



## Revisiting the Energetics of the Ocean in Boussinesq Approximation

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### ABSTRACT

Following a suggestion by Tailleux, a consistent formulation of internal energy, the first law of thermodynamics, and the thermodynamic potentials for an ocean in Boussinesq approximation with a nonlinear equation of state is given. A modification of the pressure work in the first law is the only necessary modification from which all thermodynamic potentials and thermodynamic relations follow in a consistent way. This treatment of thermodynamics allows for a closed and explicit formulation of conservation equations for dynamic and potential reservoirs of both enthalpy and internal energy, which differentiate approximately reversible from irreversible effects on internal energy, and allows for a formulation of a closed energy cycle on which energetically consistent ocean models can be based on.

### 1. Introduction

In this note, the thermodynamics and energetics of the ocean in Boussinesq approximation are revisited. The aim is to formulate a consistent energy cycle for the ocean in Boussinesq approximation with a nonlinear equation of state and to formulate explicit conservation equations of the relevant thermodynamic quantities that have been previously discussed involving the correct exchange terms and molecular fluxes of heat and salt. Such a formulation is a prerequisite for energetically consistent ocean models, both numerical and analytical ones, but it is also necessary for conceptual reasons since almost any theory of the ocean is based on the Boussinesq approximation.

In general, the total energy of the ocean is given by the sum of kinetic energy  $e_k$ , gravitational potential energy  $e_p$ , and internal energy  $u$  (e.g., [Olbers et al. 2012](#)). However, it is discussed below that a peculiar feature of the Boussinesq approximation is that the gravitational potential energy  $e_p$  turns from purely mechanical into a thermodynamic quantity and with  $u$  adds to enthalpy  $h = e_p + u$ ; that is, gravitational potential energy becomes the difference between internal energy and enthalpy. The change of  $e_p$  from a mechanical to a thermodynamic variable is because

in the Boussinesq approximation parts of the work done by compression (pressure work hereinafter) can become an irreversible exchange with internal energy related to density mixing, while the pressure work is completely reversible in the full (compressible) equations (e.g., [Olbers et al. 2012](#)). This feature complicates the interpretation of energetics in the Boussinesq approximation. To account for this peculiarity, a consistent formulation of the thermodynamics in the Boussinesq approximation is needed ([Tailleux 2012](#)) based on a consistent representation of the pressure work in the first law of thermodynamics.

Instead of  $e_p$  and  $u$ , another physically meaningful split of enthalpy  $h$  in the Boussinesq approximation is into potential enthalpy  $h^o$  and dynamic enthalpy  $h^d$  ([McDougall 2003](#); [Young 2010](#); [Nycander 2011](#)). This split has the advantage that it approximately differentiates between reversible energy changes, which are related to  $h^d$ , and irreversible mixing effects, which are related to  $h^o$ . It is shown below that the same split can be made for internal energy  $u$ , for which dynamic internal energy  $u^d$  also accounts for an exchange with  $e_p$ , while potential internal energy  $u^o$  is formally identical to  $h^o$ . Because of its (approximate) conservative property,  $h^o$  (or  $u^o$ ) is used as the thermodynamic state variable in addition to salinity and Boussinesq reference pressure in current ocean models ([McDougall 2003](#)). It is the aim of this note to provide closed and explicit conservation equations for all previously discussed forms of energy

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based on molecular fluxes of enthalpy and salt and kinetic energy dissipation and to provide a simple but consistent description of the energetics of the ocean in Boussinesq approximation.

## 2. Energetics of the Boussinesq approximation

To consider the thermodynamic properties of a salty water parcel in the Boussinesq approximation in the ocean, recall that full density  $\rho_{\text{full}}$  is split in the Boussinesq approximation into a constant reference density  $\rho_0$  and a perturbation  $\rho$ , with  $\rho_0 \gg \rho$ . The full pressure  $p_{\text{full}}$  is also split into a component  $p_0$  that is in hydrostatic balance with  $\rho_0$  and a perturbation pressure  $p$ :

$$\rho_{\text{full}} = \rho_0 + \rho, \quad p_{\text{full}} = p_0(z) + p, \quad p_0 = -g\rho_0 z. \quad (1)$$

The key to interpret the energetics of the Boussinesq approximation is to note that the term  $gz d\rho/dt$  that shows up in the equation for potential energy  $e_p = g\rho z/\rho_0$  (see below) corresponds to the pressure work  $p_{\text{full}} \nabla \cdot \mathbf{u}$  in the fully compressible equations (e.g., Winters et al. 1995; Tailleux 2012; Olbers et al. 2012). The pressure work can be written in the Boussinesq approximation as

$$\begin{aligned} p_{\text{full}} \nabla \cdot \mathbf{u} &= p_{\text{full}} \rho_{\text{full}} \frac{dv_{\text{full}}}{dt} = p_0 \rho_0 \frac{dv}{dt} + O(\delta^2) \\ &= gz \frac{d\rho}{dt} + O(\delta^2), \end{aligned} \quad (2)$$

with the small parameter  $\delta = \rho/\rho_0$ , where  $v_{\text{full}} = v_0 + v$  denotes the specific volume, with  $v_0 = 1/\rho_0$ , and where the compressible continuity equation  $\rho_{\text{full}} dv_{\text{full}}/dt = \nabla \cdot \mathbf{u}$  was used in the second step and with

$$\begin{aligned} v_{\text{full}}/v_0 &= 1 + v/v_0 = 1/(1 - v/v_0) + O(\delta^2) \\ \rightarrow v\rho_0 &= -\rho/\rho_0 + O(\delta^2). \end{aligned} \quad (3)$$

