The Evolution of Nonlinear Wave Statistics through a Variable Medium

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ABSTRACT

In coastal areas and on beaches, nonlinear effects in ocean waves are dominated by so-called triad interactions. These effects can result in large energy transfers across the wave spectrum and result in non-Gaussian wave statistics, which is important for coastal wave propagation and wave-induced transport processes. To model these effects in a stochastic wave model based on the radiative transfer equation (RTE) requires a transport equation for three-wave correlators (the bispectrum) that is compatible with quasi-homogeneous theory. Based on methods developed in optics and quantum mechanics, the authors present a general approach to derive a transport equation for higher-order correlators. The principal result of this work is a coupled set of equations consisting of the radiative transfer equation with a nonlinear forcing term and a new, generalized transport equation for bispectrum. This study discusses the implications and characteristics of the resulting equations and shows that the model contains various shallow- and deep-water asymptotes for nonlinear wave propagation as special cases.

1. Introduction

As ocean waves propagate from deep water, onto the continental shelves, and toward coastal areas, their propagation is affected increasingly by interaction with bathymetry and currents, the transition from dominant resonant four-wave interactions to near-resonant three-wave (or triad) interactions and the transformation of organized wave motion into turbulence, heat, and sound in the breaking process close to shore. The ability to model these processes and their effects on wave statistics both in deep and shallow water is important to, for example, the modeling of mixing and circulation processes in the upper ocean (e.g., Craik and Leibovich 1976; McWilliams and Restrepo 1999), marine weather predictions and safety (e.g., Cavaleri et al. 2007), and the driving of coastal circulation and transport processes (e.g., Komar 1998; Dean and Dalrymple 2002).

Great advances have been made in operational wave prediction models (e.g., Hasselmann et al. 1988; Tolman 1991; Komen et al. 1994; Booij et al. 1999; Cavaleri et al. 2007) to the point that they are now routinely used for global and regional applications, either as stand-alone or coupled to atmosphere, climate, or coastal transport and circulation models. Invariably, such operational wave models are based on some form of the radiative transfer equation (RTE), which transports the variance density spectrum $E(\mathbf{k}, \mathbf{x}, t)$ through geographical space ($\mathbf{x} = [x_1, x_2]^{\mathrm{T}}$), wavenumber space ($\mathbf{k} = [k_1, k_2]^{\mathrm{T}}$), and time t, and can be written as

$$\partial_t E + \mathbf{c}^x \cdot \nabla_{\mathbf{x}} E + \mathbf{c}^k \cdot \nabla_{\mathbf{k}} E = S. \qquad (1)$$

Here, $\partial_{\{\ldots\}}$ is shorthand for partial differentiation with respect to the subscript variable $\nabla_{\mathbf{x}} \equiv [\partial_{x_1}, \partial_{x_2}]^{\mathrm{T}}$ and $\nabla_{\mathbf{k}} \equiv [\partial_{k_1}, \partial_{k_2}]^{\mathrm{T}}$. The left-hand side of Eq. (1) represents the conservation of wave energy in a slowly varying medium, with $\mathbf{c}^x = [c^{x_1}, c^{x_2}]^{\mathrm{T}}$ and $\mathbf{c}^k = [c^{k_1}, c^{k_2}]^{\mathrm{T}}$ denoting transport velocities through geographic and spectral space, respectively, and the forcing term $S(\mathbf{k}, \mathbf{x}, t)$ on the right of Eq. (1) represents source term contributions to account for nonconservative (whitecapping, depth-induced breaking, generation by wind, etc.) and nonlinear processes (i.e., resonant quadruplet wave–wave interactions; Hasselmann 1962; Janssen 2009a).

The RTE is based on the premise that the wave field is statistically quasi-homogeneous and Gaussian (e.g., Komen et al. 1994). These assumptions are reasonable for open-ocean wave generation and propagation, where the wave field evolves principally because of the effects of wind, whitecapping (dissipation), and resonant four-wave

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nonlinearity. However, in shallow water, inhomogeneous effects (due to interaction with currents and topography and localized dissipation) can be important (e.g., Smit and Janssen 2013, hereinafter SJ13; Smit et al. 2015a,b, hereinafter SJH15a,b, respectively). Further, because of the transition to weakly dispersive wave motion, triad wave–wave interactions approach resonance, which can result in considerable deviations from Gaussian statistics (e.g., Herbers et al. 2003). For instance, strong nonlinear effects are seen in the development of the characteristic sawtooth wave shapes at the onset of breaking (e.g., Elgar and Guza 1985), resulting in enhanced values of the skewness and asymmetry of the waves. These nonlinear interactions are important since they result in rapid energy transfers away from the spectral peak toward both higher and lower frequencies, where the wave energy either continues as much longer time-scale (infragravity) motions, partially reflected and radiated back into ocean basins (e.g., Aucan and Ardhuin 2013), or is shifted to higher frequencies (harmonics) and dissipated in the breaking process (e.g., Herbers et al. 2000; Smit et al. 2014).

To incorporate statistical inhomogeneity in a stochastic wave model, SJ13 derived a generalized form of the RTE using an approach inspired by methods developed in optics and quantum mechanics (e.g., Wigner 1932; Bremmer 1973; Bastiaans 1979; Cohen 2010). The generalized RTE in SJ13 accounts for the generation and propagation of cross correlations in the wave field, thus allowing the modeling of statistical wave interference associated with strong refraction and diffraction in coastal areas (see SJ13; SJH15a,b). Although the derivation by SJ13 is quite general, and can in principle be used to develop transport equations for any correlator (including higher-order correlators), the model in SJ13 assumes linear wave dynamics, and consequently—while allowing for the development of the heterogeneous statistics—the wave field remains strictly Gaussian.

Statistical models for shallow-water nonlinearity require a transport equation for the bispectrum, which—for arbitrary wave fields and two-dimensional medium variations—is not available and difficult to derive. As a consequence, existing models consider special cases such as the evolution of nonlinear statistics of forward-propagating waves over one-dimensional topography (e.g., Agnon and Sheremet 1997; Herbers and Burton 1997; Eldeberky and Madsen 1999; Herbers et al. 2003; Janssen et al. 2008) or a forward-scattering approximation over weakly two-dimensional topography (Janssen et al. 2008). Such models can provide considerable insight in the nature of shallow-water nonlinearity in statistical models and are useful in studying various aspects of nonlinear wave propagation. However, because of their additional constraints on the spectral bandwidth of the wave field and/or the dimensionality of the medium variations, they are not compatible with operational models based on the RTE. As a consequence, and in part also because of constraints on computational capabilities, shallow-water nonlinearity in operational models has generally at best been crudely parameterized (e.g., Eldeberky 1996; Becq-Girard et al. 1999; Booij et al. 1999). A principal difficulty that has hampered the development of a general transport equation for higher-order correlations (or cross correlations for that matter) that is compatible with the RTE framework is that it is not clear how to identify a conserved property for which to derive a conservation equation (see, e.g., Salmon 1998).

The objective of this work is to incorporate shallow-water nonlinearity in quasi-homogeneous theory by developing a transport equation for the three-wave correlator (or bispectrum), which is both consistent and compatible with the RTE. The focus in this paper is on the development of the general theory, presentation of the main theoretical results, and discussion on how it includes earlier models as special cases. To do this, and to be consistent with the RTE, we assume from the outset that the wave field is quasi homogeneous (thus omitting statistical inhomogeneity; see SJ13). Also, in this paper we do not consider numerical implementation and do not go into various aspects that are important for a numerical evaluation of such a model, such as the closure approximation (e.g., Herbers et al. 2003; Janssen 2006), dissipation (e.g., Smit et al. 2014), or other source terms (e.g., Cavaleri et al. 2007). To keep the discussion focused; these aspects will be considered separately in a following publication. Rather, in what follows, we present the main theoretical development and validate the deterministic and statistical evolution equations by comparing them to known theoretical expressions in appropriate limits. We start in section 2 with the nonlinear deterministic framework (based on Zakharov 1968; Krasitskii 1994), derive statistical moments for the spectrum (RTE with nonlinear forcing term) and bispectrum (biradiative transfer equation) in section 3, discuss various limits and special cases in section 4, and sum up our main findings (section 5).

2. Deterministic evolution of weakly nonlinear waves over topography

We consider surface gravity wave motion through an inviscid and incompressible fluid and describe the flow using the potential flow equations (e.g., Mei et al. 2005) formulated in terms of the free-surface elevation $\eta(\mathbf{x}, t)$ and the velocity potential $\mathbf{\Phi}(\mathbf{x}, z, t)$. Here, t is time, and z is the vertical coordinate. Further, for convenience mean water level is at $z = 0$, the (impermeable) bottom is at $z = -h(\mathbf{x})$, with h the mean water depth, and the location of the free surface is $z = \eta(\mathbf{x}, t)$. To describe the propagation of random waves through a region of uniform depth, it is convenient to consider the problem in wavenumber space. Therefore we define a Fourier transform pair with respect to $\mathbf{x}$ for η and the velocity potential at the free surface $\boldsymbol{\phi}(\mathbf{x}, t) = \boldsymbol{\Phi}(\mathbf{x}, \eta, t)$ as

$$\hat{\eta}(\mathbf{k}, t) = \mathscr{F}^{+}_{\mathbf{x}\to\mathbf{k}}[\eta(\mathbf{x}, t)] \quad \hat{\boldsymbol{\phi}}(\mathbf{k}, t) = \mathscr{F}^{+}_{\mathbf{x}\to\mathbf{k}}[\boldsymbol{\phi}(\mathbf{x}, t)].$$

Further, we denote Fourier transformation with respect to $\mathbf{x}$ as

$$\hat{f}(\mathbf{k}) = \mathscr{F}^{+}_{\mathbf{x}\to\mathbf{k}}[f(\mathbf{x})] = \frac{1}{(2\pi)^2}\int f(\mathbf{x})\exp(-i\mathbf{k}\cdot\mathbf{x})\,d\mathbf{x}, \quad \text{and}$$

$$f(\mathbf{x}) = \mathscr{F}^{-}_{\mathbf{k}\to\mathbf{x}}[\hat{f}(\mathbf{k})] = \int \hat{f}(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{x})\,d\mathbf{k},$$

where (for future reference) we denote composite transforms as

$$\mathscr{F}^{s_1,s_2}_{\mathbf{x}_1,\mathbf{x}_2\to\mathbf{k}_1,\mathbf{k}_2}[f(\mathbf{x}_1,\mathbf{x}_2)] = \mathscr{F}^{s_1}_{\mathbf{x}_1\to\mathbf{k}_1}\left\{\mathscr{F}^{s_2}_{\mathbf{x}_2\to\mathbf{k}_2}[f(\mathbf{x}_1,\mathbf{x}_2)]\right\},$$

where the transforms are assumed to exist in the context of generalized functions (Strichartz 1993).

