Revisiting the Generation of Internal Waves by Resonant Interaction with Surface Waves

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ABSTRACT

Two surface waves can interact to produce an internal gravity wave by nonlinear resonant coupling. The process has been called spontaneous creation (SC) because it operates without internal waves being initially present. Previous studies have shown that the generated internal waves have high frequency close to the local Brunt–Väisälä frequency and wavelengths that are much larger than those of the participating surface waves, and that the spectral transfer rate of energy to the internal wave field is small compared to other generation processes. The aim of the present analysis is to provide a global map of the energy transfer into the internal wave field, which is found to be about $10^{23}$ TW in total, based on a realistic wind-sea spectrum (depending on wind speed), mixed layer depths, and stratification below the mixed layer taken from a state-of-the-art numerical ocean model. Unlike previous calculations of the spectral transfer rate based on a vertical mode decomposition, the authors use an analytical framework that directly derives the energy flux of generated internal waves radiating downward from the mixed layer base. Since the radiated waves are of high frequency, they are trapped and dissipated in the upper ocean. The radiative flux thus feeds only a small portion of the water column, unlike in cases of wind-driven near-inertial waves that spread over the entire ocean depth before dissipating. The authors also give an estimate of the interior dissipation and implied vertical diffusivities due to this process. In an extended appendix, they review the modal description of the SC interaction process, completed by the corresponding counterpart, the modulation interaction process (MI), where a preexisting internal wave is modulated by a surface wave and interacts with another one. MI establishes a damping of the internal wave field, thus acting against SC. The authors show that SC overcomes MI for wind speeds exceeding about 10 m s$^{-1}$.

1. Introduction

Internal gravity waves have a major share of the energy contained in oceanic motions. Important sources are waves radiating out of the mixed layer because of near-inertial motions excited by wind fluctuations (see, e.g., Alford 2001; Rimac et al. 2013) and waves generated at the ocean bottom, originating from the conversion of barotropic into baroclinic tides at submarine topography (see, e.g., Falahat et al. 2014). Other forcing functions that have been studied in recent years are lee-wave generation by mesoscale eddies or the mean flow (Nikurashin and Ferrari 2011) and dissipation of balanced flow (Molemaker et al. 2010), but there are several other processes that can generate internal waves (e.g., Olbers 1983), among which is the forcing by nonlinear coupling to wind-generated surface waves, the topic of the present study. If internal waves break, their energy is partly used to mix the mean stratification. Although this diapycnal mixing is weak in the interior, such breaking internal waves generate large amounts of potential energy that then drives large-scale oceanic motions. Unlike the internal wave energy generated by near-inertial motions in the mixed layer and the tides that have low frequencies and propagate through the entire water column, internal waves generated by resonant interactions of surface waves are of high frequency.
and thus likely to be trapped in the upper ocean. The interaction of surface and internal waves thus could contribute to mixing just below the mixed layer.

Since the stress (vertical flux of horizontal momentum) induced by linear surface waves is zero (see, e.g., Hasselmann 1970), nonlinear effects need to be considered, and the evaluation of the transfer of energy from surface to internal waves enters the field of nonlinear resonant interactions among the wave branches. The process has been investigated before, however, with disparate and partly inconclusive results. Restricting the discussion to studies that deal with the interactions of waves in random fields (see, e.g., Hasselmann 1966, 1967), only a few remain: Kenyon (1968) computed the transfer from an observed swell spectrum to the first baroclinic wave mode in shallow water and concluded that the observed level of internal wave energy cannot be explained by the process. Olbers and Herterich (1979, hereinafter OH79) analyzed the transfer to deep-water baroclinic wave modes from wind-sea spectra as a function of wind speed and for different stratifications. They find that high-frequency internal waves are excited with wavelengths that are large compared to the ones of the surface waves, but at a rate that is small in general. Only for the case of a shallow mixed layer, high wind speeds, and strong stratification below the mixed layer, internal waves are generated at the level of 0.1 mW m\(^{-2}\). This level is typically a lower limit of the energy transfer by near-inertial motions in the mixed layer (see, e.g., Rimac et al. 2016) and far below the transfer rates by the tides (see, e.g., Falahat et al. 2014).

Watson (1990, hereinafter W90; 1994, hereinafter W94) has analyzed surface-internal wave interactions in a more general setting. W90 applies the modal interaction framework for idealized conditions, and W94 presents a calculation of energy transfers for the North Pacific. As OH79, Watson considers what he calls spontaneous creation (SC)—two surface waves generate an internal wave—and he also discusses a modulation interaction process (MI), where a preexisting internal wave is modulated by a surface wave and interacts with another surface wave. MI is included in the general framework of OH79 but was not evaluated. It was first studied in detail by Dysthe and Das (1981). By this interaction, following W90 and W94, the internal mode loses energy to the surface waves. The SC process, analyzed in the present study, cannot be separated in Watson’s calculations from the MI process, but from his results we note that the MI dominates SC up to a critical wind speed. In strong wind regimes, the SC process overcomes the MI process. Note that this regime (high winds, low mixed layer depths) is the one where the SC surface-internal wave interactions become relevant at all, such that we focus on the spontaneous creation process in the main part of the present study but consider both processes in appendix C. The relative importance of SC and MI interactions can be assessed easily, realizing that the cross section of the interactions is identical when formulating the scattering in terms of the action spectra. We show in appendix C that the ratio of MI to SC is \((U_{crit}/U)^2\), with a critical wind speed around 10–15 m s\(^{-1}\), depending on the Brunt–Väisälä frequency and the parameters of the surface and internal wave spectra.

Although the flux by interaction with surface waves is lower than other sources of internal waves in the main pycnocline even for strong winds, it might still be important since it is likely to be dissipated just below the mixed layer. We thus revisit the process with the aim to provide a globally integrated estimate of the flux and its interior dissipation.

In section 2 we start by outlining the surface layer model with nonlinear forcing due to surface waves with a spectral and statistical treatment of the interaction. We derive the coupling conditions to the ocean interior and the spectral energy flux into the internal wave field at the mixed layer base in general form. Section 3 shows the dependency of the energy flux on the parameters of a simple factorized wind-sea spectrum, the mixed depth, and the Brunt–Väisälä frequency below the mixed layer base. We present a parameterization of the energy transfer rate in terms of the local wind speed, mixed layer depth, and the Brunt–Väisälä frequency, resulting from a numerical evaluation of the spectral transfer integral. A global estimate of the flux is provided, and its interior dissipation is based on realistic parameters. The last section contains a summary and conclusions.