In the compressible equations, the pressure work connects kinetic energy with internal energy, but it drops in the kinetic energy equation in the Boussinesq approximation since the continuity equation becomes  $\nabla \cdot \mathbf{u} = 0$ . The pressure work shows up instead in the conservation equation for gravitational potential energy ( $e_p = g\rho z/\rho_0$ ) in the Boussinesq approximation, while it is not present in the gravitational potential energy ( $e_p = gz$ ) of the compressible equations. This feature of the Boussinesq approximation links internal energy and potential energy, and it becomes therefore necessary to carefully rederive the formulation of the thermodynamics in this approximation.

### a. Thermodynamics in the Boussinesq approximation

Recall that the thermodynamics of the compressible equations are based on the first and second law of

thermodynamics for salty water parcels. The combination of both laws yields the Gibbs relation, which can be interpreted as the total differential change of internal energy by changes in its canonical state variables entropy  $\eta$ , salinity  $S$ , and specific volume  $v_{\text{full}}$ . The other thermodynamic potentials—enthalpy, Gibbs function, and Helmholtz-free energy—with different sets of (three) canonical state variables out of  $\eta$ ,  $S$ ,  $v_{\text{full}}$ , and temperature  $T$  follow from appropriate Legendre transformations (Olbers et al. 2012). The analogous concept starting from the first law is used here to derive the thermodynamics of the Boussinesq approximation. The pressure work is modified in the first law for a fluid parcel in Boussinesq approximation. The other thermodynamic potentials are then derived by Legendre transformations from this approximate first law, rather than by applying the Boussinesq approximation directly to the exact expressions for the potentials.

Tailleux (2012) suggests making an analogous replacement to Eq. (2) in the first law of thermodynamics for a salty water parcel in Boussinesq approximation, that is, to replace the pressure work  $p_{\text{full}} dv_{\text{full}}$  in the compressible equations with  $gz d\rho/\rho_0$  in the Boussinesq approximation, which yields

$$du = \delta Q - gz d\rho/\rho_0 + (\partial h/\partial S)_{S, p_{\text{full}}} dS, \quad (4)$$

where  $u$  denotes the internal energy for the Boussinesq approximation, and  $h$  is enthalpy, also for the Boussinesq approximation. While  $\delta Q$  denotes the heat exchanges at constant pressure  $p_{\text{full}}$  of the thermodynamic system under consideration with the surrounding environment, the second term in Eq. (4) denotes the pressure work, and in the last term the energy shifts through changes in the salinity  $S$  due to a mass exchange with the surrounding environment. This mass exchange is also assumed to take place at constant pressure  $p_{\text{full}}$  such that the derivative of  $h$  with respect to  $S$  is taken at constant pressure  $p_{\text{full}}$  and temperature  $T$  (Olbers et al. 2012). The replacement of the pressure work is the only necessary modification of the energetics in Boussinesq approximation. The thermodynamic potentials and all thermodynamic relations including the equation of state follow now in the same way as for the compressible equations.

The quantity  $\delta Q$  is related by the second law of thermodynamics to changes in entropy  $\eta$  of the system by  $d\eta = \delta Q/T + (\partial \eta/\partial S)_{T, p_{\text{full}}} dS$ . The second law states that the extensive state variable entropy  $\eta$  changes because of the heat exchange  $\delta Q$  but also because of salinity changes due to the mass exchange. The derivative of  $\eta$  with respect to  $S$  is again taken at constant pressure

$p_{\text{full}}$  and temperature  $T$ . Combining both relations yields the Gibbs relation in the Boussinesq approximation:

$$du = Td\eta + \mu dS - gz d\rho/\rho_0, \quad (5)$$

with the relative chemical potential of seawater  $\mu = (\partial h/\partial S)_{T,p_{\text{full}}} - T(\partial\eta/\partial S)_{T,p_{\text{full}}}$ .<sup>1</sup> The Gibbs relation [Eq. (5)] states that the canonical state variables for the thermodynamic potential  $u$  in the Boussinesq approximation are  $u = u(S, \eta, \rho$  or  $v)$ . The difference from the internal energy of the compressible equations  $u_{\text{full}}$  is that instead of the total specific volume  $v_{\text{full}}$ , the perturbation  $v$  is used as a state variable. The other thermodynamic potentials are now given by an appropriate Legendre transformation using Eq. (5). Thus, instead of setting  $h_{\text{full}} = u_{\text{full}} + p_{\text{full}}v_{\text{full}}$ , as for the compressible equations,  $h = u + gz\rho/\rho_0$  is set for the enthalpy in the Boussinesq approximation, and  $g_{\text{ibbs}} = h - T\eta$  is set for the Gibbs function. With Eq. (5) this gives

$$\begin{aligned} dh &= Td\eta + \mu dS + gpdz/\rho_0, \\ dg_{\text{ibbs}} &= -\eta dT + \mu dS + gpdz/\rho_0. \end{aligned} \quad (6)$$

