Lateral Friction in Reduced-Gravity Models: Parameterizations Consistent with Energy Dissipation and Conservation of Angular Momentum

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ABSTRACT

The *f*-plane reduced-gravity model has been extended with the parameterization of lateral friction in the momentum equations. The parameterization should preferably fulfill some requisites. One of them is that in the absence of external torques the change in angular momentum should be determined by boundary conditions alone. Internal torques should balance in the angular momentum budget. This requirement is fulfilled when the parameterization in the vertically integrated momentum equations is the divergence of a symmetric stress tensor. These equations solve for the mean transport, which includes implicitly the eddy-induced contribution. Another requirement on the parameterization of lateral stress follows from considering that it should imply kinetic energy dissipation. Both requirements fail with a commonly used parameterization and are fulfilled with the one proposed by C. Schär and R. B. Smith, which also is in near agreement with derivations of the shallow-water equations via vertical integrations of the Navier–Stokes equations. Here, the authors show two other parameterizations that are consistent with the angular momentum and energy requirements. One of the parameterizations follows from the symmetric component of a stress tensor in agreement with the parameterization shown by P. R. Gent to be energetically consistent. The other parameterization is related to the so-called biharmonic dissipation. In general, the difficulty for friction parameterizations is on the energy dissipation requirement, because the one on angular momentum is easily fulfilled.

1. Introduction

The aim or favorable characteristic of a simplified model should be to reproduce physical features known to be satisfied in more general and comprehensive models. Any buildup of a simplified model toward a ‘‘better’’ version should preferably not disrupt the physical properties that are satisfied in both general and simplified models. We say preferably because there might be some cases where a justification is at hand. This better model ends up halfway between the very simplified and general comprehensive models.

The conservation of energy and angular momentum are basic physical principles satisfied in comprehensive general models of nature. The shallow-water and reduced-gravity models in the *f* plane are simplified fluid dynamical models that reproduce these basic principles in its plain frictionless version, but the addition of lateral diffusion in the momentum equations might produce, depending on the parameterizations, inconsistencies. Shchepetkin and O’Brien (1996) analyze some consequences on the choice of the lateral frictional parameterization, and Gent (1993) shows a particular form that is consistent energetically, meaning that its inclusion in a closed domain, with either no-slip or free-slip boundary conditions, implies no energy increase for any state of the system. Schär and Smith (1993) deduce a form that is consistent with both principles. Warnings on the consequences over angular momentum have been profusely stated (e.g., Shchepetkin and O’Brien 1996; Wajsowicz 1993; Griffies 2004; Schär and Smith 1993). Here, we follow a parallel line of thought as Gent (1993) did for the energy budget but adding the principle of angular momentum conservation. The requirement for a reduced-gravity model to be consistent with the conservation of vertical angular momentum L is that its rate of change (i.e., dL/dt) should depend on the torques exerted by forces on the boundary and on the fluid interchange with the exterior for any volume in question:

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TABLE 1. A set of parameterizations of lateral friction for reduce gravity models. The frictional stress tensor shown in the fourth column applies to the notation of Eq. (1). The last two columns indicate if the parameterization is consistent (OK), inconsistent (No), or indeterminate (?) with respect to the dissipation of energy and conservation of angular momentum.

Case	Coefficient units	Θ_1 Θ_2	$\begin{bmatrix} s_{11} & s_{21} \\ s_{12} & s_{22} \end{bmatrix}$	Energy dissipation	Angular momentum
I	$m^2 s^{-1}$	$\nabla^2 u$ $\nabla^2 v$	—	No	No
II	$m^2 s^{-1}$	$h^{-1}\nabla \cdot (h\nabla u)$ $h^{-1}\nabla \cdot (h\nabla v)$	$h\begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix}$	OK	No
III	$m^2 s^{-1}$	$h^{-1}\nabla^2(hu)$ $h^{-1}\nabla^2(hv)$	$\begin{bmatrix} (hu)_x - (hv)_y & (hu)_y + (hv)_x \\ (hu)_y + (hv)_x & (hv)_y - (hu)_x \end{bmatrix}$	No	OK
IV	$m^3 s^{-1}$	$h^{-1}\nabla^2 u$ $h^{-1}\nabla^2 v$	$\begin{bmatrix} u_x - v_y & v_x + u_y \\ u_y + v_x & v_y - u_x \end{bmatrix}$	OK	OK
V	$m^2 s^{-1}$	$h^{-1}[\nabla \cdot h\nabla u + \mathbf{k} \cdot \nabla h \times \nabla v]$ $h^{-1}[\nabla \cdot h\nabla v - \mathbf{k} \cdot \nabla h \times \nabla u]$	$h\begin{bmatrix} u_x - v_y & v_x + u_y \\ u_y + v_x & v_y - u_x \end{bmatrix}$	OK	OK
VI	$m^3 [AB]^{-1} s^{-1}$	$h^{-1}[\nabla \cdot A\nabla(Bu) + \mathbf{k} \cdot \nabla A \times \nabla(Bv)]$ $h^{-1}[\nabla \cdot A\nabla(Bv) - \mathbf{k} \cdot \nabla A \times \nabla(Bu)]$	$A\begin{bmatrix} (Bu)_x - (Bv)_y & (Bu)_y + (Bv)_x \\ (Bu)_y + (Bv)_x & (Bv)_y - (Bu)_x \end{bmatrix}$	?	OK
VII	$m^5 s^{-1}$	$-h^{-1}\nabla^4 u$ $-h^{-1}\nabla^4 v$	$-\begin{bmatrix} \nabla^2(u_x - v_y) & \nabla^2(v_x + u_y) \\ \nabla^2(u_y + v_x) & \nabla^2(v_y - u_x) \end{bmatrix}$	OK	OK