### 2. Theoretical framework

We abandon here the baroclinic vertical mode framework used by OH79, W90, W94, and others and instead directly calculate the energy flux of internal waves radiating from the surface mixed layer into the interior stratified ocean. Radiation physics rather than modes, similar to our approach, are also considered by Moehlis and Llewellyn Smith (2001) for a wind-driven case. We believe that this is a much simpler framework than the one by OH79 and W90, since the flux can be interpreted as being induced by “surface wave pumping,” generated by the wave-induced vertical velocity in the surface layer, much like “Ekman pumping” is induced by the divergence of the wind stress and “inertial pumping” is induced by the divergence of wind-generated, near-inertial waves in the mixed layer (Gill 1984). In fact, our formulation would allow us to treat these surface-related
processes in a similar way to the surface wave forcing we consider here. The flux, by a radiation condition, depends only on the Brunt–Väisälä frequency just below the mixed layer but is independent of specific baroclinic mode structure, unlike OH79. On the other hand, our analysis ignores the (small) part of the surface wave forcing below the mixed layer, which we believe is unimportant, since amplitudes of surface waves rapidly decay with depth.\(^1\)

When forcing the modal system with finite depth and specified stratification below the mixed layer by time-dependent pumping at the top, a pressure field is set up and allows for resonance with the applied pumping. The resonant pressure field will project on the vertical modes and the resulting energy flux (vertical velocity \(\times\) pressure) will be due to the resonating modes and must therefore depend on their structure, as can be seen in the results of OH79. This is not the case in our semi-infinite system where the flux is independent on the interior stratification and depth. Therefore, we cannot expect a complete match of both approaches, but we expect close correspondence. We compare radiation and modal computations in appendix C.

a. The mixed layer and forcing by surface gravity waves

We consider a mixed layer from the ocean surface to a depth \(z = -d\) that has constant density and resides on top of a stratified layer where internal wave propagation becomes possible with a nonzero Brunt–Väisälä frequency \(N(z)\). The surface wave forcing will be integrated from the surface to \(z = -d\), such that this depth range should contain all (or most) of the surface wave forcing. Since the forcing is quadratic in the vertical structure function of the surface waves, the forcing indeed diminishes very rapidly with increasing depth. The part of the forcing that reaches into the stratified interior will not be considered here. The equations of motion for the surface layer are written in the form

\[
\frac{\partial \mathbf{u}}{\partial t} + f \mathbf{u} + \nabla p = \mathbf{X} = -\nabla \cdot \mathbf{u} \mathbf{u}' - \frac{\partial}{\partial z} w' \mathbf{u}' ,
\]

(1)

\[
\frac{\partial w}{\partial t} + \frac{\partial p}{\partial z} = Z = -\nabla \cdot \mathbf{u} w' - \frac{\partial}{\partial z} \mathbf{w} w' , \quad \text{and}
\]

(2)

\[
\nabla \cdot \mathbf{u} + \frac{\partial w}{\partial z} = 0 ,
\]

(3)

where \(\mathbf{u}\) is the horizontal velocity, \(p\) is the pressure (scaled by the constant density), and \(w\) is the vertical velocity. All vectors are horizontal, and the vector \(\mathbf{u}\) denotes an anticlockwise rotation of the vector \(\mathbf{u}\) by \(90^\circ\). The terms \((\mathbf{X}, Z)\) on the right-hand side are the forcing functions (stress divergence) for horizontal and vertical momentum, induced by surface waves. The surface wave fields are marked by the index \(s\). At the surface \(z = \xi\) we consider the kinematic and the dynamic boundary condition for a free surface. Expansion of the conditions around the mean sea level \(z = 0\) yields

\[
\frac{\partial \xi}{\partial t} - w = Y = -u' \cdot \nabla v' \bigg|_{z=0} + \xi' \frac{\partial w'}{\partial z} \bigg|_{z=0} \quad \text{and}
\]

(4)

\[
p - gz = P = -\xi' \frac{\partial p'}{\partial z} \bigg|_{z=0} ,
\]

where the nonlinear terms \(Y\) and \(P\) are again induced by surface waves, restricted here to the quadratic terms of the expansion about the mean sea surface \((g\) is gravity acceleration). A solution is sought now in spectral space where, for example, the vertical velocity is represented as

\[
\tilde{w}(\mathbf{k}, \omega, z) = \frac{1}{(2\pi)^3} \int d^2 x \int dt w(\mathbf{x}, z, t) e^{-i(\mathbf{k} \cdot \mathbf{x} - \omega t)} ,
\]

(5)

with the complex wave amplitude \(\tilde{w}\), the wavenumber \(\mathbf{k}\), and the frequency \(\omega\). Similar representations hold for the other variables. All (positive and negative) frequencies are considered, but since \(\tilde{w}^*(\mathbf{k}, \omega) = \tilde{w}(-\mathbf{k}, -\omega)\), the analysis may be restricted to nonnegative frequencies. Here and in the following, the complex conjugate is denoted by an asterisk. After the spectral transformation, we follow OH79 and Olbers (1986) to combine (1)–(4) to a single equation for \(\tilde{w}(\mathbf{k}, \omega)\):

\[
\frac{\partial^2 \tilde{w}}{\partial z^2} - q^2 \tilde{w} = -\frac{\hat{Q}}{\omega^2 - f^2} ,
\]

(6)

\[
\hat{Q} = -(\omega k + i f k) \frac{\partial \mathbf{X}}{\partial z} + i \omega k^2 \tilde{Z} ,
\]

where the surface boundary condition at \(z = 0\),

\[
\frac{\partial \tilde{w}}{\partial z} - \gamma \tilde{w} = -\frac{\hat{Q}_0}{\omega^2 - f^2} ,
\]

(7)

\[
\hat{Q}_0 = k^2 (i \omega \tilde{P} - g \tilde{Y}) - (\omega k + i f k) \cdot \mathbf{X}(z = 0)
\]

where \(q^2 = k^2 \omega^2/(\omega^2 - f^2)\), \(\gamma = g k^2/(\omega^2 - f^2)\). Note that \(\hat{Q}\) derives from the wave-induced stress and \(\hat{Q}_0\) from the wave-induced forcing at the upper boundary. Given the forcing, two boundary conditions are required to solve

\(\footnotesize{1}\)

Consider, for instance, a wind speed of 10 m s\(^{-1}\), for which the mean wavelength of a saturated wind sea is 83 m (see below) and \(d = 50\) m. The forcing then decreases by a factor of about \(5 \times 10^{-4}\) below \(z = -d\), such that, indeed, almost all of the surface wave forcing is contained in the mixed layer above \(z = -d\).
(6), that is, one condition in addition to the surface condition given by (7). This additional condition will be obtained by considering the matching to the lower stratified layer that allows for internal wave propagation and thus radiation of the energy generated by the forcing.

b. Matching the mixed layer forcing to the interior

To match the mixed layer field to the interior, we take continuity of $\dot{w}$ and $\partial \dot{w}/\partial z$ at the mixed layer base $z = -d$. In the stratified layer below the mixed layer, a perturbation with wavevector $k$ and frequency $\omega$ is an internal wave with a vertical velocity $\dot{w}_{\text{sw}} = \dot{w}_{\text{sw}}(k, \omega, z)$ governed by

$$\frac{\partial^2 \dot{w}_{\text{sw}}}{\partial z^2} + m^2(z)\dot{w}_{\text{sw}} = 0,$$

(8)

with the vertical wavenumber $m$ given by the local dispersion relation

$$m^2(z) = k^2(N^2(z) - \omega^2) / (\omega^2 - f^2).$$

The internal wave propagates in a variable environment with the Brunt–Väisälä frequency profile $N(z)$, where the phase of $\dot{w}_{\text{sw}}$ is exp $\pm id(\omega - \omega_c)z$ such that $m = \omega / \omega_c$. We must require a positive vertical wavenumber $m$ to have downward group velocity, which is the radiation condition. The matching conditions are

$$\dot{w} = \dot{w}_{\text{sw}} \quad \text{and} \quad \frac{\partial \dot{w}}{\partial z} = \frac{\partial \dot{w}_{\text{sw}}}{\partial z} \quad \text{at} \quad z = -d.$$  

(10)

c. Forcing by an ensemble of surface gravity wave

The forcing terms $\hat{Q}$ and $\hat{Q}_0$ are now established for a given ensemble of surface waves. The vertical velocity of the surface wave field is expressed as

$$\dot{w}(x, z, t) = \sum_{\mu = \pm} \int d^2k a_{\mu}(k)e^{i(kx - \omega t)}\phi_\mu(z),$$

