effects of strong forcing on the dynamics observed. A short conclusion is found in section 5.

2. Rigid-lid potential vorticity model

a. Nonlinear model

The forced–dissipative evolution of vertical vorticity \( \omega \) in a rigid-lid model is governed by the following dimensional system of equations:

\[
\frac{\partial \omega}{\partial t} + \nabla \cdot \left( \frac{f + \omega}{H(x)} \right) = \text{curl} \mathbf{F}(x, t) - \frac{(f + \omega)}{H(x)} \alpha D_E \omega \quad \text{and} 
\]

\[
\omega = \nabla \cdot \left[ \frac{1}{H(x)} \nabla \Psi \right], 
\]

where the horizontal coordinates are \( x = (x, y)^T \), \( t \) is time, \( \nabla = (\partial_x, \partial_y)^T = (\partial \partial_x, \partial \partial_y)^T \), the Jacobian \( J(P, Q) = \partial_x P \partial_y Q - \partial_y P \partial_x Q \) for functions \( P = P(x, y, t) \) and \( Q = Q(x, y, t) \), the Coriolis parameter \( f = f(x, y) \), the transport streamfunction is \( \Psi \), total depth \( H(x) \), forcing curl \( \mathbf{F} \), and Ekman layer depth \( D_E = \sqrt{v/\Omega} \) with (effective) viscosity \( \nu \). For the laboratory case \( f = f_0 = 2\Omega \) is constant with \( \Omega \), as the rotation rate of the domain, whereas for the planetary case a \( \beta \)-plane approximation is used (e.g., Pedlosky 1987) and \( f = f_0 + \beta y \) for constant \( \beta \). The forcing either relates to the wind stress or the differential velocity of the rigid lid, provided by Ekman suction or pumping, respectively (e.g., Pedlosky 1987, 1996). Hereafter, unless otherwise indicated, we consider the laboratory case with

\[
\text{curl} \mathbf{F}(x, t) = \frac{(f + \omega)}{H} \frac{1}{2} D_E \omega_T, 
\]

in which the vorticity at the top rigid lid

\[
\omega_T = \partial_x \nu_T - \partial_y \mu_T
\]
is related to the velocity \( \mathbf{v}_T(x, y, t) = (u_T, v_T)^T \) of the driven rigid lid. Ekman damping is assumed to be valid in the transition region between a quasigeostrophic deep ocean and the ageostrophic coastal zone, even though the topography is rapidly changing from open ocean to coastal zone. For the rigid-lid case \( \alpha = 1 \), it results in twice the amount of Ekman damping relative to the case with a free surface for which \( \alpha = \frac{1}{2} \). The above system can be derived using classical methods (Pedlosky 1987; Cenedese et al. 2007).

The dimensional equations are scaled with radius \( R \) of the cylindrical domain, a typical depth \( H_0 \) of the domain, and the Coriolis parameter \( f_0 \) (in which starred variables are dimensional):

\[
t^* = \frac{t}{f_0}, \quad (x^*, y^*) = R(x, y), \quad \Psi^* = \Psi R^2 H_0 f_0, 
\]

and

\[
\omega^* = f_0 \omega, \quad H^* = H_0 H, \quad f^* = f_0 f. 
\]

The potential vorticity of the fluid is defined by

\[
\xi = \frac{(f + \omega)}{H}. 
\]

The evolution of this potential vorticity weighted by the depth \( H \) follows by scaling with (4) and rewriting the nondimensionalized form of the system (1) into

\[
H \frac{\partial \xi}{\partial t} + \nabla \cdot (U \xi) = \kappa \left( \frac{1}{2} \omega_T + \alpha f - \alpha H \xi \right) 
\]

\[
= \kappa \left( \frac{1}{2} \omega_T + \alpha f - \alpha \xi \right), 
\]

\[
U = \nabla \cdot \Psi, \quad \text{and} 
\]

\[
\nabla \cdot \left[ \frac{1}{H(x)} \nabla \Psi \right] = H \xi - f 
\]

with transport velocity \( U \), two-dimensional curl operator \( \nabla \times = (-\partial_y, \partial_x)^T \), and parameter

\[
\kappa = \frac{D_E}{H_0} = \frac{\sqrt{\nu / \Omega}}{H_0}. 
\]

A cylindrical domain \( \Omega \) is considered with \( \Psi = 0 \) at the boundary \( \partial \Omega : r = R \) and initial conditions \( \xi = \xi(x, y, 0) \); we also used \( \xi \approx 1/H \) to simplify forcing and damping terms.