In case of weakly nonlinear motion with nonlinearity parameter $\delta \ll 1$, with δ defined as the ratio between a characteristic amplitude of the waves and a representative vertical length scale of the motion, the evolution of the wave field to $O(\delta^2)$ is described by (Zakharov 1968; Krasitskii 1994)

$$(\partial_t + i\sigma_{\mathbf{k}})\hat{\zeta}'_{\mathbf{k}} = -i\int d\boldsymbol{v}[\overline{w}^{(1)}_{\mathbf{k},\boldsymbol{v}}\hat{\zeta}'_{\boldsymbol{v}}\hat{\zeta}'_{\mathbf{k}-\boldsymbol{v}} + \overline{w}^{(2)}_{\mathbf{k},\boldsymbol{v}}(\hat{\zeta}'_{\boldsymbol{v}})^*\hat{\zeta}'_{\mathbf{k}+\boldsymbol{v}} + \overline{w}^{(3)}_{\mathbf{k},\boldsymbol{v}}(\hat{\zeta}'_{\boldsymbol{v}}\hat{\zeta}'_{-\mathbf{k}-\boldsymbol{v}})^*]. \tag{2}$$

Here, $i = \sqrt{-1}$, $w^{(j)}_{\mathbf{k},\mathbf{k}'}$ are interaction coefficients (see appendix A), the superscript * denotes the complex conjugate, and $\sigma_{\mathbf{k}}$ is short for the linear dispersion relation $\sigma(|\mathbf{k}|, h) = \sqrt{g|\mathbf{k}|\tanh|\mathbf{k}|h}$ (with g as gravitational acceleration). Moreover, the wave variable $\hat{\zeta}'_{\mathbf{k}}$ is defined from $\hat{\eta}_{\mathbf{k}}$ and $\hat{\phi}_{\mathbf{k}}$ as

$$\hat{\zeta}'_{\mathbf{k}} = \hat{\eta}_{\mathbf{k}} + \frac{i\sigma_{\mathbf{k}}}{g}\hat{\phi}_{\mathbf{k}},$$

where $\hat{\zeta}'_{\mathbf{k}}$ (and similarly for $\hat{\phi}_{\mathbf{k}}$, $\sigma_{\mathbf{k}}$, etc.) is short for $\hat{\zeta}'(\mathbf{k}, t)$ with the dependence on time implied (when applicable). The variable $\hat{\zeta}'_{\mathbf{k}}$ differs from the action variable as used by Zakharov (1968) by a factor of $\sqrt{g/\sigma_{\mathbf{k}}/2}$; in the absence of currents this has no dynamical consequences but simplifies physical interpretation since $\eta = \Re[\zeta'(\mathbf{x})]$ (where $\Re[\ldots]$ denotes the real part of the argument).

To eliminate nonresonant contributions from the outset, we introduce the transformation

$$\hat{\zeta}'_{\mathbf{k}} = \hat{\zeta}_{\mathbf{k}} - \int d\boldsymbol{v}\left[\frac{\overline{w}^{(3)}_{\mathbf{k},\boldsymbol{v}}\hat{\zeta}^*_{\boldsymbol{v}}\hat{\zeta}^*_{-\mathbf{k}-\boldsymbol{v}}}{\sigma_{\mathbf{k}} + \sigma_{\boldsymbol{v}} + \sigma_{-\mathbf{k}-\boldsymbol{v}}} - \frac{\overline{w}^{(3)}_{-\mathbf{k},\boldsymbol{v}}\hat{\zeta}_{\boldsymbol{v}}\hat{\zeta}_{\mathbf{k}-\boldsymbol{v}}}{\sigma_{\mathbf{k}} + \sigma_{\boldsymbol{v}} + \sigma_{\mathbf{k}-\boldsymbol{v}}}\right],$$

so that

$$\hat{\eta}_{\mathbf{k}} = \frac{\hat{\zeta}'_{\mathbf{k}} + (\hat{\zeta}'_{-\mathbf{k}})^*}{2} = \frac{\hat{\zeta}_{\mathbf{k}} + \hat{\zeta}^*_{-\mathbf{k}}}{2},$$

and the governing equation (2), written for the transformed variable, simplifies to

$$(\partial_t + i\sigma_{\mathbf{k}})\hat{\zeta}_{\mathbf{k}} = -i\int d\boldsymbol{v}\,[w^{(1)}_{\mathbf{k},\boldsymbol{v}}\hat{\zeta}_{\boldsymbol{v}}\hat{\zeta}_{\mathbf{k}-\boldsymbol{v}} + w^{(2)}_{\mathbf{k},\boldsymbol{v}}\hat{\zeta}^*_{\boldsymbol{v}}\hat{\zeta}_{\mathbf{k}+\boldsymbol{v}}], \tag{3}$$

with the interaction coefficients $w^{(1)}_{\mathbf{k},\mathbf{k}'}$ given in appendix A.

The transformation we use here is in some ways similar (but not equivalent) to the "canonical transformation" introduced by Zakharov (1968) to eliminate nonresonant bound waves. In our case, we are interested in shallow-water regions where triad interactions approach resonance, and as a consequence the transformation cannot be applied to eliminate the first two terms on the right-hand side of Eq. (2), which can contribute near-resonant interactions (in which case the transformation would be singular). However, the last term on the right of Eq. (2) contributes strictly nonresonant components and can be effectively eliminated using the transformation, which does not affect the wave evolution to the order considered but considerably simplifies the algebraic complexity of the stochastic description of the wave field, as discussed in the following.

Finally, we assume that the medium changes slowly on a scale l_0/ε (with l_0 a characteristic wavelength and $\varepsilon \ll 1$), such that locally the wave field can continue to be considered a superposition of wave packets, where each packet j located at $\mathbf{x}_j$ has a frequency ω_j and wavenumber $\mathbf{k}_j$ that are related through the dispersion relation, $\omega_j = \sigma(\mathbf{k}_j, \mathbf{x}_j)$, which depends on $\mathbf{x}$ [through the depth $h(\mathbf{x})$]. In this case we can make use of an operator correspondence argument between conjugate variables $-i\omega \to \partial_t$, $\mathbf{x}_j \to i\nabla_{\mathbf{k}}$, and $\mathbf{k}_j \to \mathbf{k}$ to relate the local dispersion relation and the local interaction coefficient to an operator as in $\sigma(\mathbf{k}_j, \mathbf{x}_j) \to \Omega(\mathbf{k}, i\nabla_{\mathbf{k}})$ and $w^{(j)}_{\mathbf{k},\mathbf{k}',\mathbf{x}} \to W^{(j)}_{\mathbf{k},\mathbf{k}',i\nabla_{\mathbf{k}}}$, respectively, so that Eq. (2) becomes (see SJ13)

$$(\partial_t + i\Omega_{\mathbf{k},i\nabla_{\mathbf{k}}})\hat{\zeta}_{\mathbf{k}} = -i\int d\boldsymbol{v}[W^{(1)}_{\mathbf{k},\boldsymbol{v},i\nabla_{\mathbf{k}}}\hat{\zeta}_{\boldsymbol{v}}\hat{\zeta}_{\mathbf{k}-\boldsymbol{v}} + W^{(2)}_{\mathbf{k},\boldsymbol{v},i\nabla_{\mathbf{k}}}\hat{\zeta}^*_{\boldsymbol{v}}\hat{\zeta}_{\mathbf{k}+\boldsymbol{v}}]. \tag{4}$$

To ensure the correct second-order dynamics, we define the linear operators Ω and $W^{(j)}$ using the Weyl correspondence rule (see SJ13 and appendix B) as

$$\Omega_{\mathbf{k},i\nabla_\mathbf{k}} \equiv \iint \frac{d\mathbf{p}\,d\mathbf{q}}{(2\pi)^2}[\hat{\sigma}_{\mathbf{p},\mathbf{q}} \exp(-i\mathbf{p}\cdot\mathbf{k} - \mathbf{q}\cdot\nabla_\mathbf{k})], \quad \text{and} \tag{5}$$

$$W^{(j)}_{\mathbf{k},\mathbf{k}',i\nabla_\mathbf{k}} \equiv \iint \frac{d\mathbf{p}\,d\mathbf{q}}{(2\pi)^2}\hat{w}^{(j)}_{\mathbf{p},\mathbf{k}',\mathbf{q}} \exp(-i\mathbf{p}\cdot\mathbf{k} - \mathbf{q}\cdot\nabla_\mathbf{k}), \tag{6}$$

where $\hat{\sigma}_{\mathbf{p},\mathbf{q}} = \mathscr{F}^{-,+}_{\mathbf{k},\mathbf{x}\to\mathbf{p},\mathbf{q}}[\sigma_{\mathbf{k},\mathbf{x}}]$, $\hat{w}^{(j)}_{\mathbf{p},\mathbf{k}',\mathbf{q}} = \mathscr{F}^{-,+}_{\mathbf{k},\mathbf{x}\to\mathbf{p},\mathbf{q}}[w^{(j)}_{\mathbf{k},\mathbf{k}',\mathbf{x}}]$, and $\mathbf{p}$ and $\mathbf{q}$ are the conjugate space and wavenumber coordinates to $\mathbf{k}$ and $\mathbf{x}$, respectively.

In the linear approximation, Eq. (4) reproduces the usual WKB approximation (geometric optics) for waves in a slowly varying medium (see SJ13, their appendix B). In the shallow-water limit, Eq. (4) can be shown to reduce to the Kadomtsev–Petviashvili (KP) equation (Kadomtsev and Petviashvili 1970) for variable depth (Liu et al. 1985; see appendix C). Equation (4), with the operators defined as in Eqs. (5) and (6), is the deterministic starting point of this paper.

3. Transport of statistical moments

To describe the evolution of weakly nonlinear wave statistics through a variable medium, we consider the central moments of the complex wave variable ζ, which is defined as a zero-mean process, such that the lowest-order statistical moment vanishes. The nth-order central moments in spectral and physical space, respectively, are denoted as

$$\hat{\Gamma}(\mathbf{k}_1,\ldots,\mathbf{k}_n,t) = \frac{1}{2}\langle\hat{\zeta}_1\cdots\hat{\zeta}_{n-1}\hat{\zeta}^*_n\rangle$$
$$\Gamma(\mathbf{x}_1,\ldots,\mathbf{x}_n,t) = \frac{1}{2}\langle\zeta_1\cdots\zeta_{n-1}\zeta^*_n\rangle,$$

where $\hat{\zeta}_j = \hat{\zeta}(\mathbf{k}_j, t)$, $\zeta_j = \zeta(\mathbf{x}_j, t)$, $\langle\ldots\rangle$ represents ensemble averaging, and the normalizations factor ½ is introduced to ensure that the second-order autocorrelation $\Gamma(\mathbf{x}, \mathbf{x}, t)$ represents the variance of the free-surface elevation function η, that is, $\Gamma(\mathbf{x}, \mathbf{x}, t) = \langle\eta(\mathbf{x}, t)^2\rangle$. Since in the present work we consider weakly nonlinear waves in shallow water, we consider the transport of the second- and third-order moments.

Because $\hat{\Gamma}$ and Γ form a Fourier transform pair

$$\hat{\Gamma}(\mathbf{k}_1,\ldots,\mathbf{k}_n,t) = \mathscr{F}^{+,\ldots,+}_{\mathbf{x}_1,\ldots,\mathbf{x}_n\to\mathbf{k}_1,\ldots,\mathbf{k}_n}[\Gamma(\mathbf{x}_1,\ldots,\mathbf{x}_n,t)],$$

the assumption of quasi homogeneity implies that the spectral moments are densely populated along the planes $\mathbf{k}_1 + \mathbf{k}_2 + \cdots + \mathbf{k}_{n-1} - \mathbf{k}_n = 0$. Hence, it is convenient to introduce a coordinate transformation from the set of wavenumbers $\mathbf{k}_1, \ldots, \mathbf{k}_n$ to a set of (mean) wavenumbers $\mathbf{k}'_1, \ldots, \mathbf{k}'_{n-1}$ and a difference coordinate $\mathbf{u} = \mathbf{k}_1 + \mathbf{k}_2 + \cdots + \mathbf{k}_{n-1} - \mathbf{k}_n$. Specifically, for the second-order moments, we follow SJ13 and transform the coordinates to a mean $\mathbf{k} = (\mathbf{k}_1 + \mathbf{k}_2)/2$ and difference $\mathbf{u} = \mathbf{k}_1 - \mathbf{k}_2$ wavenumber. The difference wavenumber is then distributed symmetrically over the wavenumbers (i.e., $\mathbf{k}_1 = \mathbf{k} + \mathbf{u}/2$ and $\mathbf{k}_2 = \mathbf{k} - \mathbf{u}/2$), and the correlation function can be written as

$$\hat{\Gamma}_{\mathbf{k},\mathbf{u}} = \hat{\Gamma}(\mathbf{k},\mathbf{u},t) = \frac{1}{2}\langle\hat{\zeta}_{\mathbf{k}+\mathbf{u}/2}\hat{\zeta}^*_{\mathbf{k}-\mathbf{u}/2}\rangle. \tag{7}$$