(11)

with the vertical eigenfunction $\phi_\mu(z) = k^{1/2} e^{ikz}$ and $a_{\mu}(k) = \delta_{\mu, -2}(k)$ and $\omega_{\mu}(k) = -\omega_{\mu}(2) = (gk)^{1/2}$. The other surface wave fields follow from the polarization vector ($u^i$, $w^i$, $p^i$, $\xi^i$) $\sim (id^j_p k^j/k^2, \phi_{\mu}, id^j_p \omega_p/k^2, id^j_p (z = 0)/\omega_p)$ of the linear wave solution for surface waves. We denote the vertical derivative by a prime. For $\hat{Q}_0$ we find

$$\hat{Q}_0 = \sum_{12} a_{1} a_{2} \delta(\omega - \omega_1 - \omega_2),$$

(12)

where $a_{1} = \delta_{\mu_1}$ are the amplitudes of the surface waves, $j = 1, 2, \mu_j = \pm$. In expressions like (12), the index 1 stands for $(\mu_1, k_1)$, and the sum over 1 stands for the sum over $\mu_1 = \pm$ and the integral over $k_1$; the same applies for index 2. For the resonance condition $\omega = \omega_1 + \omega_2$, the coefficient in $\hat{Q}_0$ is given by

$$C_{12}^0 = i\sqrt{\frac{k_{11}k_{22}}{k_{11}k_{22}}} \left( 1 - \frac{k_{11} \cdot k_{12}}{k_{11}k_{22}} \right).$$

(13)

The forcing term $\hat{Q}$ yields a similar form as $\hat{Q}_0$, but it turns out that the corresponding coupling coefficient vanishes identically. This property is important because $\hat{Q}$ derives entirely from interior forcing. Hence, only the weak nonlinearities of the surface boundary condition generate the transfer between surface and internal waves (as in OH79, where the volume force only contributes below the mixed layer).

The ensemble of surface waves is considered as a random process with a zero mean, $\langle a_1 \rangle = 0$, and a second moment, $\langle a_1 a_2 \rangle$, where the brackets denote the statistical expectation. The second moment is given by the surface wave spectrum $F(k)$:

$$\langle a_1 a_2 \rangle = \langle a_{\mu_1}(k_1)a_{\mu_2}(k_2) \rangle = \frac{1}{2} F(k) \delta_{\mu_1, -\mu_2} \delta(k_1 + k_2).$$

(14)

The assumption here is that the surface wave field is a horizontally homogeneous stochastic process. The normalization of the eigenfunction $\phi_\mu(z)$ in (11) ensures that the total energy is given by

$$F_0 = \frac{1}{2} \left( \int \left( u^i u^i + w^i w^i + g\xi^i \xi^i \right) dz \right) = \int d^2k F(k).$$

(15)

The energy of surface waves is equally partitioned between kinetic and potential energies; thus, $F_0 = g\langle (\xi^2) \rangle$.

d. The solution in terms of the pumping velocity

We complete the problem [(6)] by requiring $\dot{w} = W$ at $z = -d$ with a yet unknown $W = W(k, \omega)$, which is a surface-wave-induced pumping velocity at the base of the mixed layer. The unknown $W = W(k, \omega)$ is determined by the matching conditions from (10). The solution $\dot{w}(z)$ of (6) with the appropriate boundary conditions in (7) and (10), and especially for $\dot{w}(z = -d) = W$, can be found in terms of the appropriate Green’s function. The analysis is detailed in appendix A, and the result for the pumping velocity is

2 This important result was pointed out by an anonymous reviewer. Up to then we used that $\hat{Q}$ vanishes on the resonance surface.
\[ W(k, \omega) = -\frac{\tilde{Q}_0}{\omega^2 - f^2} \frac{1 - \gamma d}{\lambda \sinh(qd) - \cosh(qd)} \]

with the coefficients

\[
\tilde{\alpha} = \frac{\lambda q d \cosh(qd) - 1 - qd \sinh(qd)}{\lambda \sinh(qd) - \cosh(qd)} \quad \text{and} \quad \tilde{\beta} = \frac{\sinh(qd) + \lambda(1 - \cosh(qd))}{\lambda \sinh(qd) - \cosh(qd)},
\]

where \( \lambda = \frac{\gamma q}{g k} = g k [\omega (\omega^2 - f^2)^{1/2}] \approx g k / \omega^2 \). We use now the expression for \( \tilde{Q}_0 \), resulting from (12), in (16) and obtain \( W \) as

\[ W(k, \omega) = \sum_{12} D_{12} a_{12} \delta(k - k_1 - k_2) \delta(\omega - \omega_1 - \omega_2), \]

(18)

where \( D_{12} \) is the interaction coefficient, characterizing the interaction of the waves with labels 1 and 2. In practice we have \( \lambda \gg 1 \), and we arrive for this limit at

\[
D_{12} = D_{kk',kk''} = \frac{i \sqrt{\lambda_{kk',kk''} \omega^2 g}}{\omega \cosh(qd) + i(N_d^2 - \omega^2)^{1/2} \sinh(qd)} \left( 1 - \frac{k_1 \cdot k_2}{k_1^2} \right),
\]

(19)

where \( N_d = N(z = -d) \) is the Brunt–Väisälä frequency just below the mixed layer base. The indices in \( D_{kk',kk''} \) should indicate that the triad sum of the wave vectors is zero, and similarly for the frequencies due to the \( \delta \) functions in (18).

Writing (18) in dimensionless form by normalizing the velocities \( W \) and \( \sqrt{\omega / k} \) by the phase speed \( \omega / k \), one finds that the dimensionless coupling coefficient \( D_{12} \sqrt{k_1 k_2} \omega / k \) is of order \( \omega^2 g k \), which is typically \( O(10^{-4}) \). It is therefore the large gap between the internal wave frequency \( \omega \) and the surface wave frequency \( \sqrt{g k} \) that generates small coefficients and thus weak coupling. Note that the small ratio \( \omega / \sqrt{g k} \) enters the coupling coefficient in a quadratic way and the transfer integral (see below) as fourth power.

The pumping induced by \( W(k, \omega) \) at the mixed layer base establishes the generation of internal waves that radiate downward from the surface layer into the interior. The statistical properties of the surface wave field imply the vanishing of the mean pumping velocity, \( \langle W(k, \omega) \rangle = 0 \). The radiation flux, however, depends on the second moment of the pumping velocity, as derived in the following section.

e. The spectral energy flux at the mixed layer base

The vertical velocity \( \hat{w}_{\text{vm}} \) of the interior wave field follows from (8) with the vertical wavenumber \( m \), given by the dispersion relation (9). We consider only waves that propagate downward so that \( m \) is positive for positive \( \omega \), and thus

\[
w_{\text{vm}}(x, z, t) = \int d^2 k \int_{-\infty}^\infty d\omega a(k, \omega) e^{i(k_x x + k_z z)} d\omega d\text{m},
\]

(20)

where \( s \) is the sign of \( \omega \) and the amplitudes satisfy the reality condition \( a(k, \omega) = a^*(k', -\omega) \). We equate \( w(-d) \) with \( w_{\text{vm}} \) at \( z = -d \) (and thus neglect the tiny velocity induced by surface waves) and obtain

\[
a(k, \omega) = W(k, \omega),
\]

(21)

which is the relation between the wave amplitude and the forcing fluctuations. Only frequencies in the range \( f < \omega < N_d \) (and the negative mirror of the interval) are allowed for propagation. The response at frequencies outside this interval is evanescent (\( m \) becomes imaginary) and must be dissipated locally. This part of the response is disregarded in the present approach. It might, however, be important for mixing on the mixed layer base.