The numerical (dis)continuous Galerkin finite-element discretization used is based on formulation (6), by extending the inviscid formulation in Bernsen et al. (2006); it couples the hyperbolic potential vorticity Eq. (6a) to the elliptic Eq. (6c) for the transport streamfunction and is
advantageous for complex-shaped domains. Instead of a classical continuous finite-element method as used for the streamfunction, potential vorticity is discretized discontinuously. For smooth profiles of potential vorticity, the numerical discontinuities between elements are negligible and scale with the mesh size and order of accuracy. The numerical method conserves vorticity and energy for infinitesimally small time steps in the inviscid and unforced case, whereas enstrophy is slightly decaying for the upwind flux used. The weak formulation of this finite-element method is given in the appendix. The method provides an alternative to classical numerical methods and is well suited for complex-shaped domains, mesh, and order (h and p) refinement.

b. Linear model and hybrid Rossby-shelf modes

We have performed laboratory experiments to assess whether hybrid Rossby-shelf modes exist that couple planetary-scale Rossby modes with coastal-scale shelf modes. In the laboratory experiments, a uniformly rotating tank is used with rotation frequency $\Omega_r$. Consequently, there is no planetary variation of the background rotation in the north–south direction. We consider the Northern Hemisphere and define a north–south direction indirectly by introducing a background slope $s = s(y)$ in the y direction, which is related mathematically to the $\beta$ effect (e.g., Greenspan 1968). Hence, the nondimensional $\beta$ parameter $\beta = s/H$ at leading order for a mean depth $\bar{H}$.

FIG. 1. Sketch of laboratory domain with abrupt shelf topography, and deep interior ocean and shallow-shelf slopes mimicking $\beta$ (Bokhove and Johnson 1999).
forcing frequencies

value

H

with forcing frequency

k

Rossby-shelf modes displayed as the symmetric shelf and

r

A second-order Galerkin finite-element model (FEM) took 4671 nodes and 4590 quadrilateral elements on an unstructured mesh. In this FEM, a matrix system results from the forcing, and matrix elements \( A_{ij} \) the combination of the rotational advective and dissipative terms. To test the finite-element implementation, we successfully recovered the forced-dissipative analogs of the free planetary Rossby modes and their frequencies [appendix b(1)].

The forced–dissipative response of the laboratory ocean, described above and sketched in Fig. 1, is displayed in Fig. 2 for \( \kappa = \sqrt{\nu/\Omega H_0} = 0.042 \) and \( \Delta \theta = 2\pi \) (using laboratory values of the viscosity of water \( \nu = 10^{-6} \text{ m}^2 \text{ s}^{-1} \), \( \Omega_r = 2 \text{ s}^{-1} \), and \( H_0 = R = 0.17 \text{ m} \)). The streamfunction field at maximum response is shown in Fig. 3 at forcing frequency \( \sigma = 0.0613 \). When the experiments were performed in late 1997, Onno Bokhove (O. B.) calculated a resonance at \( \sigma = 0.0612 \) for fewer and less accurate triangular elements and used that slightly different resonant frequency in the laboratory. This small difference, however, falls within the accuracy to which the forcing frequency could be determined in the laboratory. Another resonant frequency resides at \( \sigma = 0.0878 \) (calculation FEM 2009), where O. B. found \( \sigma = 0.0871 \) earlier (calculation FEM 1997) with streamfunction fields shown in Fig. 4. The modal structure for either value—\( \sigma = 0.0878 \) (or 0.0871)—is the same, and the mode relates most clearly to the \( m = 2 \) shelf mode. In these cases, the hybrid Rossby-shelf mode is a combination of an azimuthal mode number \( m = 0 \)--Rossby mode in the deep ocean absorbing into an \( m = 2 \)--shelf mode at the western shelf, which after traversing counterclockwise along the southern shelf edge at \( R_0 = 0.8r \) radiates again into the deep-ocean “planetary” Rossby mode. Such behavior is suggested directly from the dispersion relations (A6) and (A15) of the inviscid or free planetary Rossby wave and shelf modes, plotted separately in Fig. 5. The exact solution of the planetary Rossby mode is given in (A5) and (A6) and one of the shelf modes in (A15). The modal structure of the free planetary Rossby mode clearly includes the east–west structure in its explicit \( x \) dependence, whereas the free shelf mode is well described solely in terms of polar coordinates. The spatial structure of the hybrid-shelf mode, for the case with both \( \beta \)-plane topography and a step shelf, is thus seen to correspond best with a combination of the \( m = 0 \) planetary free Rossby mode, with its explicit \( x \) and \( r \) dependence, and the \( m = 2 \) free shelf mode, with its explicit \( r \) and \( \theta \) dependence. In contrast, the modal structures of the \( m = 1 \) free Rossby mode and the \( m = 1 \) free shelf mode do not match that well, have more structure, and are therefore more affected by damping. The resonant peak expected at around \( \sigma = 0.04 \) for this \( m = 1 \) mode appears to be wiped out in Fig. 2. [See also the relevant \( m = 1 \) modes displayed in Figs. 3 and 4 in Bokhove and Johnson (1999).]