The canonical state variables for  $h$  and  $g_{\text{ibbs}}$  in the Boussinesq approximation are  $h(S, \eta, z$  or  $p_0)$  and  $g_{\text{ibbs}}(S, T, z$  or  $p_0)$ , respectively. It is important to note that the straightforward approximation  $p_{\text{full}}v_{\text{full}} = -gz + gz\rho/\rho_0 - p/\rho_0 + O(\delta^2)$  does not produce the appropriate Legendre transformation to derive the enthalpy. Instead the appropriate Legendre transformation follows from the Gibbs relation in the Boussinesq approximation [Eq. (5)].

A numerical estimate of the Gibbs function of compressible seawater is given by Feistel (2008) as  $g_{\text{ibbs}}^{\text{full}} = g_F(S, T, P)$ , with  $\partial g_{\text{ibbs}}^{\text{full}}/\partial T = -\eta$ ,  $\partial g_{\text{ibbs}}^{\text{full}}/\partial S = \mu$ , and  $\partial g_{\text{ibbs}}^{\text{full}}/\partial p_{\text{full}} = v_{\text{full}}$ . Tailleux (2012) suggested using as the Gibbs function  $g_{\text{ibbs}}$  in the Boussinesq approximation

$$g_{\text{ibbs}}(S, T, z) = g_F(S, T, p_0) + gz, \quad (7)$$

with the total differential

$$\begin{aligned} dg_{\text{ibbs}}(S, T, z) &= -\eta(S, T, p_0)dT + \mu(S, T, p_0)dS \\ &+ g[1 - v_{\text{full}}(S, T, p_0)\rho_0]dz, \end{aligned} \quad (8)$$

<sup>1</sup> In the definition for  $\mu$ , enthalpy  $h$  is differentiated with respect to  $S$  at constant pressure  $p_{\text{full}}$  and temperature  $T$ . In the following, partial differentials are always taken at constant (remaining) canonical state variables of the respective quantity. The value  $(\partial h/\partial S)_{T,p_{\text{full}}}$  in Eq. (4) and the definition of  $\mu$  are the only exceptions since the canonical state variables of  $h$  are  $S, \eta, p_0$ . The brackets at the partial derivatives will be omitted in the following.

which indeed matches to  $O(\delta^2)$  the thermodynamic relation for  $dg_{\text{ibbs}}$  in Eq. (6) using the relation Eq. (3) in the last term of Eq. (8). For this matching it is important that  $\eta, \mu$ , and  $\rho$  in Eq. (8) are evaluated at the Boussinesq reference pressure  $p_0 = -g\rho_0 z$  instead of the full pressure  $p_{\text{full}}$ .

Using Eq. (6), the equation of state for the Boussinesq approximation<sup>2</sup> becomes

$$\frac{\partial}{\partial z} g_{\text{ibbs}}(S, T, z) = g\rho/\rho_0, \quad (9)$$

where the derivative with respect to  $z$  is taken at constant salinity  $S$  and temperature  $T$ . The other thermodynamic relations follow in an analogous way.

### b. Internal energy of the Boussinesq approximation

With the consistent thermodynamics for the Boussinesq approximation outlined by Tailleux (2012), it is now possible to formulate the primitive equations using the thermodynamic potential  $g_{\text{ibbs}}$  and the Gibbs relation in the consistent form for the Boussinesq approximation:

$$\rho_0 \frac{d\mathbf{u}}{dt} + 2\rho_0 \mathbf{\Omega} \times \mathbf{u} = -\nabla p - \rho g \mathbf{k} + \nabla \cdot \mathbf{\Sigma}, \quad (10)$$

$$\nabla \cdot \mathbf{u} = 0, \quad \text{and} \quad (11)$$

$$\rho_0 \frac{dS}{dt} = -\nabla \cdot \mathbf{J}_S, \quad (12)$$

with the frictional tensor  $\mathbf{\Sigma}$ , the molecular flux of salt  $\mathbf{J}_S$ , and Earth's rotation vector  $\mathbf{\Omega}$ . For the moment, a conservation equation for entropy of the form  $\rho_0 d\eta/dt = -\nabla \cdot \mathbf{J}_\eta + \dot{\eta}$  is also added, with a molecular entropy flux  $\mathbf{J}_\eta$ , and where  $\dot{\eta}$  collects all irreversible processes that increase  $\eta$ . These processes will be specified below in Eq. (16). The conservation equation for  $\eta$  will be replaced later by an equation for internal energy or potential enthalpy.