that is, exclusively on boundary conditions. Here, the meaning of boundary conditions is not necessarily those arising from some impositions on solid boundaries but as the conditions existing on the boundary of any subdomain under consideration. There is of course the possibility of including external forces: surface and bottom stresses and their corresponding torques. Here, we limit the scope to the unforced reduced-gravity model with lateral momentum diffusion and in the absence of stresses and mass exchange at the surface and bottom. Both external torques and fluid exchange must be accounted for via lateral boundary conditions; internal torques and internal redistribution of mass should not contribute to the variation of L. This requires a symmetric viscous stress tensor. It is also customary (Batchelor 1981) to set the frictional stress tensor with null trace, in which case the "pressure" or the isotropic contribution is independent of the inclusion or absence of frictional effects, but this is an overimposition when dealing with stress tensors arising from turbulence.

In Table 1, we show six explicit parameterizations of the lateral diffusion of momentum (cases I–V and VII), out of which three (III, IV, and VII) are consistent with the conservation of angular momentum and imply, as stated by Gent (1993), energy dissipation. Notice that cases I–V collapse into the same form when the layer thickness h is assumed uniform. Case VI is a family of symmetric stress tensor parameterizations that show the difficulty of fulfilling the energy requirement. The functions A and B in parameterization VI are arbitrary scalar functions of any combination of local variables (i.e., arbitrary scalar fields). Cases III, IV, and V are particular versions of case VI. Cases II and V are close relatives; any second-rank tensor, as stress tensors are, can be uniquely decomposed in the sum of its isotropic, symmetric, and antisymmetric components, and case V corresponds to the symmetric component of case II.

Gent (1993) used the same parameterization as Gustafsson and Sundström (1978) and Bleck and Boudra (1981). Although he did not show explicitly the stress tensor, he recognized that the parameterization was the divergence of a stress tensor and decided in its favor instead of that used by Schär and Smith (1993) (case IV). In this reference, a traceless stress tensor is presented explicitly and shown to fulfill the requirements as stated here.

The 2D nondivergent flow model is slightly simpler than the reduced-gravity model. In the studies of Molenaar et al. (2004), van Heijst et al. (2003, 2006), and Clercx and van Heijst (2009) the conservation of angular momentum, as shown in those studies and stated here (i.e., depending exclusively on boundary conditions), is granted. In those studies, the parameterization in use is that of cases I–V because all of them collapse. The use of 2D nondivergent flow models avoid from the very start thickness variations in space and time, which is an ingredient that makes our analysis relevant.

Formal derivations via scaling assumptions, the law of the wall and the vertical integration of the Navier–Stokes equations, as shown by Gerbeau and Perthame

(2001), Marche (2007), and Lucas and Rousseau (2008), show forms in favor of case V. This is the situation when removing bottom stress and several other terms included in their derivations: for example, additional isotropic contributions besides pressure in the stress tensor. However, in those derivations bottom stress is the primary frictional effect from which a secondary correction appears in the form of lateral diffusion of momentum. The purpose of this study is a warning call in the direction of the title, considering particularly that eddy parameterizations of turbulence are still open to a wide set of tests.

In section 2, we set definitions and the governing equations of the reduced-gravity model with the inclusion of eddy-induced transports. Section 3 shows the consequences on energy and angular momentum of the parameterizations defined in Table 1. In section 4, we discuss the results, and the last section gives the conclusions.