We define the spectrum \( \mathcal{H}(k, \omega) \) of \( w_{\text{vm}}(z = -d) \) by

\[
\langle a(k, \omega) a^*(k', \omega') \rangle = \delta(k - k') \delta(\omega - \omega') \mathcal{H}(k, \omega),
\]

(22)

with \( \mathcal{H}(k, \omega) = \langle W(k, \omega) W^*(k, \omega) \rangle \), so that the energy of the vertical velocity of the propagating waves at the mixed layer base becomes

\[
\frac{1}{2} \langle w_{\text{vm}}^2 \rangle |_{z = -d} = \frac{1}{2} \int d^2 k \int_{f}^{N_d} d\omega [\mathcal{H}(k, \omega) + \mathcal{H}(k, -\omega)].
\]

(23)

The relation to the spectrum of the total (mechanical) energy of downward propagating waves at \( z = -d \) is thus [see, e.g., Olbers et al. 2012, their Eq. (7.22)]

\[
\mathcal{E}^< (k, \omega) = \frac{1}{2} \frac{N_d^2 - f^2}{2 \omega^2 - f^2} [\mathcal{H}(k, \omega) + \mathcal{H}(k, -\omega)].
\]

(24)

The vertical flux of energy is \( \langle w_{\text{vm}} P_{\text{vm}} \rangle |_{z = -d} \) with the spectral form \( c_{\text{vert}} \mathcal{E}^< (k, \omega) \), where

\[
c_{\text{vert}} = \frac{\partial \omega}{\partial m} = -\frac{1}{k \omega} \frac{(\omega^2 - f^2)^{3/2} (N_d^2 - \omega^2)^{1/2}}{N_d^2 - f^2}
\]

(25)

is the vertical group velocity of downward propagating waves. We thus arrive at

\[
\Phi_{\text{tot}} = \langle w_{\text{vm}} P_{\text{vm}} \rangle |_{z = -d} = \int d^2 k \int_{f}^{N_d} d\omega \Phi(k, \omega)
\]

(26)
with

$$
\Phi(k, \omega) = -\frac{1}{2} \frac{(\omega^2 - f^2)^{1/2}(N_d^2 - \omega^2)^{1/2}}{\omega k} \times [\mathcal{F}(k, \omega) + \mathcal{F}(k, -\omega)]
$$

(27)

for a general surface wave spectrum, with $D(-k, k_1, -k_2) = D_{kk_1k_2}$. We will evaluate $\mathcal{F}$ and the flux $\Phi$ in circular coordinates $(k, \varphi)$ instead of the wavenumber vector $k$. The surface wave spectrum $F(k)d^2k = F(k, \varphi)dkd\varphi$ is transformed accordingly, that is, $F(k) = F(k, \varphi)/k$. Hence,

$$
\mathcal{F}(k, \varphi, \omega) = \int dk_1d\varphi_1 |D(-k, k_1, k - k_1)|^2F(k_1, \varphi_1)F(k_2, \varphi_2)\frac{1}{2\pi}\delta(\omega - \omega_1 + \omega_2).
$$

(30)

The way to calculate the pumping velocity spectrum $\mathcal{F}$ from a general and a factorized surface wave spectrum is detailed in appendix B.

3. The energy flux for specific wind-sea spectra

In principle, it is possible to calculate the energy flux [(29)] for any given surface wave spectrum globally for any given time. However, the global distribution of the spectrum is unknown to us and is in fact not needed in that detail. Instead, we approximate the surface wave spectrum following standard methods. The surface wave spectrum $F(k, \varphi)$ is factorized with respect to wave-number and angular dependence:

$$
F(k, \varphi) = \left(F_0/k_m\right)G(k/k_m)A(\varphi, k/k_m),
$$

(31)

where $F_0 = g\langle\zeta^2\rangle$ is the total energy, $\zeta$ is the sea surface elevation due to the surface wave, and $k_m$ is a fixed wavenumber characterizing the peak of the spectrum. The function $G(x)$, describing the spectral shape, is dimensionless and normalized by integration over $x = k/k_m$, and the directional distribution $A(\varphi, x)$ is normalized to one by integration over $\varphi$. A prototype of a wind-sea spectral shape $G(x)$ is that of Pierson and Moskowitz (1964). It applies to the wind-sea part of the surface waves, that is, the part of the spectrum that is in equilibrium with the local wind. Here, we use the modification of this spectrum by Hasselmann et al. (1973), which is appropriate for a growing sea state, also called the Joint North Sea Wave Project (JONSWAP) spectrum:

$$
G(x) = (5/2)x^{-3}e^{-5/4x^2 - i\ln e\exp(-\sqrt{s(1 - 3/2)x^2})} / \nu,
$$

(32)

where $s = s_a$ for $x < 1$ and $s = s_p$ for $x > 1$, and $\nu$ is a normalization constant. The form of the Pierson-Moskowitz spectrum is obtained for $c = 1$ and $\nu = 1$, while typical parameters for the JONSWAP spectrum are $c = 3.3, s_a = 0.07, s_b = 0.09, \nu = 1.4143$ (Hasselmann et al. 1976). As directional distribution we use

$$
A(\varphi) = \frac{1}{2}\sigma \coth(\sigma\pi) \text{sech}^2(\sigma\varphi).
$$

(33)

A simple parameterization for the expectation of the mean surface displacement is $\langle(\zeta^2)\rangle = F_0g = (a/5)k_m^{-2}$ with the Phillips constant $a = 8.1 \times 10^{-3}$. The wave-number $k_m$ or the equivalent frequency $\omega_m = (gk_m)^{1/2}$ is parameterized in terms of the wind speed $U$ at 10 m height as $k_m = 0.77gU^2$, which leads to

$$
\langle(\zeta^2)\rangle = 2.8 \times 10^{-3} \left(\frac{U}{\text{m s}^{-1}}\right)^4 \text{ m}^2.
$$

(34)

These parameterizations can be derived from Hasselmann et al. (1976) and Rosenthal (1986).

With these parameterizations in terms of the wind speed, we follow OH79 and also W90 and W94, who use the Donelan spectrum (Donelan et al. 1985), an offspring of JONSWAP. We are aware that these spectra, developed for growing seas, are rather fetch limited than
equilibrium, and that the \( U^4 \) scaling of an equilibrium spectrum would not apply in these cases, but rather \( U^4 \) times dimensionless fetch \( \bar{x} = x/(U^2/g) \) (\( x \) is the actual fetch). The problem is that the fetch is basically unknown in the open ocean (\( \bar{x} \) is of order \( 10^{-4} \), large values apply to the equilibrium scaling). Since the Pierson–Moskovitz form is only rarely observed (K. Hasselmann 2015, personal communication), we decided to use the JONSWAP form together with the equilibrium wind scaling.