In case of the third-order statistics, we introduce the coordinate transformation from $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ to $\mathbf{k}, \mathbf{k}', \mathbf{u}$, where

$$\begin{bmatrix}\mathbf{k}_1\\ \mathbf{k}_2\\ \mathbf{k}_3\end{bmatrix} = \mathsf{M}\begin{bmatrix}\mathbf{k}\\ \mathbf{k}'\\ \mathbf{u}\end{bmatrix}, \quad \begin{bmatrix}\mathbf{k}\\ \mathbf{k}'\\ \mathbf{u}\end{bmatrix} = \mathsf{M}^{-1}\begin{bmatrix}\mathbf{k}_1\\ \mathbf{k}_2\\ \mathbf{k}_3\end{bmatrix}, \tag{8}$$

and

$$\mathsf{M} = \begin{bmatrix}\mathsf{I} & \mathsf{0} & \mathsf{I}\alpha_1\\ \mathsf{0} & \mathsf{I} & \mathsf{I}\alpha_2\\ \mathsf{I} & \mathsf{I} & \mathsf{I}\alpha_1 + \mathsf{I}\alpha_2 - \mathsf{I}\end{bmatrix},$$
$$\mathsf{M}^{-1} = \begin{bmatrix}\mathsf{I} - \mathsf{I}\alpha_I & -\mathsf{I}\alpha_1 & \mathsf{I}\alpha_1\\ -\mathsf{I}\alpha_2 & \mathsf{I} - \mathsf{I}\alpha_2 & \mathsf{I}\alpha_2\\ \mathsf{I} & \mathsf{I} & -\mathsf{I}\end{bmatrix}. \tag{9}$$

Here, I and $\mathsf{0}$ denote the 2 by 2 identity and zero matrices, respectively, and α_1 and α_2 are constants that determine how $\mathbf{u}$ is distributed among $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$. Here, we set $\alpha_1 = \alpha_2 = 1/3$ (appendix D), so that

$$\hat{\Gamma}_{\mathbf{k},\mathbf{k}',\mathbf{u}} = \hat{\Gamma}(\mathbf{k},\mathbf{k}',\mathbf{u},t) = \frac{1}{2}\langle\hat{\zeta}_{\mathbf{k}+\mathbf{u}/3}\hat{\zeta}_{\mathbf{k}'+\mathbf{u}/3}\hat{\zeta}^*_{\mathbf{k}+\mathbf{k}'-\mathbf{u}/3}\rangle. \tag{10}$$

For a quasi-homogeneous wave field, $\hat{\Gamma}_{\mathbf{k},\mathbf{u}}$ and $\hat{\Gamma}_{\mathbf{k},\mathbf{k}',\mathbf{u}}$ are narrowly supported along $\mathbf{u} = 0$. Consequently, we can associate with $\mathbf{u}$ a conjugate spatial variable $\mathbf{x}$, which captures the slow spatial variations of the wave statistics and describe the statistics by introducing the intermediate distribution functions

$$\mathscr{E}(\mathbf{k},\mathbf{x},t) = \mathscr{F}^{-}_{\mathbf{u}\to\mathbf{x}}[\Gamma(\mathbf{k},\mathbf{u},t)]$$
$$\mathscr{B}(\mathbf{k},\mathbf{k}',\mathbf{x},t) = \mathscr{F}^{-}_{\mathbf{u}\to\mathbf{x}}[\Gamma(\mathbf{k},\mathbf{k}',\mathbf{u},t)]. \tag{11}$$

In the quasi-homogeneous limit considered here, the distribution function $\mathscr{E}$ represents the wave spectrum and, by virtue of our choice $\alpha_1 = \alpha_2 = 1/3$, the function $\mathscr{B}$ represents the bispectrum (see appendix D), such that the local variance and the skewness of the free surface are found from the marginal distributions

$$\langle\eta^2\rangle = \int \mathscr{E}\,d\mathbf{k}, \quad \langle\eta^3\rangle = \frac{3}{2}\iint \Re(\mathscr{B})\,d\mathbf{k}\,d\mathbf{k}'.$$

a. A nonlinear radiative transfer equation

Here, we rederive the radiative transfer equation for the transport of the spectrum $\mathscr{E}$. To include nonlinear effects and develop a consistent approach that we can then also use to develop a transport for the bispectrum, we start from the deterministic Eq. (4) to develop a transport equation for $\hat{\Gamma}_{\mathbf{k},\mathbf{u}}$. Hereto we multiply Eq. (4) by $\hat{\zeta}_2^*$ multiply the equation for $\hat{\zeta}_2^*$ with $\hat{\zeta}_1$, sum both equations, and ensemble average the result. Upon transforming the resulting equation to $(\mathbf{k}, \mathbf{u})$ space, we obtain

$$(\partial_t + i\Omega_{\hat{\mathbf{D}}_k^+,\hat{\mathbf{D}}_x^+} - i\Omega_{\hat{\mathbf{D}}_k^-,\hat{\mathbf{D}}_x^-})\hat{\Gamma}_{\mathbf{k},\mathbf{u}} = -i\int(\hat{S}_{\mathbf{k},\boldsymbol{\upsilon},\mathbf{u}} + \hat{S}^*_{\mathbf{k},\boldsymbol{\upsilon},-\mathbf{u}})\,d\boldsymbol{\upsilon}, \tag{12}$$

in which $\hat{\mathbf{D}}_k^\pm = \mathbf{k} \pm \mathbf{u}/2$ and $\hat{\mathbf{D}}_x^\pm = (i/2)\nabla_k \pm i\nabla_u$. The nonlinear term is expressed in terms of the three-point correlator $\hat{\Gamma}_{\mathbf{k},\mathbf{k}',\mathbf{u}}$ as

$$\begin{aligned}\hat{S}_{\mathbf{k},\boldsymbol{\upsilon},\mathbf{u}} &= W^{(1)}_{\hat{\mathbf{D}}_k^+,\boldsymbol{\upsilon},\hat{\mathbf{D}}_x^+}\left[\exp\left(-\frac{1}{3}\mathbf{u}\cdot\nabla_\upsilon - \frac{1}{6}\mathbf{u}\cdot\nabla_k\right)\hat{\Gamma}_{\boldsymbol{\upsilon},\mathbf{k}-\boldsymbol{\upsilon},\mathbf{u}}\right]\\ &+ W^{(2)}_{\hat{\mathbf{D}}_k^+,\boldsymbol{\upsilon},\hat{\mathbf{D}}_x^+}\left[\exp\left(\frac{1}{3}\mathbf{u}\cdot\nabla_\upsilon - \frac{1}{6}\mathbf{u}\cdot\nabla_k\right)\hat{\Gamma}^*_{\boldsymbol{\upsilon},\mathbf{k},-\mathbf{u}}\right],\end{aligned}$$

where the terms between $[\ldots]$ correspond to series expansions in $\mathbf{u}$ to the first two arguments of $\hat{\Gamma}_{\boldsymbol{\upsilon}-\mathbf{u}/3,\mathbf{k}+\mathbf{u}/6-\boldsymbol{\upsilon},\mathbf{u}}$ and $\hat{\Gamma}^*_{\boldsymbol{\upsilon}+\mathbf{u}/3,\mathbf{k}-\mathbf{u}/6,-\mathbf{u}}$. We introduce these expansions because it allows us to express the result of the inverse Fourier transform from $\mathbf{u} \to \mathbf{x}$ applied on Eq. (12) directly in terms of $\mathscr{E}$ and $\mathscr{B}$:

$$(\partial_t + i\Omega_{\mathbf{D}_k^+,\mathbf{D}_x^+} - i\Omega_{\mathbf{D}_k^-,\mathbf{D}_x^-})\mathscr{E} = -2\int \Re(iS_{\mathbf{k},\boldsymbol{\upsilon},\mathbf{x}})\,d\boldsymbol{\upsilon}, \tag{13}$$

where $\mathbf{D}_k^\pm = \mathbf{k} \mp (i/2)\nabla_\mathbf{x}$, $\mathbf{D}_x^\pm = \mathbf{x} \pm (i/2)\nabla_\mathbf{k}$, and

$$\begin{aligned}S_{\mathbf{k},\boldsymbol{\upsilon},\mathbf{x}} &= W^{(1)}_{\mathbf{D}_k^+,\boldsymbol{\upsilon},\mathbf{D}_x^+}\left[\exp\left(\frac{i}{3}\nabla_x\cdot\nabla_\upsilon + \frac{i}{6}\nabla_x\cdot\nabla_k\right)\mathscr{B}_{\boldsymbol{\upsilon},\mathbf{k}-\boldsymbol{\upsilon}}\right]\\ &+ W^{(2)}_{\mathbf{D}_k^+,\boldsymbol{\upsilon},\mathbf{D}_x^+}\left[\exp\left(-\frac{i}{3}\nabla_x\cdot\nabla_\upsilon + \frac{i}{6}\nabla_x\cdot\nabla_k\right)\mathscr{B}^*_{\boldsymbol{\upsilon},\mathbf{k}}\right].\end{aligned}$$

Equation (13) transports the complete second-order statistics and includes both variance contributions and off-diagonal contributions to the covariance matrix (see SJ13). To derive a consistent approximation for the quasi-homogeneous limit, we assume that the statistics undergo $O(1)$ changes on a length scale of l_0/μ (with $\mu \ll 1$) and introduce multiple scales, with the spatial scales $\mathbf{X}_m$ and $\mathbf{X}$, and a spectral scale $\mathbf{K}$ defined as

$$\mathbf{X}_m = \varepsilon\mathbf{x}, \quad \mathbf{X} = \mu\mathbf{x} \quad \text{and} \quad \mathbf{K} = \kappa^{-1}\mathbf{k}, \tag{14}$$

where $\kappa = \Delta k/k_0$, with Δk representing a characteristic width of the spectrum and k_0 representing a characteristic wavenumber. The spatial scales $\mathbf{X}_m$ and $\mathbf{X}$ thus capture the slow variation of the medium and the wave statistics, respectively, whereas the spectral scale $\mathbf{K}$ is associated with the coherence length scale, such that the limit $\kappa \ll 1$ represents a coherent (narrowband) wave field, whereas for moderate or wideband wave fields $\kappa = O(1)$. Finally, we introduce the time scales

$$T_j = \mu^j t; \quad j = 1, 2, \ldots, \tag{15}$$

and based on the assumed quasi homogeneity and stationarity of the statistics, we let

$$\mathscr{E} = \mathscr{E}(\mathbf{K}, \mathbf{X}, T_1, T_2, \ldots), \quad \mathscr{B} = \mathscr{B}(\mathbf{K}, \mathbf{K}', \mathbf{X}, T_1, T_2, \ldots),$$

and

$$\sigma(\mathbf{k}, \mathbf{X}_m), \quad w(\mathbf{k}, \mathbf{k}', \mathbf{X}).$$

With these definitions in place, we use the infinite differential form of the operators [appendix B; Eqs. (B1) and (B2)] to express Eq. (13) in terms of the scaled variables:

$$\sum_j \mu^j\partial_{T_j}\mathscr{E} + \left[i\sigma\exp\left(\pm\beta\frac{i}{2}\overleftarrow{\nabla}_{X_m}\cdot\overrightarrow{\nabla}_K \mp \mu\frac{i}{2}\overleftarrow{\nabla}_k\cdot\overrightarrow{\nabla}_X\right)\mathscr{E} + *\right] = -2\delta\int d\boldsymbol{\upsilon}\,\Re(iS_{\mathbf{k},\mathbf{K},\boldsymbol{\upsilon},\kappa^{-1}\boldsymbol{\upsilon},\mathbf{X},\mathbf{X}_m}), \tag{16}$$