2. Governing equations and definitions

Although all the formulas can be cast in coordinate-free notation, the Cartesian system allows a simple straightforward notation for tensors, vector cross products, and the divergence and circulation theorems. The system is

$$\frac{\partial hu}{\partial t} = -\nabla\cdot(hu\mathbf{v}) + fhv - \frac{\partial \gamma}{\partial x} + \nabla\cdot\mathbf{s}_1, \qquad (1a)$$

$$\frac{\partial hv}{\partial t} = -\nabla\cdot(hv\mathbf{v}) - fhu - \frac{\partial \gamma}{\partial y} + \nabla\cdot\mathbf{s}_2, \quad \text{and} \qquad (1b)$$

$$\frac{\partial h}{\partial t} + \nabla\cdot(h\mathbf{v}) = 0, \qquad (1c)$$

where h is the layer thickness; $\mathbf{v} = u\mathbf{i} + v\mathbf{j}$ is the "equivalent" velocity (as explained below), with $\mathbf{i}$ and $\mathbf{j}$ the unit vectors in the directions parallel to the orthogonal x and y coordinate axes; t is time; $\nabla = \mathbf{i}\partial/\partial x + \mathbf{j}\partial/\partial y$ is the nabla operator, with the notation $\partial/\partial x$ for partial derivative with respect to x (and a corresponding analogous notation for y); $f = 2\Omega$ is the Coriolis parameter with Ω the planetary angular velocity; $\gamma = g'h^2/2$ is the density of potential energy per unit area, with g' being the reduced gravity; and $[\mathbf{s}_1\ \mathbf{s}_2] = \mathbf{s}$ is the frictional stress tensor due to turbulent fluctuations.

In the notation of system (1) the equivalent velocity includes the effect of eddy-induced transport. In the absence of turbulence $\mathbf{s}_1 = \mathbf{s}_2 = 0$, and the equivalent velocity coincides with the velocity, which can be called $\mathbf{v}^* = u^*\mathbf{i} + v^*\mathbf{j}$. The conventional separation into mean or large-scale and eddy or turbulent contributions with null mean are of the form $h^* = \overline{h} + h'$, $u^* = \overline{u} + u'$, and $v^* = \overline{v} + v'$ or $\mathbf{v}^* = \overline{\mathbf{v}} + \mathbf{v}'$, where the complete fields have the asterisk, the primed contributions are the turbulent fluctuations, and the overbar is the averaging operator (which fulfills, e.g., $\overline{u^*} = \overline{u}$ and $\overline{u'} = 0$). In the presence of turbulence, the eddy-induced transports arise as part of the mean fields in the form $\overline{h^*\mathbf{v}^*} = \overline{h}\ \overline{\mathbf{v}} + \overline{h'\mathbf{v}'}$; therefore, the simple redefinition

$$h\mathbf{v} \equiv \overline{h^*\mathbf{v}^*} = \overline{h}\ \overline{\mathbf{v}} + \overline{h'\mathbf{v}'} \quad \text{and} \qquad (2a)$$

$$h \equiv \overline{h} \qquad (2b)$$

is in use in the system in (1a–c). With this notation and identities implied, like $h(u - \overline{u}) = \overline{h'u'}$, it follows that

$$\mathbf{s}_1 = \overline{h'u'}\ \overline{h'\mathbf{v}'}/h - h\overline{u'\mathbf{v}'} - \overline{h'u'\mathbf{v}'} - \mathbf{i}g'\overline{h'h'}/2 \quad \text{and} \qquad (3a)$$

$$\mathbf{s}_2 = \overline{h'v'}\ \overline{h'\mathbf{v}'}/h - h\overline{v'\mathbf{v}'} - \overline{h'v'\mathbf{v}'} - \mathbf{j}g'\overline{h'h'}/2. \qquad (3b)$$

Equation (1c) can be cast showing the eddy-induced transport explicitly, as in Adcock and Marshall (2000), say

$$\frac{\partial h}{\partial t} + \nabla\cdot(\overline{h}\ \overline{\mathbf{v}}) = -\nabla\cdot\mathbf{V}^*,$$

where $\mathbf{V}^* \equiv \overline{h'\mathbf{v}'}$. However, notice that the solution of the system in (1a–c), with any closure scheme for $\nabla\cdot\mathbf{s}_1$ and $\nabla\cdot\mathbf{s}_2$, is for $h\mathbf{v}$ without actually solving any of its two additive contributions, $\overline{h}\ \overline{\mathbf{v}}$ and $\mathbf{V}^* \equiv \overline{h'\mathbf{v}'}$. Therefore, within this reduced-gravity model, there is no point of a parameterization of $\mathbf{V}^* \equiv \overline{h'\mathbf{v}'}$. The closure of the system in (1a–c) asks for parameterizations of $\mathbf{s}_1$ and $\mathbf{s}_2$ or their divergences, which are the forms that contribute in (1). As discussed by many authors (recently, Eden and Greatbatch 2008; Marshall and Adcroft 2010, and references therein), one can add a gauge field to $\mathbf{s}_1$ and $\mathbf{s}_2$ with zero divergence and determine its form based on further physical constraints.

In the simpler situation of fully 2D nondivergent flows, as in examples by van Heijst et al. (2006), $h' \equiv 0$, $\mathbf{s}_1 = -h\overline{u'\mathbf{v}'}$, $\mathbf{s}_2 = -h\overline{v'\mathbf{v}'}$, the pressure is dictated by a diagnostic equation, and there is no need of equivalent velocities because no eddy-induced transport exists (i.e., $\overline{h'\mathbf{v}'} = \mathbf{0}$).