In the nonlinear numerical simulations described in the next section, instead of the abrupt step shelf break, a smoothed shelf break is introduced between \( R_0 - \epsilon < r < R_0 + \epsilon \). The finite element mesh contains regular nodes placed at the circles with radii \( R_0 \pm \epsilon \) and \( R \); then
FIG. 3. Streamfunction field for Rossby-shelf modes over one forcing period $T$ nearby the maximum response $\sigma = 0.0613$. Parameter values $\omega_T^{\text{max}} = 2\sigma \Delta \theta = 2\sigma 2\pi$ and $\kappa = \sqrt{\nu \Omega / H_0} = 0.0042$. 
FIG. 4. Streamfunction field for Rossby-shelf modes over one forcing period $T$ at the maximum response $\sigma = 0.0878$. 
random nodes are added, subject to a minimum distance criterion outside the shelf break; subsequent triangulation yields a triangular mesh; and additional nodes are placed at the centroid and midpoints of the edges of each triangle to further divide the triangular mesh into a quadrilateral one. The shelf break contains two elements across and upon mesh refinement, the value of $e$ decreases and hence the shelf break also becomes narrower. For such smoothed topography, the linear forced–dissipative response and the streamfunction field at the maximum resonance are given in Figs. 6 and 7 for $e = 0.0314$. Relative to the abrupt shelf topography with resonant forcing frequency $s = 0.0613$, the resonant frequency in the new calculation of the linearized system has lowered by 6% to $s = 0.0577$, whereas the actual fields at resonance remain highly similar.

3. Laboratory experiments

a. Experimental setup

The laboratory tank had the following dimensions: $R^* = 16.7$ cm, $R_e = 0.8R$, and $\beta = \beta^* R^*/(2\Omega) = s/H_e = 0.3125$ for $i = 1, 2$. The topography was cut out of foam, subsequently made smooth with paste, and fit tightly into the cylindrical tank. The average shelf and interior depths were $H_1 = 0.6 \ R$ and $H_2 = 0.8 \ R$, respectively. The corresponding laboratory setup is sketched in Fig. 8. Dimensionless variables therein are defined in terms of the dimensional tank radius $R^*$ and $H_0 = R^*$. The glass plate on top of the water column was harmonically oscillating in a horizontal plane, its motion driven by a programmable stepping motor connected to the plate with a driving belt. The nondimensional azimuthal velocity of the rigid lid is compare (8), with $\Delta \theta$ the maximum angle of the lid reached over the forcing period $T = 2\pi/\sigma$. A thin horizontal light sheet had been constructed with a tungsten lamp, including a linear filament and a lens. The black background permitted optimal reflection of light from whitish Pliolite particles of diameter 150–250 $\mu$m suspended in the flow. An analog photo camera mounted above the rotating tank tracked the movement of the particles. The whole configuration was placed on a uniformly rotating table with $\Omega = 1 \ s^{-1}$. An approximate steadily oscillating state was achieved by spinning up the table for about 30 min (i.e., 30–50 forcing periods).

b. Coupled modes

Numerous experiments were carried out for a few forcing frequencies. For each experiment, streak photography was obtained with 2–4-s exposure time. The main forcing frequencies used in the laboratory experiments were calculated with a finite-element model of the linearized equations, as explained in section 2b. We report here solely four sets of experiments, deemed best for their visual resolution and the hybrid character of the Rossby-shelf mode.