Here, $\beta = \varepsilon/\kappa$, and the arrows on the operators indicate whether they act on σ (left arrows) or on $\mathscr{E}$ (right arrows). Further, we introduced a factor δ to the nonlinear term on the right-hand side (and absorbed a factor $1/\delta$ in S) to make explicit that the nonlinear term is of order $O(\delta)$ compared with the linear contributions, so that

$$\begin{aligned}\delta S_{\mathbf{k},\mathbf{K},\boldsymbol{\upsilon},\mathbf{V},\mathbf{X},\mathbf{X}_m} &= w^{(1)}_{\mathbf{k},\boldsymbol{\upsilon},\mathbf{X}_m}\exp\left[i\left(\frac{\gamma}{3}\overrightarrow{\nabla}_V - \frac{\mu}{2}\overleftarrow{\nabla}_k + \frac{\gamma}{6}\overrightarrow{\nabla}_K\right)\cdot\overrightarrow{\nabla}_x + \frac{i\beta}{2}\overleftarrow{\nabla}_{X_m}\cdot\overrightarrow{\nabla}_K\right]\mathscr{B}_{\mathbf{V},\mathbf{K}-\mathbf{V}}\\ &+ w^{(2)}_{\mathbf{k},\boldsymbol{\upsilon},\mathbf{X}_m}\exp\left[i\left(-\frac{\gamma}{3}\overrightarrow{\nabla}_V - \frac{\mu}{2}\overleftarrow{\nabla}_k + \frac{\gamma}{6}\overrightarrow{\nabla}_K\right)\cdot\overrightarrow{\nabla}_X + \frac{i\beta}{2}\overleftarrow{\nabla}_{X_m}\cdot\overrightarrow{\nabla}_K\right]\mathscr{B}^*_{\mathbf{V},\mathbf{K}},\end{aligned}$$

where $\gamma = \mu/\kappa$ and $\mathbf{V} = \kappa^{-1}\boldsymbol{v}$. The appearance of the parameters β and γ expresses that in a stochastic model the decorrelation length scale—or the memory of the wave field—is an additional spatial scale that needs to be considered. Physically, β measures the decorrelation length scale of the waves relative to the medium scales, so that for $\beta \ll 1$ the wave field has a short memory compared with the scales of the medium, and regions separated by $O(1)$ medium variations are statistically independent. Further, γ essentially is a measure comparing the variability of the mean statistics with the decorrelation length scale, so that for $\gamma \ll 1$, regions separated by $O(1)$ changes in the wave statistics are also statistically independent. In the quasi-homogeneous limit, it is assumed (often implicitly) that both γ and β are small parameters. In the present approximation, we assume that $O(\delta) = O(\varepsilon) = O(\mu)$, and assuming quasi homogeneity, we have $O(\kappa) = O(1)$, so that $O(\gamma) = O(\beta) = O(\mu)$. Moreover, we further assume that and expand the exponential operators that occur in Eq. (16) using the Taylor series, so that if we retain terms up to $O(\mu)$ and drop the scaling, the left-hand side of Eq. (16) reduces to the well-known RTE:

$$\partial_t \mathscr{E} + \mathbf{c}^x \cdot \nabla_x \mathscr{E} + \mathbf{c}^k \cdot \nabla_k \mathscr{E} = S_{\mathrm{nl3}}, \tag{17}$$

with $\mathbf{c}^x = \nabla_k \sigma$ and $\mathbf{c}^k = -\nabla_x \sigma$ and where S_{nl3} denotes the source term for the nonlinear triad (nl3) interactions. In this quasi-homogeneous limit the spectrum $\mathscr{E}$ is a positive function and can thus unambiguously be interpreted as a variance density function (see SJ13). Moreover, on the right-hand side we only need to retain the leading-order approximation, so that the nonlinear term reduces to

$$S_{\mathrm{nl3}} = 2\Im\left\{ \int [w^{(1)}_{\mathbf{k},\boldsymbol{v},\mathbf{x}} \mathscr{B}_{\boldsymbol{v},\mathbf{k}-\boldsymbol{v}} - w^{(2)}_{\mathbf{k},\boldsymbol{v},\mathbf{x}} \mathscr{B}_{\boldsymbol{v},\mathbf{k}}]\, d\boldsymbol{v} \right\}, \tag{18}$$

where $\Im\{\ldots\}$ denotes the imaginary part of the argument and depends—through the imaginary part of the bispectrum—on the higher-order statistics of the wave field. Arguably, Eq. (17), which is the principal result of this section, could have been motivated by an energy conservation argument. However, by developing a formalism based directly on the equations of motions, we can extend the same reasoning to higher-order correlators for which no a priori conservation principles are available.

b. A biradiative transfer equation for the bispectrum

Through the forcing term on right-hand side of Eq. (17), the evolution of the spectrum $\mathscr{E}$ requires knowledge of the third-order statistics as described by the bispectrum $\mathscr{B}$. This thus requires a transport equation for the bispectrum that is fully compatible with the radiative transport Eq. (17) in the sense that it does not pose any restrictions on spectral bandwidth and medium variations that are more stringent than the quasi-homogeneous limit implied by Eq. (17). To our knowledge, such a transport equation for $\mathscr{B}$ is not available and cannot be deduced from the outset through a conservation principle.

In what follows, we derive a transport equation for $\mathscr{B}$ following the same generalized approach we used for the nonlinear radiative transfer equation in the previous section. From the wave equation (4), an evolution equation for $\langle \hat{\zeta}_1 \hat{\zeta}_2 \hat{\zeta}_3^* \rangle$ is obtained as follows: multiply Eq. (4) for $\hat{\zeta}_1$ with $\hat{\zeta}_2 \hat{\zeta}_3^*$; multiply Eq. (4) for $\hat{\zeta}_2$ with $\hat{\zeta}_1 \hat{\zeta}_3^*$; multiply Eq. (4) for $\hat{\zeta}_3^*$ with $\hat{\zeta}_1 \hat{\zeta}_2$; and after summing the relations and ensemble averaging the result, we introduce the coordinate transformation equation (8) so that the result can then be expressed in terms of $\hat{\Gamma}_{\mathbf{k},\mathbf{k}',\mathbf{u}}$ as

$$(\partial_t + i\Omega_{\hat{D}^1_k,\hat{D}^1_x} + i\Omega_{\hat{D}^2_k,\hat{D}^2_x} - i\Omega^*_{\hat{D}^3_k,\hat{D}^3_x})\hat{\Gamma}_{\mathbf{k},\mathbf{k}',\mathbf{u}} = -i\hat{Q}_{\mathbf{k},\mathbf{k}'} - i\hat{\mathscr{C}}, \tag{19}$$

where

$$\begin{bmatrix} \hat{\mathscr{D}}^1_k \\ \hat{\mathscr{D}}^2_k \\ \hat{\mathscr{D}}^3_k \end{bmatrix} \equiv \mathbf{M} \begin{bmatrix} \mathbf{k} \\ \mathbf{k}' \\ \mathbf{u} \end{bmatrix}, \quad \begin{bmatrix} \hat{\mathscr{D}}^1_x \\ \hat{\mathscr{D}}^2_x \\ \hat{\mathscr{D}}^3_x \end{bmatrix} \equiv i(\mathbf{M}^{-1})^{\mathrm{T}} \begin{bmatrix} \nabla_{\mathbf{k}} \\ \nabla_{\mathbf{k}'} \\ \nabla_{\mathbf{u}} \end{bmatrix}.$$

The right-hand side of Eq. (19) depends on the fourth-order moments of the wave variable, which by decomposition of fourth-order moments can be expressed into the sum of products of lower-order moments $\hat{Q}_{\mathbf{k},\mathbf{k}'}$ and the fourth cumulant $\hat{\mathscr{C}}$ (see, e.g., Herbers and Burton 1997). The $\hat{Q}_{\mathbf{k},\mathbf{k}'}$ are

$$\hat{Q}_{\mathbf{k},\mathbf{k}'} = 2\int d\boldsymbol{v}\, [W^{(2)}_{\hat{\mathscr{D}}^1_k,\boldsymbol{v},\hat{\mathscr{D}}^1_x} \hat{R}_{\hat{\mathscr{D}}^2_k,\hat{\mathscr{D}}^1_k+\boldsymbol{v},\boldsymbol{v},\hat{\mathscr{D}}^3_k} + W^{(2)}_{\hat{\mathscr{D}}^2_k,\boldsymbol{v},\hat{\mathscr{D}}^2_x} \hat{R}_{\hat{\mathscr{D}}^1_k,\hat{\mathscr{D}}^2_k+\boldsymbol{v},\boldsymbol{v},\hat{\mathscr{D}}^3_k} - W^{(1)*}_{\hat{\mathscr{D}}^3_k,\boldsymbol{v},\hat{\mathscr{D}}^3_x} \hat{R}_{\hat{\mathscr{D}}^1_k,\hat{\mathscr{D}}^2_k,\boldsymbol{v},\hat{\mathscr{D}}^3_k-\boldsymbol{v}}],$$

where

$$\hat{R}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} = \hat{\Gamma}_{\mathbf{k}_{13},\mathbf{u}_{13}} \hat{\Gamma}_{\mathbf{k}_{24},\mathbf{u}_{24}} + \hat{\Gamma}_{\mathbf{k}_{14},\mathbf{u}_{14}} \hat{\Gamma}_{\mathbf{k}_{23},\mathbf{u}_{23}},$$

with $\mathbf{k}_{12} = (\mathbf{k}_1 + \mathbf{k}_2)/2$ and $\mathbf{u}_{12} = \mathbf{k}_1 - \mathbf{k}_2$.

To transform Eq. (19) to physical space, we apply a Fourier transform with respect to the difference wavenumber $\mathbf{u}$.

Consistent with the quasi-homogeneous limit (see SJ13), we assume that the second-order moments $\Gamma_{\mathbf{k},\mathbf{u}}$ are narrowly supported around $\mathbf{u} = 0$, so that significant contributions to $\Gamma_{\mathbf{k},\mathbf{u}}$ occur only where $\mathbf{u} \ll \mathbf{k}$, so that

$$\hat{Q}_{\mathbf{k},\mathbf{k}'} = 2\int d\boldsymbol{v}\,\{W^{(2)}_{\mathscr{D}^1_k,\mathbf{k}',\mathscr{D}^1_x}\hat{\Gamma}_{\mathbf{k}+\mathbf{k}',\boldsymbol{v}}\hat{\Gamma}_{\mathbf{k}',\mathbf{u}-\boldsymbol{v}} + W^{(2)}_{\mathscr{D}^2_k,\mathbf{k},\mathscr{D}^2_x}\hat{\Gamma}_{\mathbf{k}+\mathbf{k}',\boldsymbol{v}}\hat{\Gamma}_{\mathbf{k},\mathbf{u}-\boldsymbol{v}} - [W^{(1)*}_{\mathscr{D}^3_k,\mathbf{k},\mathscr{D}^3_x} + W^{(1)*}_{\mathscr{D}^3_k,\mathbf{k}',\mathscr{D}^3_x}]\hat{\Gamma}_{\mathbf{k},\boldsymbol{v}}\hat{\Gamma}_{\mathbf{k}',\mathbf{u}-\boldsymbol{v}}\} + O(\delta\mu). \quad (20)$$