A frequently used parameterization, which Gent (1993) proved to be inconsistent with energy requirements, replaces $\nabla\cdot\mathbf{s}_1$ and $\nabla\cdot\mathbf{s}_2$ in (1) by $\vartheta h\nabla^2 u$ and $\vartheta h\nabla^2 v$, respectively. As shown shortly, it fails the angular momentum requirement.

The energy and angular momentum budget equations are

$$\frac{dE}{dt} = -\oint_{\partial D} hb\mathbf{v}\cdot\mathbf{n}\,d\ell + \oint_{\partial D} \varepsilon\mathbf{v}_C\cdot\mathbf{n}\,d\ell + \int_D (u\nabla\cdot\mathbf{s}_1 + \upsilon\nabla\cdot\mathbf{s}_2)\,dA \quad \text{and} \tag{4a}$$

$$\frac{dL}{dt} = -\oint_{\partial D} \lambda(\mathbf{v}-\mathbf{v}_C)\cdot\mathbf{n}\,d\ell - \Omega\oint_{\partial D} hr^2(\mathbf{v}-\mathbf{v}_C)\cdot\mathbf{n}\,d\ell + \oint_{\partial D}\gamma\mathbf{r}\cdot d\mathbf{l} + \int_D [x\nabla\cdot\mathbf{s}_2 - y\nabla\cdot\mathbf{s}_1]\,dA \tag{4b}$$

where

$$E \equiv \int_D (h|\mathbf{v}|^2 + g'h^2)\,dA/2 \tag{5}$$

is the energy, $\varepsilon \equiv h|\mathbf{v}|^2/2 + g'h^2/2$ is the energy density per unit area, $b \equiv |\mathbf{v}|^2/2 + g'h$ is the Bernoulli function (notice that $hb = \varepsilon + g'h^2/2$),

$$L \equiv \int_D h(x\upsilon - yu)\,dA + \Omega\int_D hr^2\,dA \tag{6}$$

is the vertical component of angular momentum with respect to the origin of the coordinate system, $\lambda \equiv h(x\upsilon - yu)$ is the density per unit area of the relative angular momentum, the area of integration (of any subdomain in question denoted by D) might change in time and its space differential is dA, $\mathbf{n}$ is the unit vector orthogonal to the boundary contour (denoted by ∂D) pointing outwards, $\mathbf{v}_C$ is the velocity of the bounding contour, the differential along the bounding contour is $d\mathbf{l}$ with magnitude $d\ell$, and $\mathbf{r} \equiv x\mathbf{i} + y\mathbf{j}$ is the vector position of magnitude r. Looking along $d\mathbf{l}$, the domain is on the left-hand side (i.e., $\mathbf{n}\times d\mathbf{l}\|\mathbf{k} = \mathbf{i}\times\mathbf{j}$). The use of bounding contours that change in time, not fixed to the frame of reference, requires the use of Leibnitz rule

$$\frac{d}{dt}\int_D \{\cdot\}\,dA = \int_D \frac{\partial\{\cdot\}}{\partial t}\,dA + \oint_{\partial D}\{\cdot\}\mathbf{v}_C\cdot\mathbf{n}\,d\ell,$$

where, as just previously defined, $\mathbf{v}_C$ is the velocity of the bounding contour. Notice that the bounding contour ∂D and the subdomain D define each other and that $\mathbf{v}_C\cdot\mathbf{n}$ is the only component of $\mathbf{v}_C$ that is of any relevance (i.e., the other component has no effect on the time variation of ∂D or D). The area of integration is completely arbitrary [i.e., system (4) holds for all and any subdomain, either static or moving within the fluid]. The conservation equations (4a) and (4b) follow from direct substitutions of (1) in expressions like $dL/dt = \int_D (x\partial h\upsilon/\partial t - y\partial hu/\partial t)\,dA + \Omega\int_D r^2\partial h/\partial t\,dA$ (when, having $\mathbf{v}_C\cdot\mathbf{n} = 0$, the Leibnitz rule is unneeded) and the use of identities like $x\nabla\cdot(h\upsilon\mathbf{v}) - y\nabla\cdot(hu\mathbf{v}) = \nabla\cdot[h(x\upsilon - yu)\mathbf{v}]$ or the circulation theorem $\int_D (y\partial\gamma/\partial x - x\partial\gamma/\partial y)\,dA = \oint_{\partial D}\gamma\mathbf{r}\cdot d\mathbf{l}$.

The use of the origin as the reference for the arm to define the angular momentum can be relaxed to any other reference point as long as it does not change with time. In this case, $\mathbf{r}$ is the position vector minus $\mathbf{r}_O$, where $\mathbf{r}_O$ is the position of the reference point. The use of time-varying reference points adds the terms $-\mathbf{k}\cdot(d\mathbf{r}_O/dt \times \int_D h\mathbf{v}\,dA)$ and $-2\Omega\,d\mathbf{r}_O/dt\cdot\int_D h\mathbf{r}\,dA$ to (4b) on its right-hand side. In general, these additional terms cannot be cast on boundary conditions alone.