We evaluate the flux spectrum \( \Phi(k, \phi, \omega) \) from the surface wave spectrum by numerical integration. Figure 1a shows the resulting flux\(^3 \) \( \Phi(k, \omega) = \int \Phi(k, \phi, \omega) \, d\phi \), integrated over the horizontal direction, for a typical example using the JONSWAP spectrum [\( (32) \)] with an anisotropic angular distribution [\( (33) \)] with \( \sigma = 3.5 \) for a wind speed \( U = 13.7 \text{ m s}^{-1} \) where \( \langle \xi^2 \rangle = 1 \text{ m}^2 \) and \( k_m = 0.04 \text{ m}^{-1} \) (wavelength 156 m), mixed layer depth is \( d = 100 \text{ m} \), and the Brunt–Väisälä frequency at the mixed layer base is \( 0.0157 \text{ s}^{-1} \) (9 cph). The internal wave excitement is predominantly occurring for high frequencies (\( \omega \cong N_d/2 \)) and long wave lengths (\( k \ll k_m \)). Figure 1b shows the flux as density in frequency space, \( \Phi(\omega) = \int \Phi(k, \phi, \omega) \, dk \). The angular distribution is not shown. For a surface wave spectrum with narrow angular support, the internal wave radiates at almost right angles to the dominant direction of the surface waves. The total flux \( \Phi_{tot} \) integrated over all wavenumbers and frequencies for these parameters is \( 0.025 \text{ mW m}^{-2} \), a relatively small rate compared to, for example, the wind-stress-driven energy transfer to internal waves that is typically 0.1 to 1 mW m\(^{-2} \) (e.g., Rimac et al. 2013). However, as in OH79, it turns out that the flux strongly depends on the environmental parameters. Figure 2 shows the dependency of the total flux \( \Phi_{tot} \) on the wind speed \( U \), the mixed layer depth \( d \) and the Brunt–Väisälä frequency \( N_d \). Shallow mixed layers with strong Brunt–Väisälä frequency and strong winds lead to a larger excitement of internal waves. For instance, the total flux can become as large as \( 0.13 \text{ mW m}^{-2} \) for \( U = 13.7 \text{ m s}^{-1}, d = 100 \text{ m}, \text{ and } N_d = 0.026 \text{ s}^{-1} \) (15 cph). One has to keep in mind, however, that wind speed and mixed layer depth are related to each other and that stronger winds usually imply deeper mixed layers. This points toward the need to use a consistent dataset for \( U, d, \text{ and } N_d \), which we will take below from a numerical ocean model simulation.

The black lines in Fig. 2 demonstrate that

\[
\Phi_{tot}(U, d, N_d) = \Phi_0 \frac{u^7 n^4}{1 + (d/d_0)^4} \quad \text{with } u = \frac{U}{10 \text{ m s}^{-1}},
\]

\[
n = \frac{N_d}{0.0158 \text{ s}^{-1}},
\]

where \( d_0 = 150 \text{ m} \) and \( \Phi_0 = 2.83 \times 10^{-3} \text{ mW m}^{-2} \) provides a reasonable bulk fit to the numerically obtained data. This bulk fit for the flux \( \Phi_{tot} \) is motivated by a nondimensional version of the integral [\( (28) \)]. The flux spectrum can be written as \( \Phi(k, \omega) = F_0^2 \omega_m N_d g \times \Gamma(k d, \omega/N_d, \omega^2/gk) \), where \( \Gamma \) is a dimensionless function of its arguments. The dimensional factor in front of \( \Gamma \) is then proportional to \( U^7 \sim u^7 \). The proportionality \( N_d^4 \) may be
derived from the interaction coefficient; the dependence on \( d \) is a heuristic fit\(^4\) to curves in Fig. 2. In Fig. 3 all numerically evaluated data for \( F_{\text{tot}} \) from Fig. 2 are compared with the bulk fit [(35)], which shows reasonable agreement.

Analogous results are also found by OH79 using the modal framework; they analyzed the transfer to deep-water internal wave modes from wind-sea spectra as function of wind speed and stratification using different Brunt–Väisälä frequency profiles. High-frequency internal wave modes with wavelengths that are large compared to the ones at the surface waves are excited. They also find that the transfer is very small in most cases. If the mixed layer is shallow, the wind speed and the Brunt–Väisälä frequency are large, internal waves are generated at a level of 0.1 mW m\(^{-2}\). The transfer to modes of a Brunt–Väisälä frequency profile without a strong seasonal thermocline (small \( N \) under the mixed layer) is up to three orders of magnitude smaller (see Table 1 in OH79). The parametric dependence \(-U^n\) is also found. These findings are confirmed in new computations of the modal transfer for an exponential stratification with a mixed layer on top. Details are given in appendix C.

The bulk fit of (35) can now be used to estimate the global energy flux to the internal wave field. To work with a mixed layer depth that is dynamically consistent with the wind speed, we use a global ocean circulation model with high vertical resolution and high-frequency wind forcing. The ocean model is identical to the energetically consistent model used in Eden et al. (2014; their experiment CONSIST-SURF), which contains a mixed layer parameterization after Gaspar et al. (1990). Wind stress forcing and \( U \) is taken from the year 2010 of the reanalysis by Kalnay et al. (1996) with a resolution of 6 h and replaces the climatological wind stress forcing with monthly frequency in a 500-yr-long spinup period of the model. The wind stress is also used to calculate the surface boundary condition of the mixed layer model by Gaspar et al. (1990), that is, to parameterize the input of turbulent kinetic energy by the wind. More details on the model setup and performance can be found in Eden et al. (2014).

Figure 4 shows the annual mean total flux \( F_{\text{tot}} \) and its maximum during the year entering the internal wave field diagnosed from the simulated mixed layer depth and Brunt–Väisälä frequency at the mixed layer base,

---

\(^4\) As pointed out by a reviewer, the interaction coefficient should lead to an exponential dependence and, in fact, replacement of \(1/(1 + d^2/d_0^2)\) by \(\exp(-a d n)\) yields an equally good fit; however, different \( a \) are required for the range of high and low fluxes. A fit with constant \( a \) is worse than (35). Apparently, \( a \) depends on \( N_d \) and \( d \), but the functional form remains unknown and could not be retrieved.
and the wind speed $U$ taken from the reanalysis product. The largest fluxes show up in the storm track regions of the oceans; while toward the equator, the flux and its maximum almost vanish. The flux varies a lot in space and time: Fig. 5a shows a time series of the flux $F_{\text{tot}}$ in the Pacific Ocean along 160°W, demonstrating that it is the large wind speeds during fall and early winter in mid-to high latitudes in both hemispheres that drives the flux maxima. The globally integrated annual mean flux is $6.9 \times 10^{-2} \text{TW}$, and it varies from its minimum of $3.7 \times 10^{-4} \text{TW}$ during July/August to its maximum of $8.2 \times 10^{-4} \text{TW}$ in March and $1.2 \times 10^{-3} \text{TW}$ in September/October. Even the maximal globally averaged values are more than an order of magnitude smaller than the flux supported by wind fluctuations at near-inertial frequencies. In Rimac et al. (2016) the wind input to near-inertial energy in the mixed layer is estimated as 0.34 TW, of which 90% dissipate in the mixed layer and only 10% radiate out, that is, $3.4 \times 10^{-2} \text{TW}$ leave the mixed layer and are available for interior mixing.

![Fig. 4](image1)