The two sets of eight images in Figs. 9 and 10 give an impression of the flow during one forcing period of 51.3 s—that is, with dimensional frequency $\sigma^* = 0.1226 \ s^{-1}$ ($\sigma = 0.0612$). It nearly corresponds to a numerically calculated forced–dissipative resonance for a hybrid Rossby-shelf mode of the rigid-lid model (6), linearized around a state of rest (see Figs. 2 and 3). The underlying Rossby
FIG. 7. Streamfunction field for Rossby–shelf modes over one forcing period $T$ at the maximum response $\sigma = 0.0577$ for a smoothed shelf break. Parameter values $\omega_T^{\text{max}} = 2\sigma \Delta \theta = 2\sigma 2\pi$ and $\kappa = \sqrt{\nu \Pi_T / H_0} = 0.0042$. 
mode has azimuthal mode number zero, whereas the underlying shelf mode has azimuthal number \( m = 1, 2 \). The shelf break is visible as a thin whitish line at \( r = 0.8 R \). In the time sequence from top left to bottom right, a Rossby mode circulation cell in the deep interior “ocean” travels westward, where it absorbs onto the shelf and propagates counterclockwise (in the Northern Hemisphere) as a trapped shelf mode circulation cell. On the eastern boundary, this shelf mode radiates into a planetary Rossby mode. Apart from the striking qualitative resemblance with linear forced–dissipative calculations in Fig. 3, discrepancies occur in the northwest, presumably as a result of nonlinear effects, and in the east where the shelf mode disappears, presumably as a result of strong damping. The rigid lid or glass plate has rotated from zero to a relatively large angle, \( 2\pi \) and \( \pi \) (Figs. 10 and 9, respectively), and back during a period. The nonlinear oscillations in the northwest corner at \( t = 14, 42 \) s in Fig. 10 are larger under the greater forcing amplitude than at \( t = 0, 28 \) s in Fig. 9. These oscillations diminish even more when the forcing amplitude is reduced to \( \pi/2 \), which is not shown here. The experimental dilemma is that a comparison with linearized modal solutions requires a weak forcing, whereas good visualization requires strong forcing for the streak photography used. Particle image velocimetry techniques could have been used for weak forcing as well, but they were not available in 1997 at the rotating tank facilities in Woods Hole. Although the tank dimensions are not shallow, the simplifying assumption has been that the rotation is sufficiently strong to render the flow to be nearly two-dimensional outside the thin Ekman top and bottom boundary layers, the sidewall boundary layers, as well as the internal and boundary layers at the shelf break.

To compare the amplitudes observed and calculated, the streaks under 4-s exposure are compared with streak lengths in the calculation for \( \sigma = 0.00613 \) and \( \Delta \theta = \pi \) in Fig. 11. Note that the calculated streaks are weaker, about 40%, than the observed streaks. Such a difference also occurs for forcing with \( \Delta \theta = 2\pi \) in Fig. 12. The precise location of the mode around resonance, and hence its amplitude, as well as nonlinear shifts, might cause this discrepancy. The dimensionless speed at \( (x, y) = (-0.11, 0.11) \) is about 0.0276; at the southern shelf, the maximum speed is about 0.0773 in Fig. 11a. The speed at \( (x, y) = (0, 0) \) is about 0.0543; at the southern shelf, the maximum speed is about 0.1087 in Fig. 12a.

Similarly, the linear forced–dissipative mode calculated at \( \sigma = 0.0878 \) corresponds reasonably well with the observed mode for \( \sigma = 0.0871 \) (based on the FEM calculation in 1997) in Fig. 13 concerning observations for stronger forcing with \( \Delta \theta = 2\pi \).

Finally, we conclude that hybrid Rossby-shelf modes exist and can be successfully visualized and measured in the laboratory; they correspond well with linear forced–dissipative calculations at a similar frequency. Discrepancies between the experimental and numerical flow patterns are observed especially at the northwestern boundary. Additional nonlinear simulations aim to explain this discrepancy.

4. Laboratory results versus numerical simulations

Nonlinear simulations at resonance frequency \( \sigma = 0.0613 \) have been performed, starting from a state of rest and with sinusoidal forcing. The forcing period is thus 102.5 time units; in addition, \( \kappa = 0.0042 \), with a smoothed shelf break of width \( 2\varepsilon = 0.0628 \), and \( \Delta \theta = 2\pi \). The value of \( \kappa = 0.0042 \) implies that start-up transients disappear below 1% of their initial value within about 11 periods (cf. the energy and enstrophy graphs versus time in Fig. 14). We note that the solution appears quasi periodic for \( t > 800 \). Shown is the solution over period 20 in Fig. 15 (i.e., from \( t = 1947.5 \) to 2050.0). A second-order spatial and third-order temporal discretization has been used; single- and double-resolution runs have been performed with 4671 and 10 363 nodes and 4590 and 10 242 elements; the former are shown but agree well with the latter. These nonlinear simulations reveal the cause of the disturbances in the northwest corner of the domain at \( t = 14, 42 \) s in Fig. 10: a vortex starts to roll up on the northern shelf once the cell of the southern shelf mode starts to radiate into a basin Rossby mode; subsequently, the vortex gets advected counterclockwise around the domain by the basin Rossby mode and is dissipated once the new forcing cycle starts (see Figs. 15 and 16 in tandem). The simulated potential vorticity field
displayed is less smooth than the streamfunction fields and displays more structure, including a vortex shedding.