If the boundary is static (i.e., $\mathbf{v}_C\cdot\mathbf{n} = 0$) and coincides with an impermeable boundary, there is no exchange of fluid with the exterior of the domain, $\mathbf{v}\cdot\mathbf{n} = 0$ on ∂D, and several terms in (4) drop out. Likewise, if the subdomain of interest is a material blob (i.e., $\mathbf{v}_C\cdot\mathbf{n} = \mathbf{v}\cdot\mathbf{n}$), several terms cancel.

In the absence of viscous effects (i.e., with $\mathbf{s}_1 = \mathbf{s}_2 = \mathbf{0}$) and with solid boundaries on the area under consideration $dE/dt = 0$ and $dL/dt = \oint_{\partial D}\gamma\mathbf{r}\cdot d\mathbf{l}$ state the conservation laws: for example, E is invariant in time and the variation of L only depends on the external torques; those that arise from the pressure on the boundary. Molenaar et al. (2004) and van Heijst et al. (2003, 2006) show the crucial role of this pressure-dependent torque in the spinup process within a rectangular basin. Regardless of the shape, uniform pressure along the boundary implies $dL/dt = 0$. If the boundary is a circumference centered at the origin, $\mathbf{r}\cdot d\mathbf{l} = 0$ over the entire contour and even a nonuniform pressure exerts no torque.

When $\mathbf{v}_C\cdot\mathbf{n} = 0$, interpretations of terms like $L_P = \Omega\int_D hr^2\,dA = I\Omega$ or planetary angular momentum with $I \equiv \int_D hr^2\,dA$ the moment of inertia and $L_R = \int_D h(x\upsilon - yu)\,dA = \int_D \lambda\,dA$ or relative angular momentum lead to $dL_P/dt = -\Omega\oint_{\partial D} hr^2\mathbf{v}\cdot\mathbf{n}\,d\ell + 2\Omega\int_D hru_r\,dA$ and $dL_R/dt = -\oint_{\partial D}\lambda\mathbf{v}\cdot\mathbf{n}\,d\ell + \oint_{\partial D}\gamma\mathbf{r}\cdot d\mathbf{l} - f\int_D hru_r\,dA$, where u_r is the radial velocity component (i.e., $ru_r = \mathbf{r}\cdot\mathbf{v}$). Then, the area integrals cancel out when the two contributions are added. The meaning of this is straightforward when considering the situation $\mathbf{v}_C\cdot\mathbf{n} = \mathbf{v}\cdot\mathbf{n} = 0$; the redistribution of mass in the interior of the domain implies torques because of the Coriolis force (i.e., $-f\int_D hru_r\,dA$), which contributes to dL_R/dt, and, in the same amount but opposite sign, the moment of inertia rate of change contributes to dL_P/dt (i.e., $\Omega\int_D 2hru_r\,dA$).

It is worth emphasizing that it is having the symmetric stress tensor in the vertically integrated equations of motion and including the eddy-induced transport, as stated in (1)–(3), that the conservation of angular momentum is assured. This is easily proved because

$$x\nabla\cdot\mathbf{s}_2 - y\nabla\cdot\mathbf{s}_1 = \nabla\cdot(x\mathbf{s}_2 - y\mathbf{s}_1) - (\mathbf{s}_2\cdot\nabla x - \mathbf{s}_1\cdot\nabla y),$$

but

$$\mathbf{s}_2\cdot\nabla x - \mathbf{s}_1\cdot\nabla y = \mathbf{s}_2\cdot\mathbf{i} - \mathbf{s}_1\cdot\mathbf{j} = s_{21} - s_{12}; \quad \text{therefore,}$$

$$\int_D [x\nabla\cdot\mathbf{s}_2 - y\nabla\cdot\mathbf{s}_1]\,dA = \oint(x\mathbf{s}_2 - y\mathbf{s}_1)\cdot n\,d\ell - \int_D (s_{21} - s_{12})\,dA, \tag{7}$$

which states the sole dependency of (4b) on boundary conditions provided that the stress tensor is symmetric. The consistency of the conservation of angular momentum must be for all and any D; therefore, the only possibility is $s_{21} = s_{12}$ at every point. The symmetry of the stress tensor also shows up in (3).