Kinetic energy  $e_k = |\mathbf{u}|^2/2$  and gravitational potential energy  $e_p = gz\rho/\rho_0$  are readily derived from the set of equations

<sup>2</sup> Young (2010) and Tailleux (2012) argued to use the exact form

$$\partial g_{\text{ibbs}}/\partial z = g(1 - v_{\text{full}}\rho_0) = -gv\rho_0$$

as the equation of state instead of the approximate form given by Eq. (9). However, it is suggested here to ignore the difference between the definition by Young (2010) and Tailleux (2012) and Eq. (9), since it should be small and comparable to other errors in the Boussinesq approximation.

$$\rho_0 \frac{de_k}{dt} = -\nabla \cdot (\mathbf{u}p - \mathbf{u} \cdot \boldsymbol{\Sigma}) - g\rho w - \rho_0 \epsilon, \quad \text{and} \quad (13)$$

$$\rho_0 \frac{de_p}{dt} = g\rho w - \delta w, \quad \delta w = -gz \frac{d\rho}{dt}, \quad (14)$$

with kinetic energy dissipation  $\rho_0 \epsilon = \boldsymbol{\Sigma} \cdot \nabla \mathbf{u}$  and the pressure work in the Boussinesq approximation  $\delta w = -\rho_0 p_0 dv/dt$  [using Eq. (3)]. This meaning of the term  $\delta w$  is also motivated from the conservation equation for internal energy  $u$ , which follows from the Gibbs relation [Eq. (5)] for a water parcel replacing the total change in salinity  $dS$ , entropy  $d\eta$ , density  $d\rho$ , and internal energy  $du$ , with the material derivative  $d/dt$  of the respective quantity. Equation (5) becomes

$$\rho_0 \frac{du}{dt} = \delta w - \nabla \cdot \mathbf{J}_h + T\dot{\eta} + \mathbf{J}_\eta \cdot \nabla T + \mathbf{J}_S \cdot \nabla \mu, \quad (15)$$

where  $\rho_0 dS/dt$  and  $\rho_0 d\eta/dt$  were replaced with its sources and with the molecular enthalpy flux<sup>3</sup>  $\mathbf{J}_h = T\mathbf{J}_\eta + \mu\mathbf{J}_S$ . The total energy  $e_k + e_p + u$  in the Boussinesq approximation is conserved if

$$T\dot{\eta} = -\mathbf{J}_\eta \cdot \nabla T - \mathbf{J}_S \cdot \nabla \mu + \rho_0 \epsilon > 0, \quad (16)$$

where the second relation comes from the second law of thermodynamics and gives a constraint on the fluxes (which is not considered further here). Using Eq. (16) in Eq. (15) to replace  $\dot{\eta}$  reveals the conservation equation for internal energy in the Boussinesq approximation:

$$\rho_0 \frac{du}{dt} = \delta w - \nabla \cdot \mathbf{J}_h + \rho_0 \epsilon. \quad (17)$$

This conservation equation for  $u$  in the Boussinesq approximation allows for a consistent description of the energetics of ocean models based on this approximation. Total energy  $e_{\text{tot}} = e_k + e_p + u$  is conserved, such that Eqs. (13) and (14) together with Eq. (17) represent the fundamental energy cycle for the Boussinesq approximation.

*c. Dynamic and potential enthalpy*

Instead of using internal energy  $u$  as basic thermodynamic quantity, it is often more instructive to consider enthalpy, which becomes  $h = u + gz\rho/\rho_0$  in the Boussinesq

approximation as shown in section 2a. From Eq. (17), the conservation equation for  $h$  becomes

$$\rho_0 \frac{dh}{dt} = g\rho w - \nabla \cdot \mathbf{J}_h + \rho_0 \epsilon. \quad (18)$$

Note that because  $h = e_p + u$ , it is surprising that the gravitational potential energy turns into a thermodynamic property in the Boussinesq approximation; namely, it turns into the difference between enthalpy and internal energy [cf. also Oliver and Tailleux (2013) for the compressible equations]. Total energy  $e_{\text{tot}} = e_k + h$  is conserved, such that Eq. (13), together with Eq. (18), also represents a consistent energy cycle for the Boussinesq approximation. However, it is useful to split  $h$  again into different parts, namely, dynamical and potential enthalpy.

In the compressible equations, the canonical state variables of the thermodynamic potential  $h_{\text{full}}$  are salinity  $S$ , entropy  $\eta$ , and pressure  $p_{\text{full}}$ . There is, among others, the thermodynamic relation  $\partial h_{\text{full}}/\partial p_{\text{full}} = v_{\text{full}}$ , which motivated McDougall (2003), Young (2010), and Nycander (2011) to split enthalpy  $h_{\text{full}}$  into an integration constant  $h_{\text{full}}^o = h(S, \eta, p_r)$  at a chosen constant reference pressure  $p_r$ , which is called the potential enthalpy, and the dynamic enthalpy  $h_{\text{full}}^d = \int_{p_r}^{p_{\text{full}}} v_{\text{full}} dp$ , denoted by  $\tilde{h}^\dagger$  by Young (2010) and  $\tilde{h}_d$  by Nycander (2011). The constant reference pressure  $p_r$  at which  $h_{\text{full}}^o$  is evaluated is usually the surface pressure.