Upon substituting Eq. (20) into Eq. (19), ignoring $O(\delta\mu)$ and higher-order contributions, and applying the Fourier transform from $\mathbf{u}$ to $\mathbf{x}$, we obtain

$$(\partial_t + i\Omega_{\mathscr{D}^1_k,\mathscr{D}^1_x} + i\Omega_{\mathscr{D}^2_k,\mathscr{D}^2_x} - i\Omega^*_{\mathscr{D}^3_k,\mathscr{D}^3_x})\mathscr{B} = -iQ_{\mathbf{k},\mathbf{k}'} - i\mathscr{C}, \quad (21)$$

where

$$\begin{bmatrix}\mathscr{D}^1_k\\ \mathscr{D}^2_k\\ \mathscr{D}^3_k\end{bmatrix} \equiv \mathbf{M}\begin{bmatrix}\mathbf{k}\\ \mathbf{k}'\\ -i\nabla_x\end{bmatrix}, \quad \begin{bmatrix}\mathscr{D}^1_x\\ \mathscr{D}^2_x\\ \mathscr{D}^3_x\end{bmatrix} \equiv (\mathbf{M}^{-1})^{\mathrm{T}}\begin{bmatrix}i\nabla_{\mathbf{k}}\\ i\nabla_{\mathbf{k}'}\\ \mathbf{x}\end{bmatrix},$$

and where the reducible part of the nonlinear forcing term can be expressed in terms of products of $\mathscr{E}$ as in

$$Q_{\mathbf{k},\mathbf{k}'} = 2W^{(2)}_{\mathscr{D}^1_k,\mathbf{k}',\mathscr{D}^1_x}\mathscr{E}_{\mathbf{k}+\mathbf{k}'}\mathscr{E}_{\mathbf{k}'} + 2W^{(2)}_{\mathscr{D}^2_k,\mathbf{k},\mathscr{D}^2_x}\mathscr{E}_{\mathbf{k}+\mathbf{k}'}\mathscr{E}_{\mathbf{k}} - 2[W^{(1)*}_{\mathscr{D}^3_k,\mathbf{k},\mathscr{D}^3_x} + W^{(1)*}_{\mathscr{D}^3_k,\mathbf{k}',\mathscr{D}^3_x}]\mathscr{E}_{\mathbf{k}'}\mathscr{E}_{\mathbf{k}}.$$

At this point, the sets consisting of Eqs. (17) and (22) are entirely expressed in terms of $\mathscr{E}$ and $\mathscr{B}$ and a fourth-order cumulant contribution. To reduce the operators $\Omega_{\mathscr{D}^j_k,\mathscr{D}^j_x}$ on the left-hand side of Eq. (22) to a differential form, we apply the same spatial, wavenumber, and temporal scales $\mathbf{X}$, $\mathbf{X}_m$, $\mathbf{K}$, $\mathbf{k}'$, and T_j as in Eqs. (14) and (15), so that Eq. (22) can be expressed as (see appendix B)

$$\sum_j \mu^j\partial_{T_j}\mathscr{B} + i[\sigma_{\mathbf{K},\mathbf{X}_m}\exp(iF_{2,-1}) + \sigma_{\mathbf{K}',\mathbf{X}_m}\exp(iF_{-1,2}) - \sigma_{\mathbf{K}+\mathbf{K}',\mathbf{X}_m}\exp(-iF_{1,1})]\mathscr{B} = -\delta iQ - \delta i\mathscr{C}, \quad (22)$$

where F_{a_1,a_2} is short for

$$F_{a_1,a_2} = \left(\frac{\varepsilon}{6}\overleftarrow{\nabla}_{\mathbf{k}} + \frac{a_1\beta}{3}\overrightarrow{\nabla}_{\mathbf{K}} + \frac{a_2\beta}{3}\overrightarrow{\nabla}_{\mathbf{K}'}\right)\cdot\overleftarrow{\nabla}_{\mathbf{X}_m} - \frac{\mu}{3}\overleftarrow{\nabla}_{\mathbf{k}}\cdot\overrightarrow{\nabla}_{\mathbf{X}}.$$

We absorb a factor $1/\delta$ in the nonlinear terms Q and $\mathscr{C}$ to make the ordering explicit, so that

$$\delta Q = 2w^{(2)}_{\mathbf{K},\mathbf{K}',\mathbf{X}_m}\exp(iF_{2,-1})\mathscr{E}_{\mathbf{K}+\mathbf{K}'}\mathscr{E}_{\mathbf{K}'} + 2w^{(2)}_{\mathbf{K}',\mathbf{K},\mathbf{X}_m}\exp(iF_{-1,2})\mathscr{E}_{\mathbf{K}+\mathbf{K}'}\mathscr{E}_{\mathbf{K}} - 2[w^{(1)}_{\mathbf{K}'+\mathbf{K},\mathbf{K},\mathbf{X}_m} + w^{(1)}_{\mathbf{K}'+\mathbf{K},\mathbf{K}',\mathbf{X}_m}]\exp(-iF_{1,1})\mathscr{E}_{\mathbf{K}'}\mathscr{E}_{\mathbf{K}}.$$

Finally, to develop an approximation for the bispectral transport equation that is fully consistent with the radiative transfer equation [Eq. (17)], we set $O(\mu) = O(\beta) = O(\delta)$ and expand the exponential operators in a series expansion, while retaining terms up to $O(\mu)$, so that, after dropping the scaling of the coordinates, we obtain a transport equation for the bispectrum $\mathscr{B}$, which can be written as

$$(\partial_t + \overline{\mathbf{c}}^x_{\mathbf{k},\mathbf{k}'}\cdot\nabla_{\mathbf{x}} + \overline{\mathbf{c}}^k_{\mathbf{k},\mathbf{k}'}\cdot\nabla_{\mathbf{k}} + \overline{\mathbf{c}}^{k'}_{\mathbf{k},\mathbf{k}'}\cdot\nabla_{\mathbf{k}'})\mathscr{B} = (\overline{\mathbf{c}}^{kx}_{\mathbf{k},\mathbf{k}'} - i\Delta\sigma_{\mathbf{k},\mathbf{k}'})\mathscr{B} - iQ_{\mathbf{k},\mathbf{k}'} - i\mathscr{C}_{\mathbf{k},\mathbf{k}'}, \quad (23)$$

with

$$Q_{\mathbf{k},\mathbf{k}'} = 2w^{(2)}_{\mathbf{k},\mathbf{k}'}\mathscr{E}_{\mathbf{k}+\mathbf{k}'}\mathscr{E}_{\mathbf{k}'} + 2w^{(2)}_{\mathbf{k},\mathbf{k}'}\mathscr{E}_{\mathbf{k}+\mathbf{k}'}\mathscr{E}_{\mathbf{k}} - 2[w^{(1)}_{\mathbf{k}+\mathbf{k}',\mathbf{k}} + w^{(1)}_{\mathbf{k}+\mathbf{k}',\mathbf{k}'}]\mathscr{E}_{\mathbf{k}'}\mathscr{E}_{\mathbf{k}}. \quad (24)$$

In Eq. (23), the transport velocities through geographic and spectral space are

$$\overline{\mathbf{c}}^x_{\mathbf{k},\mathbf{k}'} = \frac{\mathbf{c}^x_{\mathbf{k}} + \mathbf{c}^x_{\mathbf{k}'} + \mathbf{c}^x_{\mathbf{k}+\mathbf{k}'}}{3}, \quad \overline{\mathbf{c}}^k_{\mathbf{k},\mathbf{k}'} = \mathbf{c}^k_{\mathbf{k}} - \frac{\Delta c^k_{\mathbf{k},\mathbf{k}'}}{3},$$

$$\overline{\mathbf{c}}^{k'}_{\mathbf{k},\mathbf{k}'} = \mathbf{c}^k_{\mathbf{k}'} - \frac{\Delta c^k_{\mathbf{k},\mathbf{k}'}}{3},$$

with $\Delta c^k_{\mathbf{k},\mathbf{k}'} = \mathbf{c}^k_{\mathbf{k}} + \mathbf{c}^k_{\mathbf{k}'} - \mathbf{c}^k_{\mathbf{k}+\mathbf{k}'}$. Further, on the right-hand side of Eq. (23) $\Delta\sigma_{\mathbf{k},\mathbf{k}'}$ and $\overline{\mathbf{c}}^{kx}_{\mathbf{k},\mathbf{k}'}$ are given as

$$\Delta\sigma_{\mathbf{k},\mathbf{k}'} = \sigma_{\mathbf{k}} + \sigma_{\mathbf{k}'} - \sigma_{\mathbf{k}+\mathbf{k}'} \quad \overline{\mathbf{c}}^{kx}_{\mathbf{k},\mathbf{k}'} = \frac{1}{2}\nabla_x\overline{\mathbf{c}}^x_{\mathbf{k},\mathbf{k}'},$$

which represent the frequency mismatch across the triad and a bispectral shoaling factor, respectively. The transport equation [Eq. (23)] has some similarity to the RTE, although the spectral dimensionality is increased from two to four spectral dimensions. We will refer to this transport equation as the biradiative transfer equation (bRTE). Analogous to the RTE, the left-hand side of the bRTE describes the transport of the bispectrum through geographic and spectral space. The forcing terms on the right-hand side of the bRTE [Eq. (23)], going from left to right, are associated with the variable medium (shoaling), frequency mismatch (dispersion), and wave–wave interactions (nonlinearity), respectively.

4. Discussion

The coupled equations [Eqs. (17) and (23)] are the principal results of this paper, summarized here as

$$\partial_t \mathscr{E} + \mathbf{c}^x \cdot \nabla_x \mathscr{E} + \mathbf{c}^k \cdot \nabla_k \mathscr{E} = S_{\mathrm{nl3}} + S^{\mathscr{E}}, \tag{25}$$

and

$$(\partial_t + \overline{\mathbf{c}}^x_{\mathbf{k},\mathbf{k}'} \cdot \nabla_{\mathbf{x}} + \overline{\mathbf{c}}^k_{\mathbf{k},\mathbf{k}'} \cdot \nabla_{\mathbf{k}} + \overline{\mathbf{c}}^k_{\mathbf{k},\mathbf{k}'} \cdot \nabla_{\mathbf{k}'})\mathscr{B} = (\overline{\mathbf{c}}^{kx}_{\mathbf{k},\mathbf{k}'} - i\Delta\sigma_{\mathbf{k},\mathbf{k}'})\mathscr{B} - iQ_{\mathbf{k},\mathbf{k}'} - i\mathscr{C} + S^{\mathscr{B}}, \tag{26}$$

where we have tentatively included the source terms $S^{\mathscr{E}}$ and $S^{\mathscr{B}}$, which represent other physical processes (wave breaking, generation by wind, etc.) that were not considered here. Although the RTE [Eq. (25)] is a well-known result, the transport equation for the bispectrum [Eq. (26)] is—to our knowledge—an entirely new result and allows the fully two-dimensional transportation of the bispectrum, while being consistent with the assumptions underlying the RTE (quasi-homogeneous theory) and without introducing additional constraints on bandwidth, aperture, or medium variability.

The set of Eqs. (25) and (26) is not formally closed, and to numerically evaluate the evolution of the statistics we would need to introduce some sort of closure approximation for the fourth cumulant contribution $\mathscr{C}$ (see, e.g., Herbers and Burton 1997; Kofoed-Hansen and Rasmussen 1998; Eldeberky and Madsen 1999; Herbers et al. 2003; Janssen 2006). Since in shallow water no asymptotic closure exists, and a quasi-normal approximation generally results in statistics that are too far from Gaussian (e.g., Herbers et al. 2003), some empirical closure approximation to represent the nonlinear background wave field and that acts as a damping term to allow a relaxation toward Gaussian statistics needs to be introduced (e.g., Janssen 2006). Since we are not pursuing a direct numerical evaluation of the coupled set and do not explicitly model dissipation processes (which are important for the closure approximation), we will not pursue this further here (see, e.g., Janssen 2006, for a more comprehensive discussion). In a following publication where we will discuss a full numerical implementation of the coupled equations, including other source terms for nonconservative processes and comparison with observations, this will be addressed in more detail.