5. Conclusions

Hybrid Rossby–shelf modes were shown to exist analytically and numerically in Bokhove and Johnson (1999). Based on depth-averaged potential vorticity dynamics, we showed numerically that these hybrid modes also emerged as linear forced–dissipative solutions on a laboratory scale. These hybrid modes matched the largest planetary-scale Rossby mode to a trapped shelf mode—the latter propagating around the southern shelf. The calculated frequencies of the two dominant hybrid modes were used as
driving frequency in laboratory tank experiments, based on the linearized calculations. Therein a driven lid provided the Ekman forcing. The spatial structure of the streamfunction fields in the linear calculations and the observed streamfunction fields in streak photography agreed well or reasonably well in the weaker and stronger forcing cases. Discrepancies in amplitude and structure were attributed to nonlinear effects, and nonlinear simulations of the depth-averaged flow suggested observed differences to be due to a vortex generated and shed off the northern shelf by the large westward-propagating planetary Rossby mode in the deep basin. The minimum and maximum amplitudes in the calculations of the linear and nonlinear models differed: we observed values of $-0.005$
Fig. 11. (a) Streak photography of hybrid Rossby–shelf mode observed at $t = 0$ s for a forcing period of 51.3 s (nondimensional $\sigma = 0.0612$), and maximum rigid-lid excursion $\Delta \theta = \pi$. Exposure time was 4 s. (b) Same as in (a), but for the calculated linear solution; phase shift adjusted semioptimally by eye.
and 0.009 in the former and values of $-0.014$ and $0.005$ in the latter. The simulations provided extra information on potential vorticity dynamics, which was unavailable from the laboratory observations.

Even though the topography used is still simple, the exhibited mode merging shows that the linear normal modes rapidly obtain a complicated structure. The distinction between trapped shelf modes and planetary modes becomes less clear in complex domains and is a bit artificial as both modes emerge from the spatial structure in the background potential vorticity. Vortical normal modes have recently been used to explain temporal variability in the Mascarene Basin (Weijer 2008) and in the Norwegian and Greenland gyres (LaCasce et al. 2008). Unexplored yet interesting aspects in the idealized ocean basin used by us concern the effects of a midocean ridge (cf. Pedlosky 1996) on the communication between two separate deep-ocean half basins connected only via coastal shelves and the numerical parameterization of underresolved shelf mode dynamics on the deep-ocean dynamics as a way to explore energy exchange through an effective, permeable boundary.

Acknowledgments. It is a great pleasure to acknowledge the assistance of John Salzig in the laboratory. Without his help, the experiments would have failed. The laboratory experiments were performed while O. B. was a postdoctoral scholar at the Woods Hole Oceanographic Institution (1996–97); preliminary results were posted in Bokhove (1999). The encouragements and advice of Karl Helfrich, Joe Pedlosky, and Jack Whitehead have, as always, been of great value. Jack Whitehead also kindly allowed O. B. to use the rotating table in his laboratory, and additional support came via Joe Pedlosky’s NSF Grant OCE–9901654 in 1997.

APPENDIX

(Dis)continuous Galerkin Finite Element Discretization

a. Weak formulation

A (dis)continuous Galerkin finite element method is used to discretize the following generalized system of equations:

$$
\frac{1}{A} \frac{\partial \xi}{\partial t} + \mathbf{V} \cdot (A \mathbf{U} \xi) = \kappa \left( \frac{1}{2} A \omega_r + \alpha A D - \alpha \xi \right), \quad (A1a)
$$

$$
\mathbf{U} = \nabla^\perp \Psi, \quad \text{and} \quad (A1b)
$$

with $A = A(x, y) > 0$, $B = B(x, y) \geq 0$, and $D = D(x, y)$ and for a scaling in which $D \approx D_0 = 1$ in a relevant leading-order way. Here, only a singly-connected domain $\Omega$ is considered with boundary $\partial \Omega$; it is cylindrical with radius $R$ and $\Psi = 0$ at $\partial \Omega$: $r = R$. In (A1a), we have used the approximation

$$
\kappa \xi \left( \frac{\omega_r}{2} + \alpha D - \frac{\alpha \xi}{A} \right) \approx \kappa \left( \frac{A \omega_r}{2} + \alpha A D - \alpha \xi \right)
$$

in the forcing and damping terms because $\xi \approx A$.

In Bernsen et al. (2006), a finite-element discretization is given and verified for the inviscid, unforced version of (A1) for complex-shaped, multiconnected domains. The generalized streamfunction and vorticity formulation (A1) is advantageous because it unifies several systems into one, such as the barotropic quasigeostrophic, and rigid-lid equations. A third-order Runge–Kutta discretization in time and second-, third-, or fourth-order discretizations in space are implemented and available for use. Without forcing and dissipation, discrete energy conservation is guaranteed in space, whereas the discrete enstrophy decays for the upwind numerical flux and is conserved for the central flux for infinitesimal time steps. The latter central flux is stable but yields small oscillations in combination with the third-order time integrator. When necessary, the circulation along the boundary can also be treated properly on the discrete level.