In the Boussinesq approximation the corresponding thermodynamic relation  $\partial h/\partial z = g\rho/\rho_0$  follows from Eq. (6). The split and  $h^d$  thus become

$$h(S, \eta, p_0) = h^o(S, \eta) + h^d(S, \eta, p_0 \text{ or } z),$$

$$h^d \equiv g \int_{z_r}^z \rho/\rho_0 dz', \quad h^o \equiv h(S, \eta, p_r \text{ or } z_r). \quad (19)$$

All variables are evaluated at the Boussinesq reference pressure  $p_0 = -g\rho_0 z$  instead of full pressure  $p_{\text{full}}$  when formulated in the Boussinesq approximation. The definition for  $h^d$  in Eq. (19) is identical to those of  $\tilde{h}^\dagger$  in Young (2010) and  $\tilde{h}_d$  in Nycander (2011). Taking the material derivative of  $h^o(S, \eta)$  and  $h^d(S, \eta, z)$  and using Eq. (6) yields

$$\frac{dh^o}{dt} = \mu^o \frac{dS}{dt} + T^o \frac{d\eta}{dt},$$

$$\frac{dh^d}{dt} = (\mu - \mu^o) \frac{dS}{dt} + (T - T^o) \frac{d\eta}{dt} + g\rho w/\rho_0, \quad (20)$$

with the in situ temperature as  $\partial h/\partial \eta = T$ , the relative chemical potential as  $\partial h/\partial S = \mu$ , the temperature at reference pressure as  $T^o = \partial h^o/\partial \eta = T(S, \eta, z_r)$ —usually called

<sup>3</sup>To obtain an expression for  $\mathbf{J}_\eta$ , the enthalpy flux itself can be considered, which must consist of a heat flux component at constant pressure and a salt flux component and is given by  $\mathbf{J}_h = \mathbf{J}_q + \partial h/\partial S \mathbf{J}_S = \mathbf{J}_q + (\mu + T \partial \eta/\partial S) \mathbf{J}_S$ , and thus  $\mathbf{J}_\eta = \mathbf{J}_q/T + (\partial \eta/\partial S) \mathbf{J}_S$ .

the potential temperature—and the relative chemical potential at reference pressure as  $\mu^o = \partial h^o / \partial S = \mu(S, \eta, z_r)$ . Using the salt conservation equation and the Gibbs relation for the Boussinesq approximation in the form  $T\rho_0 d\eta/dt = -\nabla \cdot \mathbf{J}_h + \mu \nabla \cdot \mathbf{J}_S + \rho_0 \epsilon$ , the conservation equation for  $h^o$  can be written as

$$\begin{aligned} \rho_0 \frac{dh^o}{dt} &= \left( \frac{T^o}{T} \mu - \mu^o \right) \nabla \cdot \mathbf{J}_S + \frac{T^o}{T} (-\nabla \cdot \mathbf{J}_h + \rho_0 \epsilon) \\ &= -\nabla \cdot \mathbf{J}_h + \rho_0 \epsilon + \delta w_{\text{McD}}, \end{aligned} \quad (21)$$

with

$$\begin{aligned} \delta w_{\text{McD}} &= -\left( \mu^o - \frac{T^o}{T} \mu \right) \nabla \cdot \mathbf{J}_S \\ &\quad - \left( 1 - \frac{T^o}{T} \right) (-\nabla \cdot \mathbf{J}_h + \rho_0 \epsilon). \end{aligned} \quad (22)$$

Subtracting Eq. (21) from Eq. (18) yields the conservation equation for  $h^d$ :

$$\begin{aligned} \rho_0 \frac{dh^d}{dt} &= -\left( \frac{T^o}{T} \mu - \mu^o \right) \nabla \cdot \mathbf{J}_S \\ &\quad + \left( 1 - \frac{T^o}{T} \right) (-\nabla \cdot \mathbf{J}_h + \rho_0 \epsilon) + g\rho w \\ &= g\rho w - \delta w_{\text{McD}}. \end{aligned} \quad (23)$$

The term  $\delta w_{\text{McD}}$  that links  $h^o$  and  $h^d$  vanishes for adiabatic motion, where  $\mathbf{J}_S = 0$ ,  $\mathbf{J}_h = 0$ , and  $\epsilon = 0$ , and remains small as long as  $T^o \approx T$  and  $\mu^o \approx \mu$ . The term  $\delta w_{\text{McD}}$  can be further manipulated by separating a flux divergence:

$$\begin{aligned} \delta w_{\text{McD}} &= -\nabla \cdot \mathbf{J}_{\text{McD}} + \delta w'_{\text{MdD}}, \\ \mathbf{J}_{\text{McD}} &= \mathbf{J}_S \left( \mu^o - \frac{T^o}{T} \mu \right) + \mathbf{J}_h \left( \frac{T^o}{T} - 1 \right) \\ \delta w'_{\text{MdD}} &= \mathbf{J}_S \cdot \nabla \left( \mu^o - \frac{T^o}{T} \mu \right) + \mathbf{J}_h \cdot \nabla \left( \frac{T^o}{T} \right) - \left( 1 - \frac{T^o}{T} \right) \rho_0 \epsilon. \end{aligned} \quad (24)$$