a. Deep-water asymptote: Forced bound waves

In deep water and most intermediate-depth regions, triad interactions are off resonant and introduce relatively small, bound wave corrections to the primary wave field. For constant-depth conditions, such bound waves are in quadrature with the forcing primary waves, so that no energy transfers occur, and the leading-order wave field is approximately Gaussian, so that the cumulant contribution can be neglected. The associated bispectrum can then be directly related to the variance spectrum of the primary waves using second-order, constant-depth wave theory (Hasselmann et al. 1963). To show that Eq. (26) correctly reproduces this asymptote, we first consider the second-order perturbation solution of our underlying deterministic evolution equation [Eq. (3)] and assume that for a constant-depth h the wave variable $\hat{\zeta}$ can be expanded as $\hat{\zeta}(\mathbf{k}, t) = {}_1\hat{\zeta}(\mathbf{k}, t)\delta + {}_2\hat{\zeta}(\mathbf{k}, t)\delta^2 + O(\delta^3)$, where ${}_n\hat{\zeta}(\mathbf{k}, t)$ denotes the nth-order solution. When substituted into Eq. (4), collecting terms of equal order, and solving to second order, we have

$$\hat{\zeta}(\mathbf{k}, t) = a_{\mathbf{k}} E^+_{\mathbf{k}} + \int d\boldsymbol{v} \left[\frac{w^{(1)}_{\mathbf{k},\boldsymbol{v}} a_{\boldsymbol{v}} a_{\mathbf{k}-\boldsymbol{v}}}{\Delta\sigma^-_{\mathbf{k}-\boldsymbol{v},\boldsymbol{v}}} E^+_{\boldsymbol{v}} E^+_{\mathbf{k}-\boldsymbol{v}} - \frac{w^{(2)}_{\mathbf{k},\boldsymbol{v}} a^*_{\boldsymbol{v}} a_{\mathbf{k}+\boldsymbol{v}}}{\Delta\sigma^-_{\mathbf{k},\boldsymbol{v}}} E^-_{\boldsymbol{v}} E^+_{\mathbf{k}+\boldsymbol{v}} \right], \tag{27}$$

where $a_{\mathbf{k}}$ are the complex amplitudes of the lowest-order wave field and where we introduced $\Delta\sigma^{\pm}_{\mathbf{k}_1,\mathbf{k}_2} = \sigma_{\mathbf{k}_1} + \sigma_{\mathbf{k}_1} \pm \sigma_{\mathbf{k}_1+\mathbf{k}_2}$ and $E^{\pm}_{\mathbf{k}} = \exp(\mp i\sigma_{\mathbf{k}} t)$. Consequently, the second-order solution for the observable free-surface variable $\hat{\eta}_{\mathbf{k}} = (\hat{\zeta}_{\mathbf{k}} + \hat{\zeta}^*_{-\mathbf{k}})/2$ reads

$$\hat{\eta}(\mathbf{k}, t) = \frac{a_{\mathbf{k}} E^+_{\mathbf{k}}}{2} + \frac{a^*_{-\mathbf{k}} E^-_{-\mathbf{k}}}{2} + \frac{1}{2} \int d\boldsymbol{v}\, (A_{\mathbf{k},\boldsymbol{v}} a_{\boldsymbol{v}} a_{\mathbf{k}-\boldsymbol{v}} E^+_{\boldsymbol{v}} E^+_{\mathbf{k}-\boldsymbol{v}} + B_{\mathbf{k},\boldsymbol{v}} a^*_{\boldsymbol{v}} a_{\mathbf{k}+\boldsymbol{v}} E^-_{\boldsymbol{v}} E^+_{\mathbf{k}+\boldsymbol{v}} + B_{-\mathbf{k},\boldsymbol{v}} a_{\boldsymbol{v}} a^*_{-\mathbf{k}+\boldsymbol{v}} E^+_{\boldsymbol{v}} E^-_{-\mathbf{k}+\boldsymbol{v}} + A_{-\mathbf{k},\boldsymbol{v}} a^*_{\boldsymbol{v}} a^*_{-\mathbf{k}-\boldsymbol{v}} E^-_{\boldsymbol{v}} E^-_{-\mathbf{k}-\boldsymbol{v}}), \tag{28}$$

with

$$A_{\mathbf{k},\boldsymbol{v}} = \frac{\overline{w}^{(1)}_{\mathbf{k},\boldsymbol{v}}}{\Delta\sigma^-_{\mathbf{k}-\boldsymbol{v},\boldsymbol{v}}} - \frac{\overline{w}^{(3)}_{-\mathbf{k},\boldsymbol{v}}}{\Delta\sigma^+_{-\mathbf{k},\boldsymbol{v}}} \quad B_{\mathbf{k},\boldsymbol{v}} = -\frac{\overline{w}^{(2)}_{\mathbf{k},\boldsymbol{v}-\mathbf{k}}}{2\Delta\sigma^-_{\mathbf{k},\boldsymbol{v}-\mathbf{k}}} - \frac{\overline{w}^{(2)}_{-\mathbf{k},\boldsymbol{v}}}{2\Delta\sigma^-_{-\mathbf{k},\boldsymbol{v}}}.$$

Equation (28) agrees with earlier results by other authors (e.g., Hasselmann 1962; and many others). This confirms that our underlying deterministic model [Eq. (3)] is consistent with second-order, constant-depth wave theory. To obtain the associated bispectrum, we substitute Eq. (27) in the three-point correlation function $\Gamma_{\mathbf{k},\mathbf{k}',\mathbf{u}}$, apply the transformation $\mathscr{B} = \mathscr{F}^{-s}_{\mathbf{u}\to\mathbf{x}}(\Gamma_{\mathbf{k},\mathbf{k}',\mathbf{u}})$, and, upon assuming a Gaussian and homogeneous wave field [so that $\langle a_{\mathbf{k}_1} a_{\mathbf{k}_2}\rangle = \delta(\mathbf{k}_1 - \mathbf{k}_2)\mathscr{E}(\mathbf{k}_1)$], the bispectrum can be written as

$$\mathscr{B} = \frac{-2}{\Delta\sigma^{-}_{\mathbf{k},\mathbf{k}'}}\{w^{(2)}_{\mathbf{k},\mathbf{k}'}\mathscr{E}_{\mathbf{k}+\mathbf{k}'}\mathscr{E}_{\mathbf{k}'} + w^{(2)}_{\mathbf{k}',\mathbf{k}}\mathscr{E}_{\mathbf{k}+\mathbf{k}'}\mathscr{E}_{\mathbf{k}} - [w^{(1)}_{\mathbf{k}+\mathbf{k}',\mathbf{k}} + w^{(1)}_{\mathbf{k}+\mathbf{k}',\mathbf{k}'}]\mathscr{E}_{\mathbf{k}'}\mathscr{E}_{\mathbf{k}}\}.$$

This expression, which is in agreement with, for example, Hasselmann et al. (1963), corresponds exactly with the homogeneous ($\nabla_x\mathscr{B} = 0$), steady-state ($\partial_t\mathscr{B} = 0$) solution of Eq. (26), namely,

$$\mathscr{B} = -Q_{\mathbf{k},\mathbf{k}'}/\Delta\sigma_{\mathbf{k},\mathbf{k}'},$$

which implies that Eq. (26) correctly reproduces the bound wave limit.

b. *Shallow-water asymptote: Near-resonant interactions*

Conventionally, statistical models for nonlinear wave propagation in shallow water are formulated in the frequency domain, with directionality usually included through some form of forward-scattering approximation, assuming either weakly two-dimensional medium variations or limiting the directional wave aperture. The nonlinear statistical model derived in this paper [Eqs. (17) and (25)] is more general and consistent with the quasi-homogeneous assumptions implied by the radiative transfer equations, without restricting the bandwidth of the wave field or the dimensionality of the slowly varying medium. To show that our result still includes the more restricted, but extensively validated, shallow-water models as a special case, we show that the shallow-water asymptote of the present model for unidirectional wave propagation is identical to the equations derived by Herbers and Burton (1997). Thereto we start by setting $\lambda = k_0h_0 \ll 1$ (with k_0 and h_0 as a characteristic wavenumber and water depth, respectively), so that the dispersion relation may be approximated up to $O(\lambda^2)$ by

$$\sigma = ck\left[1 - \frac{(kh)^2}{6}\right],$$

with $c = \sqrt{gh}$. Substituting this into the RTE and the bRTE—retaining the $O(\lambda^2)$ correction only in the resonant mismatch—and restricting the equations to 1D forward wave propagation (along the x_1 axis), the RTE [Eq. (17)] becomes

$$\partial_t\mathscr{E} + \partial_x(c\mathscr{E}) - k(\partial_x c)\partial_k\mathscr{E} = +k\frac{3}{4}\sqrt{\frac{g}{h}}\left(\int_0^k \mathscr{B}^{\mathrm{im}}_{k',k-k'}\,dk' + 2\int_0^\infty \mathscr{B}^{\mathrm{im}*}_{k',k,x}\,dk'\right). \quad (29)$$

Here, $\mathscr{E}$ and $\mathscr{B}$ contain only positive wavenumber contributions (made explicit by the definite integrals) and where spatial and wavenumber coordinates x_1, k_1, and k'_1 are denoted as, x, k, and k', respectively. In this same approximation, the bRTE [Eq. (26)] reduces to

$$\partial_t\mathscr{B} + c\partial_x\mathscr{B} - (k\partial_x c\partial_k)\mathscr{B} - (k'\partial_x c)\partial_{k'}\mathscr{B} + i\frac{1}{2}ch^2kk'(k+k')\mathscr{B} - \frac{1}{4}\sqrt{\frac{g}{h}}\partial_x h\mathscr{B} = -i\frac{3}{2}\sqrt{\frac{g}{h}}[k\mathscr{E}_{k+k'}\mathscr{E}_{k'} + k'\mathscr{E}_{k+k'}\mathscr{E}_k - (k+k')\mathscr{E}_{k'}\mathscr{E}_k], \quad (30)$$

where, to be consistent with Herbers and Burton (1997), we omitted the fourth cumulant (thus introducing a quasi-normal closure assumption). Subsequently, to express the result in the frequency domain, we introduce the coordinate transformation

$$k = \omega/c, \quad k' = \omega'/c$$

and the Jacobian transformations $\mathscr{E}(k, x, t) = c\mathscr{E}(\omega, x, t)$ and $\mathscr{B}(k, k', x, t) = c^2\mathscr{B}(\omega, \omega', x, t)$. For convenience, and to maintain the usual symmetries, we introduce $\overline{\mathscr{E}}$ and $\overline{\mathscr{B}}$ as

$$\overline{\mathscr{E}}_\omega = \frac{\mathscr{E}_\omega}{2}, \quad \overline{\mathscr{B}}_{\omega,\omega'} = \frac{\mathscr{B}_{\omega,\omega'}}{4} \quad \omega > 0, \omega' > 0$$

and define the densities for negative frequencies using the symmetry relations:

$$\overline{\mathscr{E}}_\omega = \overline{\mathscr{E}}_{-\omega}$$

$$\overline{\mathscr{B}}_{\omega,\omega'} = \overline{\mathscr{B}}_{\omega',\omega} = \overline{\mathscr{B}}^*_{-\omega,-\omega'} = \overline{\mathscr{B}}_{\omega,-\omega-\omega'} = \overline{\mathscr{B}}_{\omega',-\omega-\omega'}.$$

With this, the RTE [Eq. (29)] can be reduced to

$$\partial_t\overline{\mathscr{E}} + \partial_x(c\overline{\mathscr{E}}) = \frac{3\omega}{2h}\int \overline{\mathscr{B}}^{\mathrm{im}}_{\omega',\omega-\omega'}\,d\omega', \quad (31)$$

where the two integrals on the right-hand side of Eq. (29) are reduced to a single integral because of the symmetries in the bispectrum (e.g., Norheim et al. 1998), and the evolution equation for the bispectrum [Eq. (30)] reduces to

$$\partial_t \overline{\mathscr{B}} + c\partial_x \overline{\mathscr{B}} + \frac{3}{4}\sqrt{\frac{g}{h}}(\partial_x h)\overline{\mathscr{B}} + i\frac{h\omega\omega'(\omega+\omega')}{2g}\overline{\mathscr{B}}$$
$$= -i\frac{3}{2h}[\omega\overline{\mathscr{E}}_{\omega+\omega'}\overline{\mathscr{E}}_{\omega'} + \omega'\overline{\mathscr{E}}_{\omega+\omega'}\overline{\mathscr{E}}_{\omega} - (\omega+\omega')\overline{\mathscr{E}}_{\omega'}\overline{\mathscr{E}}_{\omega}]. \tag{32}$$

If we assume stationary conditions (i.e., $\partial_t\overline{\mathscr{E}} = \partial_t\overline{\mathscr{B}} = 0$), the set of equations is consistent with earlier results for unidirectional shallow-water wave propagation [see Herbers and Burton 1997, their Eqs. (22a) and (22b)].