The energy requirement can be restated by observing that

$$u\nabla\cdot\mathbf{s}_1 + \upsilon\nabla\cdot\mathbf{s}_2 = \nabla\cdot(u\mathbf{s}_1 + \upsilon\mathbf{s}_2) - (\mathbf{s}_1\cdot\nabla u + \mathbf{s}_2\cdot\nabla\upsilon),$$

then

$$\int_D [u\nabla\cdot\mathbf{s}_1 + \upsilon\nabla\cdot\mathbf{s}_2]\,dA = \oint(u\mathbf{s}_1 + \upsilon\mathbf{s}_2)\cdot\mathbf{n}\,d\ell - \int_D (\mathbf{s}_1\cdot\nabla u + \mathbf{s}_2\cdot\nabla\upsilon)\,dA;$$

therefore, following Gent's (1993) argument, if

$$\mathbf{s}_1\cdot\nabla u + \mathbf{s}_2\cdot\nabla\upsilon \geq 0 \tag{8}$$

throughout the domain, the kinetic energy cannot increase because of internal friction. The additional flux of energy that arises from the viscous terms usually shows in the term $u\mathbf{s}_1 + \upsilon\mathbf{s}_2$, and its effect on the energy budget depends on boundary conditions. There is also the possibility for the left-hand side of (8) to be the sum of the divergence of a flux plus a nonnegative definite quantity and therefore also fulfill the energy requirement (i.e., the fundamental requirement is the decomposition $u\nabla\cdot\mathbf{s}_1 + \upsilon\nabla\cdot\mathbf{s}_2 = \nabla\cdot\mathbf{F}_\upsilon - d^2$, where $\mathbf{F}_\upsilon$ and $d^2 \geq 0$ are the flux and dissipation of the kinetic energy density). This condition is based on the idea that the fluxes represent frictional processes. If interpreted in terms of eddy-mean flow energy exchanges, the positive semidefinite condition (8) could be relaxed (Marshall and Adcroft 2010), but that case is beyond the scope of this paper, which only intends to shed light on the properties of commonly used dissipative parameterizations in shallow-water models.

For completeness, the linear momentum budget is $d/dt\int_D h\mathbf{v}\,dA + f\mathbf{k}\times\int_D h\mathbf{v}\,dA = -\oint h\mathbf{v}(\mathbf{v}-\mathbf{v}_C)\cdot\mathbf{n}\,d\ell - \oint\gamma\mathbf{n}\,d\ell + \oint[\mathbf{i}(\mathbf{s}_1\cdot\mathbf{n}) + \mathbf{j}(\mathbf{s}_2\cdot\mathbf{n})]\,d\ell$, which shows no restriction on the stress tensor, but notice the Coriolis effect in a domain integral, which here is left on the left-hand side. The potential vorticity equation reads as

$$\frac{\partial q}{\partial t} + \mathbf{v}\cdot\nabla q = \frac{1}{h}\left[\frac{\partial}{\partial x}\left(\frac{\nabla\cdot\mathbf{s}_2}{h}\right) - \frac{\partial}{\partial y}\left(\frac{\nabla\cdot\mathbf{s}_1}{h}\right)\right],$$

where $q \equiv (f + \zeta)/h$ and $\zeta = \partial\upsilon/\partial x - \partial u/\partial y$ are the potential vorticity and the vorticity.

In the simpler 2D problem (i.e., nondivergent flow with constant and uniform h), a formal relationship between the angular momentum and integrals of vorticity forms is possible. This follows from the vorticity and boundary conditions uniquely setting the velocity field [see, e.g., Eqs. (9) and (10) in Molenaar et al. (2004) or (22)–(24) in Clercx and van Heijst (2009)]. The variable thickness or nondivergence of the flow prevents these reductions, and, to our knowledge, it is not possible in general to derive the angular momentum out of integrals of potential vorticity forms. Salmon (1998) shows how the conservation of potential vorticity is intimately related to a differential form of angular momentum, but with the arm defining the angular momentum in the same position as the column in consideration. The same is true for vorticity in general 3D flows, but again with the arm in the center of mass (Chatwinal 1973).

3. A set of parameterizations

The issue of this note is the rightmost terms of Eqs. (4a) and (4b), but such equations do not follow the most common notation; they imply the divergence of a stress tensor. In the usual notations forms as $h\mathbf{\Theta} \equiv h(\Theta_1\mathbf{i} + \Theta_2\mathbf{j})$ replace $(\nabla\cdot\mathbf{s}_1)\mathbf{i} + (\nabla\cdot\mathbf{s}_2)\mathbf{j}$; the vector $\mathbf{\Theta}$ is the frictional force per unit mass, and the momentum equations are

$$\partial u/\partial t + \mathbf{v}\cdot\nabla u - f\upsilon = -g'\partial h/\partial x + \Theta_1 \quad \text{and} \tag{9a}$$

$$\partial\upsilon/\partial t + \mathbf{v}\cdot\nabla\upsilon + fu = -g'\partial h/\partial y + \Theta_2. \tag{9b}$$

With this notation, the extreme right-hand terms in (4a) and (4b) are

$$\int_D (u\nabla\cdot\mathbf{s}_1 + \upsilon\nabla\cdot\mathbf{s}_2)\,dA = \int_D h\mathbf{v}\cdot\mathbf{\Theta}\,dA \quad \text{and} \tag{10a}$$

$$\int_D (x\nabla\cdot\mathbf{s}_2 - y\nabla\cdot\mathbf{s}_1)\,dA = \int_D h\mathbf{r}\times\mathbf{\Theta}\,dA. \tag{10b}$$

The parameterizations listed in Table 1 show the relationship for several values of $\mathbf{\Theta}$ and, with the exception of the first one, the corresponding stress tensor. Of these parameterizations, only the first one (I) does not allow to write it as the divergence of a tensor, and only parameterization II does not have a corresponding symmetric stress tensor. Forms I and II fail the principle of angular momentum conservation. Parameterizations III–VII follow from the use of a symmetric stress tensor and therefore are consistent with the principle of angular momentum conservation.