Only the component  $\delta w'_{\text{MdD}}$  contributes to the energy exchange, while the divergence of  $\mathbf{J}_{\text{McD}}$  does not contribute and only modifies or introduces molecular fluxes of  $h^o$  and  $h^d$ . McDougall (2003) and more recently Graham and McDougall (2013) show that  $\delta w_{\text{McD}}$  can be neglected in Eq. (21) to a very good approximation. Equations (21) and (23) demonstrate that  $h^d$  is changed predominantly by reversible energy exchanges, that is, the raising or sinking of water parcels within the geopotential, while  $h^o$  is changed by irreversible energy changes given by fluxes of salt and enthalpy and dissipation of kinetic energy. Total energy  $e_{\text{tot}} = e_k + h^o + h^d$  is conserved, such

that Eq. (13) together with Eqs. (21) and (23) represent another description of the energy cycle.

#### d. Dynamic and potential internal energy

To obtain a relation between dynamic enthalpy and gravitational potential energy, it is useful to consider an analogous split into a dynamical and a potential part for internal energy. In the compressible equations, the canonical thermodynamic state variables of internal energy are salinity  $S$ , entropy  $\eta$ , and specific volume  $v_{\text{full}}$ , and there is (among others) the thermodynamic relation  $\partial u_{\text{full}} / \partial v_{\text{full}} = -p_{\text{full}}$ . The dynamic internal energy is thus given in the compressible equations as  $u_{\text{full}}^d = -\int_{v_{\text{full}}^r}^{v_{\text{full}}} p_{\text{full}} dv'$ , introducing the constant (total) reference-specific volume  $v_{\text{full}}^r$ .

In Boussinesq approximation, the thermodynamic state variables using  $u$  as potential are  $S$ ,  $\eta$ , and  $\rho$  (or  $v$ ), which follows from the Gibbs relation in the form Eq. (5). The resulting thermodynamic relation  $\partial u / \partial \rho = -gz / \rho_0$  can be integrated to yield

$$\begin{aligned} u &= u^o(S, \eta) + u^d(S, \eta, \rho), \quad u^o \equiv u(S, \eta, \rho_r), \\ u^d &\equiv -g \int_{\rho_r}^{\rho} z(\rho) v_0 d\rho' = \int_{v_r}^v p_0(v') dv', \end{aligned} \quad (25)$$

introducing the constant reference-specific volume (perturbation)  $v_r$  and using  $\rho v_0 = -v \rho_0$  from Eq. (3) and  $v_r = 1/\rho_r$ . Taking the material derivatives and using Eq. (5) yields

$$\begin{aligned} \frac{du^o}{dt} &= \mu^o \frac{dS}{dt} + T^o \frac{d\eta}{dt}, \quad \frac{du^d}{dt} = (\mu - \mu^o) \frac{dS}{dt} \\ &\quad + (T - T^o) \frac{d\eta}{dt} - gz / \rho_0 \frac{d\rho}{dt}, \end{aligned} \quad (26)$$

Note that  $\mu^o$  and  $T^o$  here denote  $\mu$  and  $T$  taken at the reference density  $\rho_r$  instead of  $\mu$  and  $T$  taken at the reference depth  $p_r$  as in the previous section. From Eq. (26) the conservation equation for  $u^o$  follows immediately using the salt conservation equation and the Gibbs relation as

$$\rho_0 \frac{du^o}{dt} = -\nabla \cdot \mathbf{J}_h + \rho_0 \epsilon + \delta w_{\text{McD}}. \quad (27)$$

Subtracting this from Eq. (17), the conservation equation for  $u^d$  follows as

$$\rho_0 \frac{du^d}{dt} = \delta w - \delta w_{\text{McD}}. \quad (28)$$

Note that  $\delta w_{\text{McD}}$  is the same as in Eq. (22), but with  $\mu^o$  and  $T^o$  from Eq. (26), that is, taken at  $\rho_r$  instead of  $p_r$ . The equations for  $h^o$  and  $u^o$  are identical, as long as the reference density  $\rho_r$  is constant at the geopotential  $z_r$  or

vice versa, otherwise the small exchange term  $\delta w_{\text{McD}}$  differs slightly.<sup>4</sup> However, it was noted by McDougall (2003) that the magnitude of  $\delta w'_{\text{McD}}$  is in both cases very small, such that it appears unnecessary to differentiate between both cases in Eqs. (21) and (27). Note that  $\delta w_{\text{McD}} = 0$  for adiabatic systems also in both cases. It holds that  $e_p + u^o + u^d = h^o + h^d$  and that total energy  $e_{\text{tot}} = e_k + e_p + u^d + u^o$  is conserved, such that Eqs. (13) and (14) together with Eqs. (27) and (28) represent another description of the energy cycle.