In extension to Bernsen et al. (2006), a discontinuous Galerkin method is used to find the weak formulation of (A1a). After we multiply (A1a) with an arbitrary test function $\nu_h$, integrate over the domain $\Omega$, and use numerical fluxes between interior elements and at the boundary elements, over each element $K$ the following weak formulation is obtained:

$$
\left( \frac{\partial \xi_h}{\partial t}, \nu_h \right)_K = \left( \xi_h \nabla^\perp \Psi_h, \nabla \nu_h \right)_K
$$

$$
- \int_{\partial K} v_h^+ \tilde{\xi}_h \cdot \nabla^\perp \Psi_h \cdot \nu_h \, d\Gamma
$$

$$
+ \left[ v_h, \kappa \left( \frac{A \omega_r}{2} + \alpha A D - \alpha \xi_h \right) \right]_{\partial K}, \quad (A2)
$$

where the numerical flux $\tilde{\mathbf{F}}$ is replacing $\xi U_n$, in which $U_n$ is the component of the transport velocity $\mathbf{U}$ normal to an element face $\partial K$; $\tilde{\xi}_h$ and $\xi_h^+$ are the limit values of the vorticity just inside and outside an element face. Likewise, $v_h$ is the limit value of the test function just inside.
FIG. 12. Same as in Fig. 11, but with maximum rigid-lid excursion $\Delta \theta = 2\pi$, and exposure time was 2 s.
Moreover, \((\cdot, \cdot)_K\) is the \(L^2\) inner product over element domain \(K\). In contrast, a continuous Galerkin finite element is used to find the weak formulation of \((A1c)\). We multiply \((A1c)\) with an arbitrary test function \(w_h\), integrate over the domain \(\Omega\), and use proper boundary conditions to obtain

\[ \left(\sqrt{A} \nabla \Psi_h, \sqrt{A} \nabla w_h\right)_\Omega + \left(\sqrt{B} \Psi_h, \sqrt{B} w_h\right)_\Omega = -\left(\frac{\epsilon_h}{A}, w_h\right)_\Omega + \int_\Omega D w_h \, d\Omega + w_h|_{\Gamma_0} C \]  

(A3)

with circulation.

FIG. 13. Streak photography of hybrid Rossby-shelf modes at \(t = 0, 5, 10, 15, 20, 25, 30, \) and \(35\) s for a forcing period of \(36.1\) s (nondimensional \(\sigma = 0.0871\)), and maximum rigid-lid excursion \(\Delta \theta = 2\pi\).
b. Normal mode numerical tests

1) Linear free and forced–dissipative planetary Rossby modes

A free Rossby mode in a cylindrical domain of radius $R$ satisfies the linearized version of system (6) with constant $H = H_0 = 1, f = 1 + \beta y$, and $\kappa = 0$, or (A1) with $A = 1/H_0, B = 0, D = 1 + \beta y = D_0 + \beta y$, and $\kappa = 0$. A linear modal solution reads

$$\Psi_{lin}(r, \theta, t) = b_m J_m(k_{lm} r) \cos \left( \frac{\beta r \cos \theta}{2\omega_{lm}} + m\theta + \omega_{lm} t \right),$$

(A5)

where $b_m$ is the amplitude, $J_m$ is the Bessel function of the first kind, $r$ is the radius, $\theta$ is the azimuthal angle, and frequency

$$\omega_{lm} = \pm \frac{0.5\beta k_{lm}}{k_{lm}}.$$

(A6)

The boundary condition is satisfied because $k_{lm}$ are the zeroes of $J_m(k_{lm} R) = 0$, given the azimuthal mode number $m$ (with $l = 1, 2, \ldots, \infty$). A comparison between the linear solution (A5) for $m = 0$ is made with the nonlinear numerical solution initialized by the exact streamfunction $C(x, y, 0)$ given the initial streamfunction $C(x, y, 0) = C_{lin}(r, \theta, 0)$. An approximate $C_{lin}(x, y, 0)$ is then calculated numerically. Good agreement between the linear exact and the nonlinear numerical solutions is found. We also numerically calculated the linear forced–dissipative response of planetary Rossby modes with $\beta = 0.3125$ as a representative laboratory value and $H_1 = H_2 = R$. Some of the first few frequencies of inviscid free modes are

$$\omega_{lm} = 0.0650, 0.0408, 0.0304, 0.0245, 0.0206$$

(A7)

and correspond to solutions (A5). The larger resonant frequencies match finely with the free-wave frequencies (A7).