Because Gent (1993) showed that parameterization I is energetically inconsistent, whereas II is consistent, let us consider the extreme right term of (4a) for III–VII. Parameterizations of the type III–V are particular cases of the family described by VI, in which case Eq. (8) takes the form

$$A\{[(Bu)_x - (Bv)_y](u_x - v_y) + [(Bv)_x + (Bu)_y](v_x + u_y)\} \geq 0, \tag{11a}$$

where A and B are arbitrary local functions. This expression can be rewritten as

$$A\left\{B[(u_x - v_y)^2 + (v_x + u_y)^2] + \nabla B \cdot \nabla \frac{u^2 + v^2}{2} + \mathbf{k} \cdot \nabla B \times (v\nabla u - u\nabla v)\right\} \geq 0; \tag{11b}$$

therefore, a natural choice is $A > 0$ and $B > 0$ (or, slightly more general, of the same sign), so the first terms on the left are nonnegative. The term containing $\nabla B \cdot \nabla(u^2 + v^2)$ can easily be forced to be nonnegative, by choosing the function B to be a positive definite function of the flow speed. Nonetheless, the last term makes it quite difficult to build a nontrivial function B that satisfies (11). Using the trivial choice $B = 1$ makes Eq. (8)

$$A[(u_x - v_y)^2 + (v_x + u_y)^2] \geq 0; \tag{12}$$

hence, any $A > 0$ satisfies the energy requirement.

For the biharmonic parameterization, described in case VII of Table 1, the additional flux of energy that arises from the viscous terms is not only $u\mathbf{s}_1 + v\mathbf{s}_2$. In this case, $u\nabla \cdot \mathbf{s}_1 + v\nabla \cdot \mathbf{s}_2 = \nabla \cdot \mathbf{F}_v - d^2$, where $\mathbf{F}_v = u\mathbf{s}_1 + v\mathbf{s}_2 + \vartheta\nabla[(u_x - v_y)^2 + (v_x + u_y)^2]/2$, $d^2 = \vartheta[|\nabla(u_x - v_y)|^2 + |\nabla(v_x + u_y)|^2]$, and ϑ is the (positive) frictional coefficient. It is a parameterization consistent with energy requirements.

4. Discussion

Setting $A = 1$ and $B = 1$ in case VI corresponds to case IV, which is the form used by Schär and Smith (1993) and satisfies Eq. (8) automatically $\{$i.e., $[(u_x - v_y)^2 + (v_x + u_y)^2] \geq 0\}$. Setting $A = h$ and $B = 1$ is case V, and it also satisfies Eq. (8) [see Eqs. (11)] because h is a nonnegative function. Although Gent (1993) does not explicitly exhibits a stress tensor, the corresponding of case II implies the same parameterization. A tensor of this kind can be uniquely decomposed as the sum of an isotropic tensor, a symmetric tensor, and an antisymmetric tensor. The stress tensor of case V is the symmetric component of case II. Case III corresponds to the choice $A = 1$ and $B = h$.

Parameterizations IV and V are consistent with the principles of energy and angular momentum conservation, but case V has an awkward form in the momentum equations viscous terms; one component depends on the cross product of the thickness and the orthogonal velocity component gradients (see Table 1). This awkward dependency exists for all nontrivial choices of the function A. Only with a trivial choice of function A do the viscous terms in the momentum equations have a familiar form. Only with a trivial choice of function B is the energy requirement directly guaranteed.

Case VII is peculiar. It shows a viscous-dependent energy flux that has more terms than the immediately recognized contribution $u\mathbf{s}_1 + v\mathbf{s}_2$.

The use of a stress tensor that is a linear combination of different cases is also possible. If the individual cases involved in the linear combination satisfy the decomposition $u\nabla \cdot \mathbf{s}_1 + v\nabla \cdot \mathbf{s}_2 = \nabla \cdot \mathbf{F}_v - d^2$ and the coefficients are positive, the result is a stress consistent with energy dissipation. In the fully 2D nondivergent model, cases I–V are the same.