![Fig. 5](image2)
The spectral shape of the flux in the frequency domain, shown in Fig. 1, can be well approximated by the function

\[
\Phi(k, \varphi, \omega) dk d\varphi = n_\varphi \left( \frac{\omega^2 - f^2}{\omega} \right)^{1/2} \left( N_d^2 - \omega^2 \right)^{1/2} \times \frac{\omega^2 - f^2}{\omega^2 - f^2 + 4N_d^2},
\]

where \(N_d\) denotes the Brunt–Väisälä frequency at the base of the mixed layer and \(n_\varphi\) is a normalization factor that sets \(\int \Phi d\omega dk d\varphi = \Phi_{\text{tot}}\). Motivation to use this form comes from the group velocity in (27) and the frequency dependency of the interaction coefficient (19). Using this spectral shape, it is possible to calculate the depth \(z_0 = -b(\omega)\) below which internal wave propagation stops. This depth is given by \(\omega = N(z_0)\) where the waves at the frequency \(\omega\) have their turning point. Assuming that the flux \(\Phi(\omega)\) from the mixed layer is dissipated equally over the depth range \(b-d\), it becomes possible to calculate the total interior wave dissipation \(F_{\text{diss}}\) by integration in frequency domain. The turning point \(b\) depends on \(\omega\) but is typically only 50 to 400 m below the mixed layer (not shown) for the Brunt–Väisälä frequency profiles of the model. Figure 5b shows the annual mean interior dissipation of internal waves \(F_{\text{diss}}\) along 160°W generated by the flux \(\Phi_{\text{tot}}\) during 2010, shown as a time series in Fig. 5a. Along 160°W, the flux \(\Phi_{\text{tot}}\) is large in the northern North Pacific and the Southern Ocean compared to other longitudes. Enhanced values of \(F_{\text{diss}}\) are restricted close to the mixed layer with values between \(10^{-10}\) and \(10^{-9}\) m² s⁻³, comparable to the low range of observational estimates of mixing rates below the mixed layer (Kunze et al. 2006; Waterhouse et al. 2014). In case of strong storm events in combination with shallow mixed layers, for example, during fall in the northern North Pacific or during February to April in the Southern Ocean, the vertical maximum of \(F_{\text{diss}}\) can locally reach values up to \(10^{-8}\) to \(10^{-7}\) m² s⁻³ along the section, as seen in Fig. 5c. Using the Osborn–Cox relation \(K = \delta/(1 + \delta)F_{\text{diss}}/N^2\) (Osborn 1980) with the mixing efficiency \(\delta \approx 0.2\), it is possible to calculate the vertical diffusivity \(K\) from the internal wave dissipation. The diffusivity \(K\) calculated from the mean \(F_{\text{diss}}\) remains low, but for large fluxes \(\Phi_{\text{tot}}\), the diffusivity calculated from the instantaneous \(F_{\text{diss}}\) locally reaches values of \(10^{-5}\) to \(10^{-4}\) m² s⁻¹, as shown in Fig. 5d.

### 4. Summary and discussion

The transfer of energy from a surface wave field to internal waves below a mixed layer is revisited here, building on previous work by OH79, but using a simpler framework. It is based on matching the vertical velocity in the mixed layer generated by the nonlinear forcing of surface waves with the vertical velocity of an internal wave propagating from the mixed layer base into the interior. The generated energy flux by this radiation process can be interpreted as being induced by surface wave pumping, generated by the wave-induced vertical velocity in the surface layer. The physics is analogous to Ekman pumping, induced by the divergence of the wind stress, and inertial pumping, induced by the divergence of wind-generated near-inertial motions in the mixed layer (Gill 1984). However, unlike Ekman or inertial pumping, the surface wave pumping is due to triad interactions between two surface waves and one internal wave, like modal triad interaction of internal waves in the interior.

Since surface waves may have an order of magnitude larger energy content than internal gravity waves, such an interaction is potentially a large source for the internal wave field. The coupling of the wave types is, however, rather weak, characterized by an interaction coefficient of order \(\omega^2/\ell k\) that enters the spectral transfer integral as square. The ratio reflects the large spectral gap between the internal wave frequency \(\omega\) and the surface wave frequency \(\sqrt{\ell k}\). On the other hand, the energy flux \(\Phi_{\text{tot}}\) depends quadratically on the surface wave energy. As a result, the local wind speed \(U\) enters as the seventh power in our expression for \(\Phi_{\text{tot}}\), using standard dependency of the wind sea spectra from \(U\), while the Brunt–Väisälä frequency \(N_d\) at the mixed layer base enters the flux via its fourth power. We have derived the flux \(\Phi_{\text{tot}}\) radiating from the mixed layer base in analytical form and also fitted the parametrical dependency in terms of \(U, N_d\) at the mixed layer base, and the mixed layer depth \(d\). The basic results of OH79 are confirmed, also by new calculations of the modal energy transfer described in appendix C, which, besides the spontaneous creation process (SC) treated in the main part of this study, also includes the modulation interaction process (MI; W90, W94). We have shown that the importance of SC overcomes MI at wind speeds around 10 m s⁻¹. For lower wind speeds, both processes have only minor transfer rates at all.

Using a model-based estimate of the flux \(\Phi_{\text{tot}}\) demonstrates that, in the global integral, it is only about \(0.5–1 \times 10^{-3}\) TW, that is, two orders of magnitude smaller than the flux due to the inertial pumping. Locally in space and time, however, it can reach similar magnitudes. Since the internal waves generated by interaction with surface waves are of high frequency, their turning point lies close to the mixed layer, such that it is likely that the flux \(\Phi_{\text{tot}}\) is also dissipated close to the mixed layer. This is different from the fluxes due to inertial pumping.
and the tides that generate low-frequency waves penetrating the entire depth range of the ocean. Our estimate of the implied dissipation shows that it sometimes reaches magnitudes comparable to observational estimates close to the mixed layer, in particular during strong wind events, but in general stays below them.

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APPENDIX A

Solution for the Pumping Velocity

The solution from (6) with the boundary conditions of (7) and \( \dot{w}(-d) = W \) is written in terms of Green's function for the problem with homogeneous boundary conditions. We thus define \( \dot{w} = \dot{w} + az + b \), where \( \dot{w} \) satisfies

\[
\frac{\partial^2 \dot{w}}{\partial z^2} - q^2 \dot{w} = q^2 (az + b). \quad (A1)
\]

Note that we use \( \dot{Q} = 0 \) for the surface wave forcing. Homogeneous boundary conditions for \( \dot{w} \) are obtained for

\[
a = \frac{\gamma W + \zeta_0}{1 - \gamma d} \quad \text{and} \quad b = \frac{W + d\zeta_0}{1 - \gamma d}, \quad (A2)
\]

where \( \zeta_0 = -\dot{Q}/(\omega^2 - f^2) \). The solution of (A1) is written as

\[
\dot{w}(z) = q^2 \int_{-d}^{0} d\xi G(z, \xi)(az + b), \quad (A3)
\]

with Green's function

\[
G(z, \xi) = \frac{1}{q(\lambda \sinh qd - \cosh qd)} \left\{ \sinh q(z + d)(\cosh q\xi + \lambda \sinh q\xi) \right\} \frac{(\cosh qz + \lambda \sinh qz) \sinh q(\xi + d)}{\sinh qd} \quad \text{for} \quad -d \leq z \leq \xi \leq 0, \quad (A4)
\]

which leads to expression (16) for \( W \).

APPENDIX B

Derivation of the Spectrum of the Pumping Velocity

To calculate the energy flux \( \Phi_{\text{bot}} \) radiating out of the mixed layer, we need the spectrum of the pumping velocity \( \Phi (\mathbf{k}, \omega) \). From (18) the squared modulus of \( W \) becomes

\[
\Phi^2 (\mathbf{k}, \omega) = |W(\mathbf{k}, \omega)W(\mathbf{k}, \omega)^*| = \sum_{\mu_1\mu_2\mu_3\mu_4} D_{\mathbf{k}k_1, \mathbf{k}k_2}^{-\mu_1\mu_2}(D_{\mathbf{k}k_1, \mathbf{k}k_2}^{-\mu_1\mu_2})^* \langle a_1 a_2 a_3 a_4^* \delta(\mathbf{k} - \mathbf{k}_1, \mathbf{k}_2) \delta(\omega - \omega_1, \omega_2) \delta(\mathbf{k} - \mathbf{k}_3, \mathbf{k}_4) \delta(\omega - \omega_3, \omega_4). \quad (B1)
\]