5. Conclusions

In this work, we used an approach inspired by methods developed in optics and quantum mechanics (e.g., Wigner 1932; Bremmer 1973; Bastiaans 1979; Cohen 2010) to derive a new transport equation for three-wave correlators in a random surface gravity wave field propagating through a variable medium. From a general second-order wave equation, valid for broad-banded wave propagation in a variable medium, we formulated evolution equations for the second- and third-order correlators and demonstrated by means of a multiple-scale argument that for quasi-homogeneous wave statistics, the resulting set can be written in terms of transport equations for the spectrum (the second-order correlator) and the bispectrum (the third-order correlator). These evolution equations take the form of the conventional radiative transport equation (RTE), forced by a nonlinear term that depends on the bispectrum, and a newly derived bispectral evolution equation, which we refer to as the biradiative transfer equation (bRTE). We discuss the bRTE equation and show that, when taking the appropriate limits, it includes known deep- and shallow-water asymptotes as special cases. The bRTE, which is the principal result of this work, is completely consistent with the quasi-homogeneous limit implied by the RTE, without introducing additional constraints on bandwidth, aperture, or medium variability. Consequently, the bRTE can be readily incorporated in existing stochastic wave models to account for non-Gaussian wave statistics in shallow water, which is expected to improve shallow-water predictive capability of non-Gaussian statistics, and further understanding of nearshore nonlinear effects.

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APPENDIX A

Interaction Coefficients

The interaction coefficients $w^{(j)}_{\mathbf{k},\mathbf{k}'}$ in the deterministic equation [Eq. (2)] for the variable $\hat{\zeta}'_{\mathbf{k}}$, derived from the coefficients for the conventional action variable (see, e.g., Krasitskii 1994; Janssen 2009b), are given as

$$\overline{w}^{(1)}_{\mathbf{k},\mathbf{k}'} = \overline{w}^{-,-}_{\mathbf{k},\mathbf{k}',\mathbf{k}-\mathbf{k}'}, \quad \overline{w}^{(2)}_{\mathbf{k},\mathbf{k}'} = 2\overline{w}^{+,-}_{\mathbf{k},\mathbf{k}',\mathbf{k}+\mathbf{k}'},$$
$$\overline{w}^{(3)}_{\mathbf{k},\mathbf{k}'} = \overline{w}^{+,+}_{\mathbf{k},\mathbf{k}',-\mathbf{k}-\mathbf{k}'}, \tag{A1}$$

in which

$$\overline{w}^{s_1,s_2}_{\mathbf{k}_0,\mathbf{k}_1,\mathbf{k}_2} = \frac{g}{8\sigma_{\mathbf{k}_1}}\left[\mathbf{k}_0\cdot\mathbf{k}_1 + s_1\left(\frac{\sigma_{\mathbf{k}_0}\sigma_{\mathbf{k}_1}}{g}\right)^2\right] + \frac{g}{8\sigma_{\mathbf{k}_2}}\left[\mathbf{k}_0\cdot\mathbf{k}_2 + s_2\left(\frac{\sigma_{\mathbf{k}_0}\sigma_{\mathbf{k}_2}}{g}\right)^2\right] + \frac{g\sigma_{\mathbf{k}_0}}{8\sigma_{\mathbf{k}_1}\sigma_{\mathbf{k}_2}}\left[\mathbf{k}_1\cdot\mathbf{k}_2 + s_1 s_2\left(\frac{\sigma_{\mathbf{k}_1}\sigma_{\mathbf{k}_2}}{g}\right)^2\right],$$

where s_1 and s_2 are sign indices. The interaction coefficients on the right-hand side of Eq. (3) for the free-surface variable $\hat{\zeta}_{\mathbf{k}}$ are given as

$$w^{(1)}_{\mathbf{k},\mathbf{k}'} = \overline{w}^{(1)}_{\mathbf{k},\mathbf{k}'} - \frac{\Delta\sigma_{\mathbf{k}-\mathbf{k}',\mathbf{k}'}\overline{w}^{(3)}_{-\mathbf{k},\mathbf{k}'}}{\sigma_{\mathbf{k}} + \sigma_{-\boldsymbol{v}} + \sigma_{\mathbf{k}-\boldsymbol{v}}} \quad w^{(2)}_{\mathbf{k},\mathbf{k}'} = \overline{w}^{(2)}_{\mathbf{k},\mathbf{k}'} \quad w^{(3)}_{\mathbf{k},\mathbf{k}'} = 0, \tag{A2}$$

where we included $w^{(3)}_{\mathbf{k},\mathbf{k}'}$ to illustrate that to obtain Eq. (3) from Eq. (2) we merely need to replace the coefficients $\overline{w}^{(j)}_{\mathbf{k},\mathbf{k}'}$ with $w^{(j)}_{\mathbf{k},\mathbf{k}'}$ and reinterpret the wave variable.

APPENDIX B

Operator Definition

In this section, we first give the definition of the Weyl correspondence rule that we use to incorporate slow changes of the medium into our deterministic equation and subsequently demonstrate that from this definition the operators that account for the transport of the coupled mode spectrum and bispectrum follow. The Weyl correspondence rule assigns to a phase space symbol $s_{\mathbf{k},\mathbf{x}} = s(\mathbf{k}, \mathbf{x})$ an operator S if $\mathbf{k}$ and $\mathbf{x}$ are replaced by the operators $\boldsymbol{D}_k$ and D_x, as

$$S_{D_k,D_x} \equiv \iint \frac{d\mathbf{p}\,d\mathbf{q}}{(2\pi)^2}\hat{s}(\mathbf{p},\mathbf{q})\exp(-iD_k\cdot\mathbf{p} + i\mathbf{q}\cdot D_x), \tag{B1}$$

where $\hat{s}$ denotes the $\mathbf{k}\to\mathbf{p}$ and $\mathbf{x}\to\mathbf{q}$ Fourier transform of s and where the exponential of an operator [e.g., $\exp(\nabla_x)$] is defined by its Taylor series. To justify the transformed operators used in Eq. (16) we have to manipulate the exponential function according to[1]

$$\begin{aligned}\exp(-iD_k\cdot\mathbf{p}+i\mathbf{q}\cdot D_x)\\&=\exp\left[\frac{-\mathbf{p}\cdot\mathbf{q}}{2}\{D_k,D_x\}\right]\exp(-iD_k\cdot\mathbf{p})\exp(+i\mathbf{q}\cdot D_x)\\&=\exp\left[\frac{-\mathbf{p}\cdot\mathbf{q}}{2}\{D_x,D_k\}\right]\exp(iD_x\cdot\mathbf{q})\exp(-i\mathbf{p}\cdot D_k),\end{aligned}$$

where the extra factor containing the commutator

$$\{D_k,D_x\}=D_kD_x-D_xD_k$$

appears because in general the operators do not commute [e.g., if $D_k=k$ and $D_x=\nabla_k$ then $\{D_k,D_x\}=-1$]. To establish Eq. (16), we set $D_k=\mathbf{k}\mp i\nabla_x/2$ and $D_x=\mathbf{x}\pm i\nabla_k/2$, which permits us to write Eq. (B1) in the form

$$\begin{aligned}S_{\mathbf{k}\mp(i/2)\nabla_x,\mathbf{x}\pm(i/2)\nabla_k}&\equiv\iint\frac{d\mathbf{p}\,d\mathbf{q}}{(2\pi)^2}\hat{s}_{\mathbf{p},\mathbf{q}}\exp(-i\mathbf{k}\cdot\mathbf{p}+i\mathbf{q}\cdot\mathbf{x})\exp\left(\mp\frac{\mathbf{p}}{2}\nabla_x\mp\frac{\mathbf{q}}{2}\nabla_k\right)\\&=s(\mathbf{k},\mathbf{x})\exp\left(\mp\frac{i}{2}\overleftarrow{\nabla}_k\cdot\overrightarrow{\nabla}_x\pm\frac{i}{2}\overleftarrow{\nabla}_x\cdot\overrightarrow{\nabla}_k\right)\end{aligned}\qquad\text{(B2)}$$

which corresponds to the operator as given in Eq. (22).

because in this case the noncommuting factors exactly cancel, that is, $\{\mathbf{k},\pm(i/2)\nabla_k\}+\{\mathbf{x},\mp(i/2)\nabla_x\}=0$. To establish Eq. (22), we set

$$\begin{aligned}\mathscr{D}_k^j&=\mathbf{M}_{j,1}\mathbf{k}+\mathbf{M}_{j,2}\mathbf{k}'-i|\mathbf{M}_{j,3}|\nabla_x\\\mathscr{D}_x^j&=i\mathbf{M}_{1,j}^{-1}\nabla_k+i\mathbf{M}_{2,j}^{-1}\nabla_{k'}+\mathbf{M}_{3,j}^{-1}\mathbf{x}\end{aligned}$$

and substitute these into the definition (B1). If we expand the exponential operators and define

$$\begin{aligned}Y_j&=[(\mathbf{M}_{j,1}\mathbf{k},iM_{1,j}^{-1}\nabla_k)+(\mathbf{M}_{j,2}\mathbf{k}',i\mathbf{M}_{2,j}^{-1}\nabla_{k'})\\&\quad+(\mathbf{M}_{j,3}\mathbf{x},-i\mathbf{M}_{3,j}^{-1}\nabla_x)]\\&=-i(\mathbf{M}_{j,1}\mathbf{M}_{1,j}^{-1}+\mathbf{M}_{j,2}\mathbf{M}_{2,j}^{-1}-\mathbf{M}_{j,3}\mathbf{M}_{3,j}^{-1}),\end{aligned}$$

Eq. (B1) may be written in the form

$$\begin{aligned}S_{\mathscr{D}_k^j,\mathscr{D}_x^j}&\equiv\iint\frac{d\mathbf{p}\,d\mathbf{q}}{(2\pi)^2}\hat{s}_{\mathbf{p},\mathbf{q}}\exp[-i(\mathbf{M}_{j,1}\mathbf{k}+\mathbf{M}_{j,2}\mathbf{k}')\cdot\mathbf{p}+i\mathbf{q}\cdot\mathbf{x}]\\&\quad\times\exp\left(-Y_j\frac{\mathbf{p}\cdot\mathbf{q}}{2}\right)\exp(-|\mathbf{M}_{j,3}|\mathbf{p}\cdot\nabla_x\\&\quad-\mathbf{M}_{1,j}^{-1}\mathbf{q}\cdot\nabla_k-\mathbf{M}_{2,j}^{-1}\mathbf{q}\cdot\nabla_{k'}).\end{aligned}$$

If we integrate this expression with respect to $\mathbf{q}$ and $\mathbf{p}$, so that $\hat{s}_{\mathbf{p},\mathbf{q}}$ transforms to $s_{\mathbf{M}_{j,1}\mathbf{k}+\mathbf{M}_{j,2}\mathbf{k}',\mathbf{x}_m}$, this becomes

$$\begin{aligned}S_{\mathscr{D}_k^j,\mathscr{D}_x^j}&=s(\mathbf{M}_{j,1}\mathbf{k}+\mathbf{M}_{j,2}\mathbf{k}',\mathbf{X}_m)\exp\left[-\frac{Y_j}{2}\overleftarrow{\nabla}_\mathbf{x}\cdot\overleftarrow{\nabla}_\mathbf{k}\right.\\&\quad+i(\mathbf{M}_{1,j}^{-1}\overleftarrow{\nabla}_\mathbf{x}\cdot\overrightarrow{\nabla}_\mathbf{k}+\mathbf{M}_{2,j}^{-1}\overleftarrow{\nabla}_\mathbf{x}\cdot\overrightarrow{\nabla}_{\mathbf{k}'}\\&\quad\left.-|\mathbf{M}_{j,3}|\overleftarrow{\nabla}_\mathbf{k}\cdot\overrightarrow{\nabla}_\mathbf{x})\right],\end{aligned}\qquad\text{(B3)}$$

[1] Note that the relation only works if the commutator commutes with both operators, for example, $\{\{D_k,D_x\},D_x\}=\{D_x,(D_k,D_x\}\}$, which in the present work always is the case.