The derivation of the dynamical internal energy  $e^\ddagger$  by Young (2010) differs slightly from the present one since it relies on neglecting terms that are small in the Boussinesq approximation instead of consistent thermodynamics. The dynamic enthalpy of the compressible equations is given by

$$h_{\text{full}}^d = \int_{p_r}^{p_{\text{full}}} v_{\text{full}} dp' = (p_0 - p_r)/\rho_0 + p/\rho_0 + h^d + \int_{p_0}^{p_0+p} v dp' + O(\delta^2). \tag{29}$$

While the constant  $v_0$  is integrated in Eq. (29) to the full pressure  $p_0 + p$ , integrating  $v$  to the Boussinesq reference pressure  $p_0$  only yields (approximately)  $h^d$ . It was argued by Young (2010) that the remaining term  $\int_{p_0}^{p_0+p} v dp'$  is of the second order and can be neglected in the Boussinesq approximation. The internal energy  $u_{\text{full}} = h_{\text{full}} - p_{\text{full}}v_{\text{full}}$  of the compressible equations then becomes

$$u_{\text{full}} = h_{\text{full}} + gz - gz\rho/\rho_0 - p/\rho_0 + O(\delta^2) = h_{\text{full}}^o - p_r/\rho_0 + h^d - gz\rho/\rho_0 + O(\delta^2), \tag{30}$$

using the split  $h_{\text{full}} = h_{\text{full}}^o + h_{\text{full}}^d$  and Eq. (29) to replace  $h_{\text{full}}^d$ . Young's definitions of potential internal energy  $e_0^\ddagger = h_{\text{full}}^o - p_r/\rho_0$  and dynamic internal energy  $e^\ddagger = h^d - gz\rho/\rho_0$  in the Boussinesq approximation follow then from Eq. (30) by neglecting the terms  $O(\delta^2)$ . Using Eqs. (21) and (23), it follows that

$$\rho_0 \frac{de_0^\ddagger}{dt} = -\nabla \cdot \mathbf{J}_h + \rho_0 \epsilon + \delta w_{\text{McD}}, \quad \rho_0 \frac{de^\ddagger}{dt} = g\rho w - \delta w_{\text{McD}}, \tag{31}$$

<sup>4</sup>The term  $\delta w_{\text{McD}}$  is the largest far away from  $z_r$ , that is, at the bottom of the deep ocean if  $p_r$  denotes the surface pressure. Since the variation of  $\rho_r$  with depth due to the compressibility is one or two orders of magnitude larger than its surface variation, the difference in  $\delta w_{\text{McD}}$  using either  $z_r$  or  $\rho_r$  becomes  $O(1 - 10\%)$  at depth at maximum, that is, on the order of the Boussinesq approximation itself. The difference in the term  $\delta w'_{\text{McD}}$  will be even smaller.

with  $h^o = h_{\text{full}}^o$ , and where  $\delta w_{\text{McD}}$  is the same as in Eq. (22), that is, with  $\mu^o$  and  $T^o$  taken at reference pressure  $p_r$ . Thus, only if there are no variations of  $\rho_r$  at the reference pressure  $p_r$  or vice versa does the present definition of dynamic internal energy  $u^d$  correspond to  $e^\ddagger$  by Young (2010). The same role is played by the variable  $E_{\text{in}} = h^d - e_p$  in Nycander (2011). Since  $e_p + u^d + u^o = h^d + h^o$ , it follows that  $E_{\text{in}} = u^d + u^o - h^o$ , and thus  $E_{\text{in}} = u^d$  as long as  $h^o = u^o$ , which is only the case if there are no variations of  $\rho_r$  at the reference pressure  $p_r$  or vice versa.

### 3. Conclusions and discussion

It is the aim of this note to provide closed and explicit conservation equations for all forms of energy of a salty water parcel in Boussinesq approximation previously discussed by Young (2010) and Nycander (2011) based on molecular fluxes of enthalpy and salt and kinetic energy dissipation. The momentum, continuity, and salt equation in the Boussinesq approximation, Eqs. (10), (11), and (12), are complemented by a consistent thermodynamic equation obtained by a modification of the pressure work in the first law as suggested by Tailleux (2012). This modification leads to a thermodynamic conservation equation formulated either for internal energy  $u$  in Eq. (17) or for enthalpy  $h$  in Eq. (18) and also to a consistent equation of state for the Boussinesq approximation [Eq. (9)]. Adding either kinetic energy  $e_k$  and enthalpy  $h$ , or gravitational potential energy  $e_p$  and  $e_k$  and internal energy  $u$ , yields both a conserved total energy variable and a consistent description of the energetics of the ocean in Boussinesq approximation is obtained.