The neglect of bottom stress is an assumption of the reduced-gravity model at hand, an assumption that seems reasonable because the slippage of the active layer is over heavier fluid and not over a riverbed as in Gerbeau and Perthame (2001), Marche (2007), and Lucas and Rousseau (2008). These derivations of viscous shallow-water equations via integration of the Navier–Stokes equations show the lateral viscous terms as a secondary corrections, using the law of the wall, of the primary frictional effect: that due to bottom stress. Thus, in the context of an active layer slipping without friction over an inactive heavier layer, the form of the lateral frictional stress cannot be formally supported by those studies. Moreover, if friction is allowed between the active and inactive layers the reduced-gravity model at hand is incomplete.

The stress with the underlying fluid when the active layer is very thin or forms an edge can hardly be neglected. Therefore, under such conditions, this model becomes unrealistic (i.e., of questionable usefulness). Even more, insisting within this model in the limit of null thickness, the Θ_1 or Θ_2 terms in Eq. (9) become undefined in cases III, IV, V, and VII (all the cases with an explicitly defined symmetric stress tensor) for quite simple initial conditions. For example, consider the initial condition of a rectilinear jet in geostrophic balance, say starting with $u = 0$, $fv = g'\partial h/\partial x$, null derivatives in the y direction, and imposing an edge (i.e., $h \rightarrow 0$) at say $x = 0$ with the active layer extending at $x > 0$. In all these cases, computing Θ_2 requires terms like $(\partial^n v/\partial x^n)/h$ with $n \geq 1$ or $(\partial h/\partial x)(\partial v/\partial x)/h$, but in the geostrophic balance $\partial h/\partial x \propto v$; therefore, even with several loops of the l'Hospital rule, the lower derivative of h persists in the denominator and indicates the indefiniteness (i.e., a division of a finite term by a null term in the limit $x \rightarrow 0^+$). Schär and Smith (1993) also point to this indefiniteness.

An issue not fully treated here but of importance is the resulting stress on the boundary. In the simple case of a circular boundary and using case VI, it follows that

$$\oint (x\mathbf{s}_2 - y\mathbf{s}_1)\cdot\mathbf{n}\,d\ell = \int_0^{2\pi} rA\left(\frac{\partial Bu_\theta}{\partial r} - \frac{Bu_\theta}{r} + \frac{1}{r}\frac{\partial Bu_r}{\partial \theta}\right)d\theta, \tag{13}$$

where u_θ and u_r are the azimuthal and radial velocity components. If the boundary is impermeable $u_r = 0$ and $dL/dt = \int_0^{2\pi} rA(\partial Bu_\theta/\partial r - Bu_\theta/r)\,d\theta$ [see Eqs. (4b), (7), and (13)]. On solid boundaries, the common condition is that of no slip, in which case if the boundary is motionless $dL/dt = \int_0^{2\pi} rA\partial Bu_\theta/\partial r\,d\theta$, and $dL/dt = 0$ if the solid boundary rotates such that $\partial Bu_\theta/\partial r - Bu_\theta/r = 0$. This last equation is then the stress-free condition for circular solid boundaries (Jiménez Domínguez 2008). For $B = 1$ (cases IV and V), it means a boundary in solid-body rotation with the immediate fluid. The stress-free condition implies that the viscous torque is null.

4. Conclusions

As shown by many authors, the consistency with the principle of angular momentum conservation implies restrictions on the allowed frictional parameterizations that are easy to fulfill (i.e., a symmetric stress tensor). Here, we show that the freedom to choose the stress tensor is limited, given the restriction coming from energy considerations. A broad family of tensors, which depends on two arbitrary weighting functions, as depicted by case VI in Table 1, shows that, to assure consistency with the energy dissipation, (i) function B must be a trivial constant function; (ii) function A is quite free, with the only requirement of definite sign; but (iii) any nontrivial function A imposes a rather awkward friction form in the momentum equations. Even very simple choices of a function A, as the resulting form ($A = h$) in the symmetric component of the stress tensor proposed by Gent (1993), produces an unusual frictional term (see case V).

The very common parameterization of friction, case I, fails to requirements regarding energy dissipation (Gent 1993) and angular momentum. The close relative to case I that is consistent with energy dissipation as shown by Gent (1993) is case II, but it is not consistent with the principle of angular momentum. Case II can easily be transformed to case V, which is consistent with both energy and angular momentum but at the expense of rather awkward viscous terms in the momentum equations. The parameterization used by Schär and Smith (1993), which in case IV is simple and fulfils the requirements with energy and angular momentum.

The biharmonic parameterization (case VII) is consistent with the angular momentum conservation principle and with energy dissipation. The parameterization of the eddy-induced transport within this reduced-gravity model is unnecessary, because the actual variables to be solved are the full transports, without solving separate contributions.

Proper parameterization of Reynolds stresses in ocean models remains an unsolved issue (Griffies 2004; Marshall and Adcroft 2010; Eden and Greatbatch 2008). Our purpose here has been simply to draw attention to the properties of some commonly used horizontal viscosity parameterizations employed in layer models, and we have not dwelled into the physical basis that relate eddy stresses with the formulas given in Table 1. Users of these simple models should favor the use of formulas that preserve basic physical principles such as the ones discussed here.

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