The fourth-order moment can be broken into products of second-order moments by the Gaussian assumption, \( \langle a_1 a_2 a_3 a_4^* \rangle = \langle a_1 a_2 \rangle \langle a_3 a_4^* \rangle + \langle a_1 a_3 \rangle \langle a_2 a_4^* \rangle + \langle a_1 a_4 \rangle \langle a_2 a_3^* \rangle \). The first term does not contribute here because \( \langle a_1 a_2 \rangle = \langle a_1 a_2^* \rangle \) implies from (14) that \( \mathbf{k}_1 + \mathbf{k}_2 = 0 \), and for \( \mathbf{k} = 0 \) the interaction coefficient vanishes. For the second term we have \( 1 = 3, 2 = 4 \), and for the third \( 1 = 4, 2 = 3 \). Since \( D \) is symmetric in the last two arguments, we obtain for these terms the same result (this yields a factor 2). Inserting the surface wave spectrum, we arrive at

\[
\Phi^2 (\mathbf{k}, \omega) = \frac{1}{2} \sum_{\mu_1\mu_2} |D_{\mathbf{k}k_1, \mathbf{k}k_2}^{-\mu_1\mu_2}|^2 F_1 F_2 \delta(\mathbf{k} - \mathbf{k}_1, \mathbf{k}_2) \delta(\omega - \omega_1, \omega_2). \quad (B2)
\]

The sum over \( \mu_1 \) and \( \mu_2 \) must be written out. On the resonance surface where the interaction coefficient is given by (19), there is no dependence of \( D \) on the frequency indices \( \mu_1 \) and \( \mu_2 \). The expression is needed for positive and negative \( \omega \).
Consider first $\omega > 0$. The term from $\mu_1 = \mu_2 = -$ does not contribute because the resonance condition cannot be satisfied. Hence, using the convention of (14), we find

$$\mathcal{H}\left(\mathbf{k}, \omega\right) = \frac{1}{2} \int d^2k_1 \left[ d^2k_2 F(k_1)F(k_2) \right. $$

$$\times \{ \left| D_{++}^{++} \right|^2 \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) \delta(\omega - \omega_1 - \omega_2) + 2 \left| D_{+-}^{-+} \right|^2 \delta(\mathbf{k} - \mathbf{k}_1 + \mathbf{k}_2) \delta(\omega - \omega_1 + \omega_2) \}, \quad (B3)$$

where the surface wave spectrum is introduced and the frequencies $\omega, \omega_1,$ and $\omega_2$ are now to be taken as positive. The first term in the curly brackets will not contribute because the condition $\omega = \omega_1 + \omega_2$ cannot be satisfied for surface waves that have frequencies $\omega_n \gg \omega$. For negative $\omega$ the combination $\mu_1 = \mu_2 = +$ does not contribute, and we obtain a similar expression for $\mathcal{H}\left(\mathbf{k}, -\omega\right)$. It turns out that the squared interaction coefficient does not depend on the sign of $\omega$ and the frequency indices $\mu_1$ and $\mu_2$, so that $\mathcal{H}\left(\mathbf{k}, \omega\right) = \mathcal{H}\left(\mathbf{k}, -\omega\right)$. The spectrum is then expressed by (28). For the evaluation we factorize the surface wave spectrum as in (31). The integral then finally becomes

$$\mathcal{H}(k, \varphi, \omega) = \left( \frac{F_0}{k_m} \right)^2 \int_{k_{min}}^{k_{max}} dk_1 [D(-\mathbf{k}, \mathbf{k}_1, -\mathbf{k}_2) \right|^2 G(x_1)G(x_2) \frac{1}{k_2} \left. \frac{d\omega_2}{d\varphi_1} \right]^{-1} \sum_{\text{sign}(\varphi - \varphi_0)} A(\varphi_1, x_1)A(\varphi_2, x_2). \quad (B4)$$

Further details, such as the functions $k_{1, min}, k_{1, max}, \varphi_1, \varphi_2$ and $d\omega_2/d\varphi_1$ and the method of integration (the integrand becomes singular on the boundaries of integration), are found in OH79. We evaluate this form of the integral numerically to calculate the energy flux spectrum and the total flux $\Phi_{tot}$ radiating out of the mixed layer in (26) and (27).

### APPENDIX C

#### The Modal Treatment of SC and MI Mechanisms

In this section we briefly review the modal transfer between internal waves (IW) and surface waves (SW) of OH79 and compare with W90 and W94. The modal analysis of OH79 results in the transfer rate for the spontaneous creation process (SC) to mode $n$, given by

$$S_{SC}(k) = \int d^2k_1 F(k_1)F(k_1 - k)T \delta(\omega_1 - \omega_2 - \omega), \quad (C1)$$

with the scattering cross section

$$T = 2\pi(1 + \cos \alpha)^2 \left[ \int_{-\infty}^{0} dz \frac{N^2(z)}{g} \phi_{k_0}(z)\phi_1(z)\phi_2(z) \right.$$

$$\left. + (\phi_{k_0}^* / g)\phi_1^* \phi_2^* \right]^{2}$$

The modulation interaction process (MI) is then represented by (see OH79 or W94)

$$S_{MI}(k) = E_n(k) \int d^2k_1 [\omega_2 F(k_1) - \omega_1 F(k_1 - k)] T \delta(\omega_1 - \omega_2 - \omega) = \lambda_n(k)E_n(k). \quad (C2)$$

The IW mode is normalized according to $\int N^2 \phi_k \phi_{k'} dz = \omega^2$ and the SW mode is given by $\phi_\xi = \sqrt{k} e^{ikz}$. Variable $\alpha$ is the angle between $\mathbf{k}$ and $\mathbf{k}_1$. As in the main text, $F(\mathbf{k})$ is the SW energy spectrum with integral $g(\xi^2)$ (contrary to Watson’s notation, where $F$ denotes the SW action spectrum). We have repeated an SC case from OH79 with constant Brunt–Väisälä frequency and found reasonable agreement.

The new calculations for SC and MI reported are for an exponential Brunt–Väisälä frequency $N(z) = N_0 \exp[(z + d)/b]$ with a mixed layer $0 > z > -d$ on top, where $N(z) = 0$. In all applications in this section, we take $d = 100$ m and $b = 1300$ m. The IW eigenfunctions are exponential $\text{Asinh} kz$ in the mixed layer and Bessel functions $\mathcal{J}_\kappa(\xi) = B[J_\kappa(\xi) - Y_\kappa(\xi)J_\kappa(\xi)/Y_\kappa(\xi)]$ with $\xi = kbN(z)/\omega, \kappa = kb$, and $\xi_2 = \xi(-H) = \kappa N(-H)/\omega$. The amplitudes $A$ and $B$ are related by continuity. The dispersion relation $(N_0/\omega)\tan \kappa k d \mathcal{J}_\kappa(\xi) = -\mathcal{J}_\kappa(\xi_1)$ with $\xi_1 = \xi(-d) = \kappa N_0/\omega$ follows by continuity of the derivative of the mode and is solved numerically. For this exponential model, the cross section becomes
Fig. C1. Transfer for SC mechanism for an anisotropic JONSWAP spectrum with $\sigma = 3.5$ and $U = 13.7 \text{ m s}^{-1}$, and a Brunt–Väisälä frequency $N_0 = 0.0157 \text{ s}^{-1}$ (same conditions as for Fig. 1). (a) The SC transfer as function of scaled wavenumber for modes 1 (blue) and higher. (b) The modal dependence of the flux, integrated over wavenumbers. (c) Dispersion relation of the first 10 modes.

$$T = 2\pi (1 + \cos \alpha)^2 k_1 k_2 B^2 g^4 \left[ -\frac{\mathcal{T}_\kappa (\xi_1)}{k \sinh kd} + \frac{b}{k^2} e^{-\eta d} \xi_1 I_{12} \right]^2,$$

(C3)

where

$$B^2 = \frac{\kappa^2 / b}{I(\xi_1) - I(\xi_2)}$$

with

$$I(\xi) = \int d\xi \xi \mathcal{T}_\kappa(\xi).$$

(C4)

and

$$I_{12} = \int_{\xi_1}^{\xi_2} d\xi (\xi \xi_1)^{b+k_1+1} \mathcal{T}_\kappa(\xi).$$

(C5)

The normalization integral $I$ of modes is obtained analytically, whereas the integral $I_{12}$ must be calculated numerically. The corresponding term in the cross section is found to be generally very small and negligible. The cross section given in W94 [their Eq. (26)], though OH79 is used, looks at first fundamentally different; however, implementing Watson’s different mode normalization and some reasonable approximations [$\cos \alpha \approx 0, \omega_1 - \omega_2 \approx \epsilon_g \cdot (k_1 - k_2)$, where $\epsilon_g$ is the group velocity, $k_1 \approx k_2$] we find agreement, except that our cross section is 2 times the one of Watson.