2) Linear free shelf mode

In the nonlinear numerical simulations, we wish to avoid a discontinuous profile of $A = A(x, y)$ or the depth $A = 1/H$. Instead of a discontinuous step in the depth, we consider the continuous axisymmetric depth profile

$$H(r, \theta) = \begin{cases} H_1, & r \geq R_y + \epsilon \\ H_2 + \frac{1}{2}(H_1 - H_2)(r - R_y + \epsilon)/\epsilon, & R_y - \epsilon < r < R_y + \epsilon, \\ H_2, & r \leq R_y - \epsilon \end{cases}$$

(A8)
FIG. 15. Streamfunction over forcing period 20; \( \sigma = 0.0613, \kappa = 0.0042, \epsilon = 0.0314, \) and \( \Delta \theta = 2\pi \).
FIG. 16. Same as in Fig. 15, but for potential vorticity.
with $H_1 < H_2$, the shelf break radius $R_s < R$, where the depth changes suddenly, and $\epsilon \ll 1$. A matched asymptotic solution (cf. Bokhove and Vanneste 2001, unpublished manuscript) to the linearized version of (6) is sought for $D = f_0 = 1$, $B = 0$, and $A = 1/H$—that is, a solution of

\[
\partial_t \left[ \frac{1}{r} \partial_r \left( r \frac{\partial \Psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Psi}{\partial r^2} \right] + \frac{1}{r} \left( \frac{\partial \Psi}{\partial r} \frac{\partial H^{-1}}{\partial \theta} - \frac{\partial \Psi}{\partial \theta} \frac{\partial H^{-1}}{\partial r} \right) = 0. \tag{A9}
\]

Outer solutions in the regions $r \leq R_s - \epsilon$ and $r \geq R_s + \epsilon$ will satisfy the linearized Eq. (A9) with depth (A8) in the step shelf limit $\epsilon \rightarrow 0$. Inner solutions are valid in the transition region $R_s - \epsilon < r < R_s + \epsilon$, and a suitable sum of inner and outer solutions will provide the entire asymptotic solution.

Outer solutions satisfy, to all orders, $\nabla^2 \Psi = 0$ with $\Psi(R, \theta, t) = 0$; hence, it concerns the real part of

\[
\left\{ \begin{array}{ll} 
\Psi_1(r, \theta, t) &= \sum_{m \neq 0} a_m \left( \frac{r}{R} \right)^m \left( \frac{R_s}{R} \right)^m e^{i(m \theta + \omega t)} \quad r > R_s, \\
\Psi_2(r, \theta, t) &= \sum_{m \neq 0} c_m r^m \left( \frac{R_s}{R} \right)^m e^{i(m \theta + \omega t)} \quad r \leq R_s,
\end{array} \right.
\tag{A10}
\]

with coefficients $a_m$, $c_m$, frequency $\omega$, and azimuthal wavenumber $m$.

For the inner solution, a stretched coordinate $\zeta = (r - R_s)/\epsilon$ is introduced. Subsequently, the inner expansion

\[ \Psi = \Psi(R_s, \theta) + \epsilon \phi^{(1)}(\zeta, \theta) + \cdots \tag{A11} \]

is substituted in (A9) and evaluated at leading order in $\epsilon$, giving the leading-order equation

\[ i \omega \frac{\partial}{\partial \zeta} \left( \frac{1}{H} \frac{\partial \phi^{(1)}}{\partial \zeta} \right) - \frac{1}{R_s} \frac{\partial \Psi}{\partial \theta} (R_s, \theta) \frac{\partial H^{-1}}{\partial \zeta} = 0. \tag{A12} \]

Because the outer expansion in the inner region satisfies

\[ \Psi(r, \theta) = \Psi(R_s, \theta) + \epsilon \frac{\partial \Psi}{\partial r} (R_s, \theta) + \cdots \]

and the inner one (A11), we obtain together with the continuity requirement the following inner boundary conditions at $\zeta = \pm 1$:

\[ \Psi_1(R_s, \theta) = \Psi_2(R_s, \theta) \quad \text{and} \quad \frac{\partial \Psi_1}{\partial r} (R_s, \theta) = \frac{\partial \Psi_2}{\partial r} (R_s, \theta) \tag{A13} \]

The integration of (A12) from $\zeta = -1$ to $\zeta = 1$ using (A13) yields the dispersion relation for the outer expansion

\[ \omega \left[ \frac{1}{H_1} \frac{\partial \Psi_1}{\partial r} (R_s, \theta) - \frac{1}{H_2} \frac{\partial \Psi_2}{\partial r} (R_s, \theta) \right] \]

\[ - \frac{m}{R_s} \Psi(R_s, \theta) \left( \frac{1}{H_1} - \frac{1}{H_2} \right) = 0. \tag{A14} \]