The definition of dynamic and potential enthalpy ( $h^d$  and  $h^o$ ) (McDougall 2003; Young 2010; Nycander 2011) or dynamic and potential internal energy ( $u^d$  and  $u^o$ ) further allows to approximately separate the irreversible (given by  $-\nabla \cdot \mathbf{J}_h + \rho_0 \epsilon$ ) from reversible changes (given by  $g\rho w$ ) of  $h$  or  $u$ . This yields a simple and consistent description of the energetics of the ocean in the Boussinesq approximation given by

$$\rho_0 \frac{de_k}{dt} = -\nabla \cdot (\mathbf{u}p - \mathbf{u} \cdot \boldsymbol{\Sigma}) - g\rho w - \rho_0 \epsilon, \tag{32}$$

$$\rho_0 \frac{de_p}{dt} = g\rho w - \delta w \quad \text{and} \quad \rho_0 \frac{du^d}{dt} = \delta w - \delta w_{\text{McD}} \quad \text{or} \tag{33}$$

$$\rho_0 \frac{dh^d}{dt} = g\rho w - \delta w_{\text{McD}},$$

$$\rho_0 \frac{dh^o}{dt} = -\nabla \cdot \mathbf{J}_h + \rho_0 \epsilon + \delta w_{\text{McD}}. \tag{34}$$

For simplicity, it is assumed here in Eq. (33) that  $\delta w_{\text{McD}}$  is identical using reference pressure  $p_r$  or reference density  $\rho_r$  for the definition of  $\mu^o$  and  $T^o$  in Eq. (22) and that consequently  $h^o = u^o$ . Considering only the gravitational potential energy  $e_p$  and potential enthalpy  $h^o$  is not sufficient to describe a consistent energy cycle, as already pointed out by Young (2010) and Nycander (2011), since  $e_p$  also has an exchange with dynamic internal energy  $u^d$  because of the remaining compressibility in the Boussinesq approximation given by  $\delta w = -gzdp/dt$ , which originates from  $-p_{\text{full}} \nabla \cdot \mathbf{u}$  in the fully compressible equations, directly linking there kinetic energy  $e_k$  with  $u^d$ . Since  $\nabla \cdot \mathbf{u}$  is set to zero in the Boussinesq approximation,  $\delta w$  connects  $e_p$  and  $u^d$  instead. Moreover, the fully reversible pressure work of the compressible equations becomes partly irreversible in the Boussinesq approximation. This is the main effect of the Boussinesq approximation on the energetics.

It was proposed by McDougall (2003) to neglect  $\delta w_{\text{McD}}$  in Eq. (34) and to use  $h^o$  (or  $u^o$ ) as a (conserved) state variable, which was named Conservative Temperature by McDougall (scaled by an arbitrary constant heat capacity, which is suppressed here), and to express density as  $\rho(S, h^o, p_0)$ . Approximate forms for this equation of state are given by, for example, Jackett et al. (2006) and IOC et al. (2010). To obtain a consistent energy cycle, that is, a closed conservation equation for total energy, it then appears tempting to neglect  $\delta w_{\text{McD}}$  in Eq. (33) for both  $u^d$  and  $h^d$  as well, since otherwise an unbalanced energy source or sink shows up by the term  $\delta w_{\text{McD}}$ . However, while  $\delta w_{\text{McD}}$  is very small for the conservation of  $h^o$ , this is not necessarily the case for dynamic enthalpy  $h^d$ . McDougall (2003) also suggested neglecting the dissipation  $\rho_0 \epsilon$  in Eqs. (21) and (27), since it is very small, which also appears formally invalid in the present context because energy conservation would again be violated.

This means that while the dissipation  $\rho_0 \epsilon$  and also  $\delta w_{\text{McD}}$  are unimportant for the large reservoir of  $h^o$  or  $u^o$ , these terms can be important for the mechanical energy and in consequence also for  $h^d$  and  $u^d$ . This is because internal energy  $u$  is in general much larger than mechanical energy  $e_k$  or  $e_p$  for a fluid in the Boussinesq approximation, such that the irreversible conversions between these forms of energy might be essential for the mechanical but become insignificant for the internal energy. Note that the argument concerning the neglect of  $\delta w_{\text{McD}}$  and  $\rho_0 \epsilon$  holds for both the Boussinesq approximation and the fully compressible equations. In the latter,  $\delta w_{\text{McD}}$  also shows up<sup>5</sup> and

is also related to the separation between reversible and irreversible energy conversions by the definition of dynamic and potential internal energy or enthalpy that is only approximate. Nevertheless, a closed and consistent energy cycle can be defined for the Boussinesq approximation as for fully compressible equations based on the full internal energy  $u$  or enthalpy  $h$  as outlined above.

Instead of splitting the energy variables  $h$  and  $u$  into dynamic and potential reservoirs, it can also be useful to consider the total energy  $e_k + h$  in other ways. One of such alternatives is the concept of available potential energy, originally proposed by Lorenz (1955) and stated in precise form by Winters et al. (1995). Tailleux (2013) recently discussed a generalization of the concept in the Boussinesq approximation, introducing a sorted reference state of enthalpy  $h_r$  and deviations from that ( $h - h_r$ ). These deviations play the role of available potential energy and should be used to replace the approximate original definition by Lorenz (1955) that is often used for diagnostics.

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<sup>5</sup>The only differences besides the role and definition of the pressure work  $\delta w$  in the fully compressible equations are the replacement of  $\rho_0$  with full density  $\rho_{\text{full}}$  in the conservation equations and the evaluation of all quantities at  $p_{\text{full}}$  instead of  $p_0$ .

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