APPENDIX C

Reduction to a KP-Like Equation over Topography

To establish that Eq. (4) applies when second-order nonlinear effects are significant, we wish to demonstrate that our deterministic equation set in wavenumber space describes the propagation of weakly dispersive (in frequency and direction) shallow-water waves, where dispersive effects are balanced by weak nonlinearity. For a homogeneous medium, for which Eq. (2) applies, Onorato et al. (2004) demonstrated that when restricted to unidirectional, shallow-water waves, the Korteweg de Vries (KdV) equation follows (Korteweg and De Vries 1895). Here, we not only wish to extend the argument to include weakly directional waves leading to the KP equation (Kadomtsev and Petviashvili 1970), but our principle goal is to verify that our operator correspondence argument in this limit reduces to a KP-like equation as given by Liu et al. (1985) that includes the effect of variable topography. To demonstrate this, we apply the operator correspondence argument to the original Eq. (2) (thus retaining the off-resonant contributions explicitly) and note that the linear dispersive operator then can be written in the form

$$\begin{aligned}\Omega_{\mathbf{k},i\nabla_\mathbf{k}}&=\iint\frac{d\mathbf{p}\,d\mathbf{q}}{(2\pi)^2}\left[\hat{\sigma}_{\mathbf{p},\mathbf{q}}\exp\left(-i\mathbf{p}\cdot\mathbf{k}+\frac{i}{2}\mathbf{p}\cdot\mathbf{q}\right)\exp(+\mathbf{q}\cdot\nabla_\mathbf{k})\right]\\&=\int d\mathbf{q}\hat{\sigma}_{\mathbf{k}-\mathbf{q}/2,\mathbf{q}}\exp(\mathbf{q}\cdot\nabla_\mathbf{k}),\end{aligned}$$

where the additional factor $\exp(i\mathbf{p}\cdot\mathbf{q}/2)$ occurs in the first line because $\mathbf{k}$ and $\nabla_\mathbf{k}$ do not commute (see appendix A). Let a typical wavenumber be given by $\mathbf{k}_w=(k_p,k_l)$, where k_p denotes the wavenumber in the principle direction, and k_l is the lateral wavenumber, so that $k_l/k_p = O(\lambda) \ll 1$. Moreover, the depth is characterized by h_m where it is assumed that $|\mathbf{k}_w|h_m \sim k_p h_m = O(\lambda) \ll 1$. In this case, the dispersion relation that occurs in the symbol of the dispersive operator can be approximated up to $O(\lambda^3)$ as

$$\sigma_{\mathbf{k},\mathbf{x}} = k_1 c\left[1 + \left(\frac{1}{2}\frac{k_2}{k_1}\right)^2 - \frac{(k_1 h)^2}{6}\right], \qquad \text{(C1)}$$

with $c = \sqrt{gh}$. Moreover, assuming that the variation in the topography occurs on a slow scale such that the ratio ε between a typical wavenumber of the topography q_m and k_w is expected to be small, we can approximate the factor $\hat{\sigma}_{\mathbf{k}-\mathbf{q}/2,\mathbf{q}}$ by a first-order Taylor series around $\mathbf{k}$, so that the linear dispersive operator can be approximated as

$$\Omega_{\mathbf{k},i\nabla_k} = \int d\mathbf{q}\left[\hat{\sigma}_{k,q} - \frac{q_1}{2}\hat{c}(\mathbf{q})\right]\exp(\mathbf{q}\cdot\nabla_{\mathbf{k}}), \qquad \text{(C2)}$$

where we set $O(\varepsilon) = O(\lambda^2)$ and consequently approximated the first-order Taylor term $\mathbf{q}/2 \cdot \nabla_k \hat{\sigma}$, using only the lowest-order term of the dispersion relation.

Because nonlinearity is assumed small, the RHS of Eq. (4) is considered as a small perturbation of $O(\delta)$ to the linear equation. If $O(\delta) = O(\lambda^2)$ it is then sufficient to retain an (λ^0) approximation to W in the shallow-water limit:

$$W^{(j)}_{\mathbf{k},\mathbf{k}',i\nabla_{\mathbf{k}}} = \int d\mathbf{q}\hat{w}^{(j)}_{\mathbf{k},\mathbf{k}',\mathbf{q}}\exp(\mathbf{q}\cdot\nabla_{\mathbf{k}}),$$

where the coupling coefficients take the simple form

$$w^{(1)}_{\mathbf{k},\mathbf{k}',\mathbf{x}} = w^{(2)}_{\mathbf{k},\mathbf{k}',\mathbf{x}}/2 = w^{(3)}_{\mathbf{k},\mathbf{k}',\mathbf{x}} = \frac{3k_1}{8}\sqrt{\frac{g}{h}}.$$

To obtain a spatial description, we apply the Fourier transform over Eq. (4), exchanging the wavenumber coordinate $\mathbf{k}$ with a spatial coordinate $\mathbf{x}$. However, to obtain a conventional partial differential equation, we first remove the k_1^{-1} factor in Eq. (C1)—which originates from the small aperture approximation and would introduce a pseudo-differential operator $\partial_{x_1}^{-1}$—by multiplication of the equation with k_1 and then apply the transform to obtain an equation in terms of $\zeta(\mathbf{x}, t)$. If we multiply the result by ½, and add the conjugate equation, we recover an equation in terms of the free-surface variable $\eta = (\zeta + \zeta^*)/2$:

$$\partial_{x_1}\left(\partial_t + \sqrt{gh}\partial_{x_1} + \frac{g}{4\sqrt{h}}\partial_{x_1}h + \sqrt{gh}\,\frac{h^2}{6}\partial^3_{x_1}\right)\eta(\mathbf{x},t) + \frac{1}{2}\sqrt{gh}\partial^2_{x_2}\eta(\mathbf{x},t) = \partial_{x_1}\left[\frac{3}{4}\sqrt{\frac{g}{h}}\partial_{x_1}\eta^2(\mathbf{x},t)\right], \qquad \text{(C3)}$$

which when neglecting variability in the lateral direction reduces to a KdV equation over variable topography and to the order considered is equivalent to a generalization of the KP equation to variable topography[2] [Liu et al. 1985, their Eq. (4.4)].

APPENDIX D

Determination of α_1 and α_2

To derive the evolution equation for the bispectrum [Eq. (26)], we assumed (see section 3) that the difference wavenumber $\mathbf{u}$ is distributed symmetrically across the wavenumbers $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$. Specifically, in the definition

$$\hat{\Gamma}(\mathbf{k},\mathbf{k}',\mathbf{u}) = \frac{1}{2}\langle\hat{\zeta}_{\mathbf{k}+\alpha_1\mathbf{u}}\hat{\zeta}_{\mathbf{k}'+\alpha_2\mathbf{u}}\hat{\zeta}^*_{\mathbf{k}+\mathbf{k}'-(\alpha_1+\alpha_2-1)\mathbf{u}}\rangle, \qquad \text{(D1)}$$

we set $\alpha_1 = \alpha_2 = 1/3$. Formally this choice is arbitrary and guided mostly by convenience. Any other choice, as long as α_1 and α_2 are both $O(1)$, would likewise result in a consistent description of the evolution of the wave statistics. However, by defining $\alpha_1 = \alpha_2 = 1/3$, the interpretation of the resulting bispectrum is simplified. First, for any choice where $\alpha_1 = \alpha_2$, $\mathscr{B}$ remains identical under permutation of the wavenumbers (i.e., $\mathscr{B}_{\mathbf{k},\mathbf{k}'} = \mathscr{B}_{\mathbf{k}',\mathbf{k}}$), so that $\mathscr{B}$ can be interpreted as the local correlation between three waves with wavenumber $\mathbf{k}$, $\mathbf{k}'$, and $\mathbf{k} + \mathbf{k}'$. Second, the specific choice $\alpha_1 = \alpha_2 = 1/3$ is convenient since it allows us to relate $\mathscr{B}$ to the third-order statistics of an observable wave variable (the free-surface elevation η). After all, from the relation $\hat{\eta}_{\mathbf{k}} = (\hat{\zeta}_{\mathbf{k}} + \hat{\zeta}^*_{-\mathbf{k}})/2$, we see that

$$\hat{\Gamma}^\eta(\mathbf{k},\mathbf{k}',\mathbf{u}) = \frac{1}{2}\langle\hat{\eta}_{\mathbf{k}+\alpha\mathbf{u}}\hat{\eta}_{\mathbf{k}'+\alpha\mathbf{u}}\hat{\eta}^*_{\mathbf{k}+\mathbf{k}'-(2\alpha-1)\mathbf{u}}\rangle \qquad \text{(D2)}$$

can be expressed as a three-point correlator of ζ:

$$\hat{\Gamma}^\eta_{\mathbf{k},k',\mathbf{u}} = \frac{1}{4}(\Gamma_{\mathbf{k},\mathbf{k}',\mathbf{u}} + \Gamma_{-\boldsymbol{\kappa}^-,\mathbf{k},\mathbf{u}} + \Gamma_{-\boldsymbol{\kappa}^-,\mathbf{k}',\mathbf{u}} + \Gamma^*_{-\mathbf{k},-\mathbf{k}',-\mathbf{u}} + \Gamma^*_{\boldsymbol{\kappa}^+,-\mathbf{k},-\mathbf{u}} + \Gamma^*_{\boldsymbol{\kappa}^+,-\mathbf{k}',-\mathbf{u}}),$$

with $\boldsymbol{\kappa}^\pm = \mathbf{k} + \mathbf{k}' \pm (1-3\alpha)\mathbf{u}$. Consequently, if $\alpha = 1/3$, $\boldsymbol{\kappa}^\pm$ becomes independent of $\mathbf{u}$, so that—after Fourier transformation—$\mathscr{B}^\eta$ can be explicitly expressed in terms of $\mathscr{B}$:

[2] The slight differences that occur in the coefficients are due to terms that are $O(\lambda)^3$ or greater and are thus neglected in the present approximation.

$$\mathscr{B}^{\eta}_{\mathbf{k},\mathbf{k}'} = \frac{1}{4}\Big(\mathscr{B}_{\mathbf{k},\mathbf{k}'} + \mathscr{B}_{-\mathbf{k}-\mathbf{k}',\mathbf{k}} + \mathscr{B}_{-\mathbf{k}-\mathbf{k}',\mathbf{k}'} + \mathscr{B}^{*}_{-\mathbf{k},-\mathbf{k}'} + \mathscr{B}^{*}_{\mathbf{k}+\mathbf{k}',-\mathbf{k}} + \mathscr{B}^{*}_{\mathbf{k}+\mathbf{k}',-\mathbf{k}'}\Big). \quad \text{(D3)}$$

And so nonlinear bulk statistics of the free-surface elevation, such as skewness, can be expressed directly in terms of $\mathscr{B}$:

$$\text{Skewness} = \frac{\iint \mathscr{B}^{\eta}_{\mathbf{k},\mathbf{k}'}\, d\mathbf{k}\, d\mathbf{k}'}{\left(\int \mathscr{E}_{\mathbf{k}}\, d\mathbf{k}\right)^{3/2}} = \frac{3\iint \Re(\mathscr{B}_{\mathbf{k},\mathbf{k}'})\, d\mathbf{k}\, d\mathbf{k}'}{2\left(\int \mathscr{E}_{\mathbf{k}}\, d\mathbf{k}\right)^{3/2}}. \quad \text{(D4)}$$

For any other choice of α_1 and α_2 [with the constraint that both are $O(1)$], the same results could be obtained but the expressions would be more convoluted.

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