In rewritten form and by using $\Psi_1(R_s, \theta) = \Psi_2(R_s, \theta)$ and (A10), the outer solutions therefore become

\[ \Psi_1(R_s, \theta) = \Psi_2(R_s, \theta) \quad \text{and} \quad \frac{\partial \Psi_1}{\partial r} (R_s, \theta) = \frac{\partial \Psi_2}{\partial r} (R_s, \theta) \tag{A15a} \]

A first integration of (A12) from $\zeta = -1$ to $\zeta < 1$ or $\zeta > -1$ to $\zeta = 1$, using (A13), followed by a second integration, gives

\[
\phi^{(1)} = \frac{\partial \Psi_2}{\partial r} (R_s, \theta) \zeta + \left[ \frac{1}{H_2} \frac{\partial \Psi_2}{\partial r} (R_s, \theta) \right]
\]

\[ - \frac{m}{R_s \epsilon H_2} \Psi(R_s, \theta) \left( \frac{H_1}{H_2} - \frac{1}{H_2} \right) (\zeta + 1)^2 + c_1. \tag{A16} \]

At $\zeta = -1$, using (A16), we find

\[ \phi^{(1)}(-1, \theta) = - \frac{\partial \Psi_2}{\partial r} (R_s, \theta) + c_1 \tag{A17} \]
and, likewise, at $\zeta = 1$,
\[
\phi^{(1)}(1, \theta) = \frac{\partial \Psi}{\partial r}(R_s, \theta) + c_1, \quad (A18)
\]
where we used the dispersion relation (A14) to find the relation at $\zeta = 1$. Considering the inner and outer expansions in the inner region, we choose $c_1 = 0$; it also explains why the outer solution is chosen to hold at all orders.

The uniformly valid matched asymptotic solution consists of the mean of inner and outer solutions (A16) with $c_1 = 0$ and (A15):

\[
\Psi_{\text{Uniform}} = \begin{cases} 
\Psi_1 \\
\Psi(R_s, \theta) + \frac{1}{2} (r - R_s) \frac{\partial \Psi}{\partial r}(R_s, \theta) + \frac{1}{2} \left( \frac{\partial \Psi}{\partial r}(R_s, \theta) (r - R_s) \\
+C_c + \frac{1}{H_2} \frac{\partial \Psi}{\partial r}(R_s, \theta) - \frac{m}{R_s \omega H_2^2} \Psi(R_s, \theta) \left[ \frac{H_1 - H_2}{4} (r - R_s + \epsilon)^2 \right] \right),
\end{cases}
\]

We initialized the nonlinear numerical simulation with the potential vorticity based on the asymptotic solution (A19) of the streamfunction $\Psi = \Psi_{\text{Uniform}}$ at $t = 0$. The agreement between the asymptotic linear solution and the nonlinear numerical solution of the streamfunction was quite reasonable.

\[
H(r, \theta) = \begin{cases} 
H_{1s} = H_{1s} - s_{1} y \\
H_{2s} = \frac{1}{2} (H_{1s} - H_{2s})(r - R_s + \epsilon) / \epsilon \\
H_{2s} = H_{2s} - s_{2} y \\
\end{cases}
\]

with $H_{1s} < H_{2s}$, $s_{1} = \beta H_{1s}$, and $\epsilon \ll 1$ for $i = 1, 2$. The simulated streamfunction fields displayed in Fig. 15 over forcing period 20 have been verified to equal the ones in forcing period 19. Parameter values are $\sigma = 0.0613$, $\kappa = 0.0042$, $\epsilon = 0.0314$, and $\Delta \theta = 2\pi$.

3) **FORCED–DISSIPATIVE HYBRID-SHELF MODES**

In the nonlinear numerical simulations, we use $A = A(x, y) = 1/H(x, y)$, $D(x, y) = f_0 = 1$. Instead of a discontinuous step in the depth, we consider a depth profile

\[
H(r, \theta) = \begin{cases} 
H_{1s} = H_{1s} - s_{1} y \\
H_{2s} = \frac{1}{2} (H_{1s} - H_{2s})(r - R_s + \epsilon) / \epsilon \\
H_{2s} = H_{2s} - s_{2} y \\
\end{cases}
\]

with $H_{1s} < H_{2s}$, $s_{1} = \beta H_{1s}$, and $\epsilon \ll 1$ for $i = 1, 2$. The simulated streamfunction fields displayed in Fig. 15 over forcing period 20 have been verified to equal the ones in forcing period 19. Parameter values are $\sigma = 0.0613$, $\kappa = 0.0042$, $\epsilon = 0.0314$, and $\Delta \theta = 2\pi$.

**REFERENCES**