The program to compute the scattering integrals (C1) and (C2) is similar to the one used for the radiation flux form [(B4)] of SC. We compute $S_n^{\text{SC}}$ and $\lambda_n = S_n^{\text{ML}}/E_n$ in polar coordinates $(k, \varphi)$. For SC we display $\Phi_n(k) = \int dk S_n^{\text{SC}}(k, \varphi)$ and $\phi_n = \int dk \Phi_n(k)$. The case shown in Fig. C1 is for the same parameters as Fig. 1. The figure also shows the dispersion relation $\omega_n(k)$ of the first 10 modes obtained during the numerical integration. The radiation form yielded a rate $0.025 \text{ mW m}^{-2}$ (see section 3), the modal form yields $0.022 \text{ mW m}^{-2}$ to the first mode and $0.025 \text{ mW m}^{-2}$ as sum over the 10 modes calculated. Such a perfect agreement is fortuitous; for example, for the same conditions except that $N_0 = 0.0052 \text{ s}^{-1}$, we find $4.5 \times 10^{-4} \text{ mW m}^{-2}$ for the radiative form and $1.8 \times 10^{-4} \text{ mW m}^{-2}$ for the modal form.

Since we will consider the isotropic Garrett and Munk (Garrett and Munk 1972; Munk 1981) spectrum for the
IW energy $E_n(k, \varphi) = E_n(k)/2\pi$, it is sufficient to look for
\(\lambda_n(k) = \int d\varphi \lambda_n(k, \varphi)\). The modal MI transfer is then
completely described by $S_n^\text{MI} = E_n(k)\lambda_n(k)/2\pi$ and
\(S_n^\text{MI} = \int dk E_n(k)\lambda_n(k)/2\pi\). We use the Garrett and Munk
model (GM76; Cairns and Williams 1976) form of the IW spectrum (integrated
over direction) as a function of $n$ and $k$,

\[
E_n(k) = \hat{E} H(n) \frac{2}{\pi} \frac{f N_0^2 m^2}{(N_0^2 k^2 + f^2 m^2)\sqrt{k^2 + m^2}} \quad \text{with}
\]

\[
H(n) = \frac{H_*}{n_*^2 + n^2}, \quad \text{(C6)}
\]

where we assume the WKB form $m = n\pi/b$ for the vertical wavenumber
and $\hat{E} = 4 m^3 s^{-2} = 4 \times 10^3 J m^{-2}$ for the total IW energy. The modal part $H(n)$ is normalized
($H_* = 2.13$ for $n_* = 3$). We show MI transfers in Fig. C2
for $U = 13.7 m s^{-1}$ and $N_0 = 0.0157 s^{-1}$. In the third panel
of the figure, we show the modal energy distribution integrated over all
wavenumbers (+) and integrated for the wavenumber range participating in the SW–IW interaction
(stars and line). As in most other cases considered for the present study, we find $\lambda_n$ to be negative so that IWs are
damped and $S_n^\text{MI}$ is a transfer from IW to SW as in W90
and W94. For the high wind speed and Brunt–Väisälä frequency, the MI transfer shown in Fig. C2 is much
smaller than the corresponding SC transfer: MI has $10^{-3}$
$mW m^{-2}$ for the 10 modes and SC has $0.025 mW m^{-2}$. For
low wind speeds the behavior reverses, as demonstrated in
Fig. C3, showing transfer rates for SC and MI for wind
speeds from 5 to $20 m s^{-1}$ and two values of the Brunt–
Väisälä frequency.

We summarize these results:

- the SC and MI transfer is mainly in mode 1;
- $\lambda_n$, and hence the MI transfer, is overall negative (not
so at very large $N_0$ and very small $U$, not shown) and
IW loses energy to SW;
- the SC transfer behaves as $-U^2$ and MI as $-U^2$, so SC
overcomes MI at a critical wind speed (around
$10 m s^{-1}$ for standard GM);
- both the SC and MI transfer strongly increase with $N_0$,
SC roughly as $N_0^4$ (not shown); and

\[\text{FIG. C2. As in Fig. C1, but for MI mechanism. (a) The decay time for the MI transfer as a function of scaled wavenumbers for modes 1}
\]
The relative importance of SC and MI interactions can be easily assessed from the scattering integrals (C1) and (C2). One obtains for the ratio of MI terms and SC terms

$$\frac{\text{MI}}{\text{SC}} \simeq \frac{E_n \omega_k F_1 - \omega_l F_2}{F_1 F_2} = \frac{E_n A_1 - A_2}{\omega_n A_1 A_2},$$

(C7)

where $E_n$ is the IW spectral energy level and $A_j = F_j / \omega_j$, $j = 1, 2$ is the corresponding SW spectral action level at the respective interacting wavenumbers and frequencies. We proceed with $A_1 - A_2 \simeq k \partial A_1 / \partial k_1$ (because $k_1 \gg k$) and use the steep descent of the SW spectrum $F_j = (F_0 / k_m) \mu (k_1 / k_m)^{3/2}$, $\mu = 5/2 \nu$ (see section 3) to evaluate the derivative. We arrive at $(A_1 - A_2) / (A_1 A_2) \simeq (7/2) k \omega_m / (\mu F_0) (k_1 / k_m)^{5/2}$, which enters the following estimate with $k_1 \simeq k_m$. The IW energy level and frequency is taken in the high-frequency limit $E_n \simeq EH(n) \omega_m / (f / N_0) k_1^2$, $\omega_n \simeq N_0 k / N_0$ with $m_n = n \pi / b$, which then leads to

$$\frac{\text{MI}}{\text{SC}} \simeq \frac{7}{2} \mu \frac{f \omega_m^2}{\nu} \frac{\omega_n}{F_0},$$

(C8)

The final step is implementation of the wind speed parameterizations $\omega_m = 0.88 g / U$, $F_0 = g \beta U_4$ (see section 3), which yields

$$\frac{\text{MI}}{\text{SC}} \simeq \frac{7 \times 0.88}{\mu \nu} \frac{f \omega_m}{\beta \omega_n} U^{-5} = \left(\frac{U_{\text{crit}}}{U}\right)^5.$$  

(C9)

The standard GM parameters and $\beta$ from (34) yield $U_{\text{crit}} = 10 \text{ m s}^{-1}$ for the first mode. Consistent with the results in Fig. C3, the critical speed depends on the mode number and increases with it. For the scenario of W90 and W94, with a higher IW energy and smaller $\beta$, we find a larger $U_{\text{crit}} = 14 \text{ m s}^{-1}$.

REFERENCES


